CREAT. MATH. INFORM.
21 (2012), No. 1, 103 - 114

Online version at `http://creative-mathematics.ubm.ro/`
Print Edition: ISSN 1584 - 286X Online Edition: ISSN 1843 - 441X

Further properties of (γ, γ')-preopen sets

N. RAJESH and V. VIJAYABHARATHI

ABSTRACT. In [Carpintero, C., Rajesh, N. and Rosas, E., *On a class of (γ, γ')-preopen sets in a topological space*, Fasciculi Mathematici, **46** (2011), 25–36], the authors introduced the notion of (γ, γ')-preopeness and investigated its fundamental properties. In this paper, we investigate some more properties of this type of open set.

1. INTRODUCTION

Generalized open sets play a very important role in General Topology and they are now the research topics of many topologists worldwide. Indeed a significant theme in General Topology and Real Analysis concerns the various modified forms of continuity, separation axioms etc. by utilizing generalized open sets. Kasahara [2] defined the concept of an operation on topological spaces and introduce the concept of γ-closed graphs of a function. Umehara et. al. [4] introduced the notion of $\tau_{(\gamma,\gamma')}$ which is the collection of all (γ, γ')-open sets in a topological space (X, τ). In [1] the authors, introduced the notion of (γ, γ')-preopeness and investigated its fundamental properties. In this paper, we investigate some more properties of this type of open set.

2. PRELIMINARIES

Definition 2.1. Let (X, τ) be a topological space. An operation (γ, γ') [2] on the topology τ is function from τ on to power set $\mathcal{P}(X)$ of X such that $V \subset V^\gamma$ for each $V \in \tau$, where V^γ denotes the value of γ at V. It is denoted by $\gamma : \tau \to \mathcal{P}(X)$.

Definition 2.2. Let (X, τ) be a topological space. An operation γ on τ is said to be regular [2] if for every open neighborhood U and V of each $x \in X$, there exists an open neighborhood W of x such that $U^\gamma \cap V^\gamma \supset W^\gamma$.

Definition 2.3. A subset A of a topological space (X, τ) is said to be (γ, γ')-open set [4] if for each $x \in A$ there exist open neighborhoods U and V of x such that $U^\gamma \cup V^{\gamma'} \subset A$. The complement of (γ, γ')-open set is called (γ, γ')-closed. $\tau_{(\gamma,\gamma')}$ denotes set of all (γ, γ')-open sets in (X, τ).

Definition 2.4. [4] Let A be subset of a topological space (X, τ) and γ, γ' be operations on τ. Then

(i) the $\tau_{(\gamma,\gamma')}$-closure of A is defined as intersection of all (γ, γ')-closed sets containing A. That is, $\tau_{(\gamma,\gamma')}\text{-Cl}(A) = \{F : F \text{ is } (\gamma, \gamma')\text{-closed and } A \subset F\}$.
(ii) the $\tau_{(\gamma,\gamma')}$-interior of A is defined as union of all (γ, γ')-open sets contained in A. That is, $\tau_{(\gamma,\gamma')}\text{-Int}(A) = \{U : U \text{ is } (\gamma, \gamma')\text{-open and } U \subset A\}$.

Definition 2.5. A subset A of a topological space (X, τ) is said to be

Received: 17.10.2010. In revised form: 19.12.2011. Accepted: 31.12.2011
2010 *Mathematics Subject Classification.* 54A05, 54A10.
Key words and phrases. *Topological spaces, (γ, γ')-open set, (γ, γ')-preopen set.*

(i) (γ,γ')-regular open [3] if $\tau_{(\gamma,\gamma')}$-$\mathrm{Int}(\tau_{(\gamma,\gamma')}$-$\mathrm{Cl}(A)) = A$.
(ii) (γ,γ')-preopen [1] if $A \subset \tau_{(\gamma,\gamma')}$-$\mathrm{Int}(\tau_{(\gamma,\gamma')}$-$\mathrm{Cl}(A))$.
(iii) (γ,γ')-dense [1] if $\tau_{(\gamma,\gamma')}$-$\mathrm{Cl}(A)) = X$.

The complement of a (γ,γ')-preopen set is called a (γ,γ')-preclosed set. The family of all (γ,γ')-preopen (resp. (γ,γ')-preclosed) sets of (X,τ) is denoted by (γ,γ')-$PO(X)$ (resp. (γ,γ')-$PC(X)$). The family of all (γ,γ')-preopen sets of (X,τ) containing the point x is denoted by (γ,γ')-$PO(X,x)$.

Definition 2.6. [1] Let A be subset of a topological space (X,τ) and γ, γ' be operations on τ. Then

(i) the $\tau_{(\gamma,\gamma')}$-preclosure of A is defined as intersection of all (γ,γ')-preclosed sets containing A. That is, $\tau_{(\gamma,\gamma')}$-$p\,\mathrm{Cl}(A) = \{F : F$ is (γ,γ')-preclosed and $A \subset F\}$.
(ii) the $\tau_{(\gamma,\gamma')}$-preinterior of A is defined as union of all (γ,γ')-preopen sets contained in A. That is, $\tau_{(\gamma,\gamma')}$-$p\,\mathrm{Int}(A) = \{U : U$ is (γ,γ')-preopen and $U \subset A\}$.

Theorem 2.1. [1] *Let A be subset of a topological space (X,τ) and γ, γ' be operations on τ. Then*

(i) A is (γ,γ')-preopen if and only if $A = \tau_{(\gamma,\gamma')}$-$p\,\mathrm{Int}(A)$.
(ii) A point $x \in \tau_{(\gamma,\gamma')}$-$p\,\mathrm{Cl}(A)$ if and only if $U \cap A \neq \emptyset$ for every $U \in (\gamma,\gamma')$-$PO(X,x)$.
(iii) $\tau_{(\gamma,\gamma')}$-$p\,\mathrm{Cl}(A)$ is the smallest (γ,γ')-preclosed subset of X containing A.
(iv) A is (γ,γ')-preclosed if and only if $A = \tau_{(\gamma,\gamma')}$-$p\,\mathrm{Cl}(A)$.
(v) $\tau_{(\gamma,\gamma')}$-$p\,\mathrm{Int}(X\backslash A) = X\backslash\tau_{(\gamma,\gamma')}$-$p\,\mathrm{Cl}(A)$.
(vi) $\tau_{(\gamma,\gamma')}$-$p\,\mathrm{Cl}(X\backslash A) = X\backslash\tau_{(\gamma,\gamma')}$-$p\,\mathrm{Int}(A)$.

Definition 2.7. [1] Let (X,τ) be a topological space and γ, γ' be operations on τ. Then a subset A of X is said to be (γ,γ')-pre g.closed (written as (γ,γ')-pg.closed) set if $\tau_{(\gamma,\gamma')}$-$p\,\mathrm{Cl}(A) \subset U$ whenever $A \subset U$ and U is (γ,γ')-preopen.

Theorem 2.2. [1] *Let (X,τ) be a topological space and γ, γ' be operations on τ. Then subset A of X is (γ,γ')-pg.closed, then $\tau_{(\gamma,\gamma')}$-$p\,\mathrm{Cl}(A) \setminus A$ does not contain any nonempty (γ,γ')-preclosed set.*

3. Properties of (γ,γ')-preopen sets

Through this paper, the operators γ and γ' are defined on (X,τ) and the operators β and β' are defined on (Y,σ).

Theorem 3.3. *For any subset of a space (X,τ) the following are equivalent:*

(i) $S \in (\gamma,\gamma')$-$PO(X)$.
(ii) There is a (γ,γ')-regular open set $G \subset X$ such that $S \subset G$ and $\tau_{(\gamma,\gamma')}$-$\mathrm{Cl}(S) = \tau_{(\gamma,\gamma')}$-$\mathrm{Cl}(G)$.
(iii) S is the intersection of a (γ,γ')-regular open set and a (γ,γ')-dense set.
(iv) S is the intersection of a (γ,γ')-open set and a (γ,γ')-dense set.

Proof. (i)⇒ (ii): Let $S \in (\gamma,\gamma')$-$PO(X)$. Then $S \subset \tau_{(\gamma,\gamma')}$-$\mathrm{Int}(\tau_{(\gamma,\gamma')}$-$\mathrm{Cl}(S))$. Let $G = \tau_{(\gamma,\gamma')}$-$\mathrm{Int}(S)$. Then G is (γ,γ')-regular open with $S \subset G$ and $S = G$.

(ii)⇒ (iii): Let $D = S \cup (X\backslash G)$. Then D is (γ,γ')-dense and $S = G \cap D$.

(iii)⇒ (iv): This is trivial.

(iv)⇒ (i): Suppose $S = G \cap D$ with G is (γ,γ')-open and D (γ,γ')-dense. Then $S = G$, hence $S \subset G \subset \tau_{(\gamma,\gamma')}$-$\mathrm{Cl}(G) = S$. □

Theorem 3.4. *If every subset of X is either (γ,γ')-open or (γ,γ')-closed, then every (γ,γ')-preopen set in X is (γ,γ')-open.*

Proof. Let A be a (γ,γ')-preopen in X. If A is not (γ,γ')-open, then A is (γ,γ')-closed by hypothesis. Hence $A=\tau_{(\gamma,\gamma')}$-Cl$(A)$, and $\tau_{(\gamma,\gamma')}$-Int$(\tau_{(\gamma,\gamma')}$-Cl$(A))=\tau_{(\gamma,\gamma')}$-Int$(A)$ is a proper subset of A. Thus, $A\not\subseteq\tau_{(\gamma,\gamma')}$-Int$(\tau_{(\gamma,\gamma')}$-Cl$(A))$, so that A is not (γ,γ')-preopen, contradiction. □

Theorem 3.5. *Let (X,τ) be a topological space in which every (γ,γ')-preopen set in X is (γ,γ')-open. Then each singleton in X is either (γ,γ')-open or (γ,γ')-closed.*

Proof. Let $x\in X$, and suppose that $\{x\}$ is not (γ,γ')-open. Then $\{x\}$ is not (γ,γ')-preopen. Hence $\{x\}\not\subseteq\tau_{(\gamma,\gamma')}$-Int$(\tau_{(\gamma,\gamma')}$-Cl$(\{x\})$, so that $\tau_{(\gamma,\gamma')}$-Int$(\tau_{(\gamma,\gamma')}$-Cl$(\{x\})=\emptyset$. We have that $\tau_{(\gamma,\gamma')}$-Int$(\tau_{(\gamma,\gamma')}$-Cl$(X\backslash\{x\})\supset\tau_{(\gamma,\gamma')}$-Int$(\tau_{(\gamma,\gamma')}$-Cl$(X\backslash(\tau_{(\gamma,\gamma')}$-Cl$(\{x\})))=\tau_{(\gamma,\gamma')}$-Int$(X\backslash(\tau_{(\gamma,\gamma')}$-Int$(\tau_{(\gamma,\gamma')}$-Cl$(\{x\}))))=X\supset X\backslash\{x\}$. Thus, $X\backslash\{x\}$ is (γ,γ')-preopen and hence (γ,γ')-open. Therefore, $\{x\}$ is (γ,γ')-closed. □

Theorem 3.6. *For a topological space (X,τ) and γ, γ' be regular operations on τ, the following are equivalent:*

(i) Every (γ,γ')-preopen set is (γ,γ')-open.
(ii) Every (γ,γ')-dense set is (γ,γ')-open.

Proof. (i)⇒ (ii): Let A be a (γ,γ')-dense subset of X. Then $\tau_{(\gamma,\gamma')}$-Int$(\tau_{(\gamma,\gamma')}$-Cl$(A)=X$, so that $A\subset\tau_{(\gamma,\gamma')}$-Int$(\tau_{(\gamma,\gamma')}$-Cl$(A))$ and A is (γ,γ')-preopen. Hence A is (γ,γ')-open.

(ii)⇒ (i): Let B be a (γ,γ')-preopen subset of X, so that $B\subset\tau_{(\gamma,\gamma')}$-Int$(\tau_{(\gamma,\gamma')}$-Cl$(B))=G$, say. hen $\tau_{(\gamma,\gamma')}$-Cl$(B)=\tau_{(\gamma,\gamma')}$-Cl$(G)$, so that $\tau_{(\gamma,\gamma')}$-Cl$(X\backslash G)\cup B)=\tau_{(\gamma,\gamma')}$-Cl$(X\backslash G)\cup\tau_{(\gamma,\gamma')}$-Cl$(B)=(X\backslash G)\cup\tau_{(\gamma,\gamma')}$-Cl$(G)=X$, and thus $(X\backslash G)\cup B$ is (γ,γ')-dense in X. Thus, $(X\backslash G)\cup B$ is (γ,γ')-open. Now, $B=(X\backslash G)\cup B)\cap G$, the intersection of two (γ,γ')-open sets is (γ,γ')-open ([4], Proposition 2.7), so that B is (γ,γ')-open. □

Theorem 3.7. *(X,τ) is a topological space in which every subset is (γ,γ')-preopen if and only if every (γ,γ')-open set in (X,τ) is (γ,γ')-closed.*

Proof. Let G be (γ,γ')-open. Then $X\backslash G=\tau_{(\gamma,\gamma')}$-Cl$(X\backslash G)$ which is (γ,γ')-preopen, so that $\tau_{(\gamma,\gamma')}$-Cl$(X\backslash G)\subset\tau_{(\gamma,\gamma')}$-Int$(\tau_{(\gamma,\gamma')}$-Cl$(\tau_{(\gamma,\gamma')}$- Cl$(X\backslash G)))=\tau_{(\gamma,\gamma')}$-Int$(\tau_{(\gamma,\gamma')}$-Cl$(X\backslash G))$ $=\tau_{(\gamma,\gamma')}$-Int$(X\backslash G)$. Thus, $X\backslash G=\tau_{(\gamma,\gamma')}$-Int$(X\backslash G)$, so that $X\backslash G$ is (γ,γ')-open, and G is (γ,γ')-closed. Conversely, let A be any subset of X. Then $X\backslash\tau_{(\gamma,\gamma')}$-Cl$(A)$ is (γ,γ')-open, and hence (γ,γ')-closed. Thus, $X\backslash\tau_{(\gamma,\gamma')}$-Cl$(A)=\tau_{(\gamma,\gamma')}$-Cl$(X\backslash\tau_{(\gamma,\gamma')}$-Cl$(A))=X\backslash\tau_{(\gamma,\gamma')}$-Int$(\tau_{(\gamma,\gamma')}$-Cl$(A))$, so that $A\subset\tau_{(\gamma,\gamma')}$-Cl$(A)=\tau_{(\gamma,\gamma')}$-Int$(\tau_{(\gamma,\gamma')}$-Cl$(A))$, and hence A is (γ,γ')-preopen. □

Theorem 3.8. *Let (X,τ) be a topological space, G be a (γ,γ')-open subset of X and b be a point of $\tau_{(\gamma,\gamma')}$-Cl$(G)\backslash G$. Then $\{b\}$ is not (γ,γ')-preopen in (X,τ).*

Proof. Suppose $\{b\}$ is (γ,γ')-preopen, so that $\{b\}\subset\tau_{(\gamma,\gamma')}$-Int$(\tau_{(\gamma,\gamma')}$-Cl$(\{b\}))$. Thus, $G\cap\tau_{(\gamma,\gamma')}$-Int$(\tau_{(\gamma,\gamma')}$-Cl$(\{b\}))\neq\emptyset$. Let $c\in G\cap\tau_{(\gamma,\gamma')}$-Int$(\tau_{(\gamma,\gamma')}$-Cl$(\{b\}))$, so $c\in\tau_{(\gamma,\gamma')}$-Cl$(\{b\})$ and hence $\{b\}\cap(G\cap\tau_{(\gamma,\gamma')}$-Int$(\tau_{(\gamma,\gamma')}$-Cl$(\{b\}))\neq\emptyset$. This contradicts the fact that $\{b\}\cap G=\emptyset$. Hence $\{b\}$ is not (γ,γ')-preopen. □

Theorem 3.9. *Let (X,τ) be a topological space, G be a (γ,γ')-regular open subset of X and b be a point of $\tau_{(\gamma,\gamma')}$-Cl$(G)\backslash G$. Then $G\cup\{b\}$ is not (γ,γ')-preopen in (X,τ).*

Proof. We have, since $\tau_{(\gamma,\gamma')}$-Cl$(\{b\})\subset\tau_{(\gamma,\gamma')}$-Cl$(G)$, so that $\tau_{(\gamma,\gamma')}$-Int$(\tau_{(\gamma,\gamma')}$-Cl$(G\cup\{b\}))=\tau_{(\gamma,\gamma')}$-Int$(\tau_{(\gamma,\gamma')}$-Cl$(G)\cup\tau_{(\gamma,\gamma')}$-Cl$(\{b\}))=\tau_{(\gamma,\gamma')}$-Int$(\tau_{(\gamma,\gamma')}$-Cl$(G)=G$, and thus $G\cup\{b\}\not\subseteq\tau_{(\gamma,\gamma')}$-Int$(\tau_{(\gamma,\gamma')}$-Cl$(G\cup\{b\}))$. Hence $G\cup\{b\}$ is not (γ,γ')-preopen. □

Theorem 3.10. *Let (X, τ) be a topological space. Then every singleton of X is either (γ, γ')-open or (γ, γ')-preclosed.*

Proof. If $\{x\}$ is not (γ, γ')-open, then $\tau_{(\gamma,\gamma')}$-$\mathrm{Int}(\{x\}) = \emptyset$. Thus, $\tau_{(\gamma,\gamma')}$-$\mathrm{Cl}(\tau_{(\gamma,\gamma')}$-$\mathrm{Int}(\{x\})) = \emptyset$; hence $\{x\}$ is (γ, γ')-preclosed. The proof of the second part is straightforward. □

Theorem 3.11. *If A is a (γ, γ')-preopen and (γ, γ')-pg.closed subset of (X, τ), then A is (γ, γ')-preclosed.*

Proof. Since A is (γ, γ')-preopen and (γ, γ')-pg.closed, $\tau_{(\gamma,\gamma')}$-$p\,\mathrm{Cl}(A) \subset A$ and hence $\tau_{(\gamma,\gamma')}$-$p\,\mathrm{Cl}(A) = A$. This implies that A is (γ, γ')-preclosed by Theorem 2.1 (iv). □

Theorem 3.12. *If A is a (γ, γ')-pg.closed subset of (X, τ) such that $A \subset B \subset \tau_{(\gamma,\gamma')}$-$p\,\mathrm{Cl}(A)$, then B is also (γ, γ')-pg.closed subset of (X, τ).*

Proof. Let U be a (γ, γ')-preopen set in (X, τ) such that $B \subset U$. Then $A \subset U$. Since A is (γ, γ')-pg.closed, then $\tau_{(\gamma,\gamma')}$-$p\,\mathrm{Cl}(A) \subset U$. Now, since $\tau_{(\gamma,\gamma')}$-$p\,\mathrm{Cl}(A)$ is (γ, γ')-preclosed, $\tau_{(\gamma,\gamma')}$-$p\,\mathrm{Cl}(B) \subset \tau_{(\gamma,\gamma')}$-$p\,\mathrm{Cl}(\tau_{(\gamma,\gamma')}$-$p\,\mathrm{Cl}(A)) = \tau_{(\gamma,\gamma')}$-$p\,\mathrm{Cl}(A) \subset U$. Therefore, B is also a (γ, γ')-pg.closed. □

Theorem 3.13. *A set A in a topological space (X, τ) is (γ, γ')-pg.open if and only if $F \subset \tau_{(\gamma,\gamma')}$-$p\,\mathrm{Int}(A)$ whenever F is (γ, γ')-preclosed in (X, τ) and $F \subset A$.*

Proof. Let A be (γ, γ')-pg.open. Let F be (γ, γ')-preclosed and $F \subset A$. Then $X \setminus A \subset X \setminus F$, where $X \setminus F$ is (γ, γ')-preopen. (γ, γ')-pg.closedness of $X \setminus A$ implies $\tau_{(\gamma,\gamma')}$-$p\,\mathrm{Cl}(X \setminus A) \subset X \setminus F$. By Theorem 2.1, $X \setminus \tau_{(\gamma,\gamma')}$-$p\,\mathrm{Int}(A) \subset X \setminus F$. That is, $F \subset \tau_{(\gamma,\gamma')}$-$p\,\mathrm{Int}(A)$. Conversely, Suppose if F is (γ, γ')-preclosed and $F \subset A$ implies $F \subset \tau_{(\gamma,\gamma')}$-$p\,\mathrm{Int}(A)$. Let $X \setminus A \subset U$ where U is (γ, γ')-preopen. Then $X \setminus U \subset A$ where $X \setminus U$ is (γ, γ')-preclosed. By supposition, $X \setminus U \subset \tau_{(\gamma,\gamma')}$-$p\,\mathrm{Int}(A)$. That is, $X \setminus \tau_{(\gamma,\gamma')}$-$p\,\mathrm{Int}(A) \subset U$. By Theorem 2.1, $\tau_{(\gamma,\gamma')}$-$p\,\mathrm{Cl}(X \backslash A) \subset U$. This implies $X \setminus A$ is (γ, γ')-pg.closed and hence A is (γ, γ')-pg.open. □

Theorem 3.14. *If $\tau_{(\gamma,\gamma')}$-$p\,\mathrm{Int}(A) \subset B \subset A$ and A is (γ, γ')-pg.open, then B is (γ, γ')-pg.open.*

Proof. Easily follows from Theorems 2.1 and 3.12. □

Theorem 3.15. *If a set A is (γ, γ')-pg.open in a topological space (X, τ), then $G = X$ whenever G is (γ, γ')-preopen in (X, τ) and $\tau_{(\gamma,\gamma')}$-$p\,\mathrm{Int}(A) \cup X \backslash A \subset G$.*

Proof. Suppose that G is (γ, γ')-preopen and $\tau_{(\gamma,\gamma')}$-$p\,\mathrm{Int}(A) \cup X \backslash A \subset G$. Now $X \backslash G \subset \tau_{(\gamma,\gamma')}$-$p\,\mathrm{Cl}(X \backslash A) \cap A = \tau_{(\gamma,\gamma')}$-$p\,\mathrm{Cl}(X \backslash A) \backslash X \backslash A$. Since $X \backslash G$ is (γ, γ')-preclosed and $X \backslash A$ is (γ, γ')-pg.closed, by Theorem 2.2, $X \backslash G = \varnothing$ and hence $G = X$. □

Proposition 3.1. *Let (X, τ) be a topological space and $A, B \subset X$. If B is (γ, γ')-pg.open and if $A \supset \tau_{(\gamma,\gamma')}$-$p\,\mathrm{Int}(B)$, then $A \cap B$ is (γ, γ')-pg.open.*

Proof. Since B is (γ, γ')-pg.open and $A \supset \tau_{(\gamma,\gamma')}$-$p\,\mathrm{Int}(B)$, $\tau_{(\gamma,\gamma')}$-$p\,\mathrm{Int}(B) \subset A \cap B \subset B$. By Theorem 3.14, $A \cap B$ is (γ, γ')-pg.open. □

Proposition 3.2. *Let the family (γ, γ')-$PO(X)$ of all (γ, γ')-preopen subsets of (X, τ) be closed under finite intersections i.e., let (γ, γ')-$PO(X)$ be the topology on X. If A and B are (γ, γ')-pg.open in (X, τ), then $A \cap B$ is (γ, γ')-pg.open.*

Proof. Let $X \setminus (A\cap B) = (X\backslash A)\cup(X\backslash B)\subset U$, where U is (γ,γ')-preopen. Then $X\setminus A\subset U$ and $X\setminus B\subset U$. Since A and B are (γ,γ')-pg.open, $\tau_{(\gamma,\gamma')}$-$p\,\mathrm{Cl}(X\backslash A)\subset U$ and $\tau_{(\gamma,\gamma')}$-$p\,\mathrm{Cl}(X\backslash B)\subset U$. By hypothesis, $\tau_{(\gamma,\gamma')}$-$p\,\mathrm{Cl}((X\setminus A)\cup(X\setminus B)) = \tau_{(\gamma,\gamma')}$-$p\,\mathrm{Cl}(X\setminus A)$ $\cup\,\tau_{(\gamma,\gamma')}$-$p\,\mathrm{Cl}(X\setminus B)\subset U$. That is, $\tau_{(\gamma,\gamma')}$-$p\,\mathrm{Cl}(X\setminus(A\cap B))\subset U$. This shows that $A\cap B$ is (γ,γ')-pg.open. □

Theorem 3.16. *If $A\subset X$ is (γ,γ')-pg.closed, then $\tau_{(\gamma,\gamma')}$-$p\,\mathrm{Cl}(A)\setminus A$ is (γ,γ')-pg.open.*

Proof. Let A be (γ,γ')-pg.closed. Let F be a (γ,γ')-preclosed set such that $F\subset\tau_{(\gamma,\gamma')}$-$p\,\mathrm{Cl}(A)\setminus A$. Then by Theorem 2.2 F=∅. So, $F\subset\tau_{(\gamma,\gamma')}$-$p\,\mathrm{Int}(\tau_{(\gamma,\gamma')}$-$p\,\mathrm{Cl}(A)\setminus A)$. This shows $\tau_{(\gamma,\gamma')}$-$p\,\mathrm{Cl}(A)\setminus A$ is (γ,γ')-pg.open. □

Definition 3.8. A topological space (X,τ) with operations γ and γ' on τ is called (γ,γ')-preregular if for each (γ,γ')-preclosed set F of X not containing x, there exists disjoint (γ,γ')-preopen sets U and V such that $x\in U$ and $F\subset V$.

The following examples show that regularity and (γ,γ')-preregularity are independent concepts.

Example 3.1. Let $X=\{a,b,c\}$ and τ be the discrete topology on X. Let $\gamma:\tau\to\mathcal{P}(X)$ and $\gamma':\tau\to\mathcal{P}(X)$ be operations defined as follows: for every $A\in\tau$

$$A^{\gamma}=\begin{cases}\{a\} & \text{if A = \{a\}},\\ A\cup\{c\} & \text{if A} \neq \text{\{a\}}\end{cases}$$

$$A^{\gamma'}=\begin{cases}A & \text{if} \neq \text{\{a\}},\\ \mathrm{Cl}(A) & \text{if A = \{a\}}.\end{cases}$$

Then this space regular but not (γ,γ')-preregular.

Example 3.2. Let $X=\{a,b,c\}$ and $\tau=\{\emptyset,X,\{a\},\{c\},\{a,c\}\}$. Let $\gamma:\tau\to\mathcal{P}(X)$ and $\gamma':\tau\to\mathcal{P}(X)$ be operations defined as follows: for every $A\in\tau$

$$A^{\gamma}=\begin{cases}A & \text{if a} \in \text{A},\\ A\cup\{a\} & \text{if a} \notin \text{A}.\end{cases}$$

$$A^{\gamma'}=\begin{cases}A & \text{if c} \in \text{A},\\ A\cup\{c\} & \text{if c} \notin \text{A}.\end{cases}$$

Then this space (γ,γ')-preregular but not regular.

Theorem 3.17. *The following are equivalent for a topological space (X,τ) with operations γ and γ' on τ :*

(i) *X is (γ,γ')-preregular.*
(ii) *For each $x\in X$ and each $U\in(\gamma,\gamma')$-$PO(X,x)$, there exists a $V\in(\gamma,\gamma')$-$PO(X,x)$ such that $x\in V\subset\tau_{(\gamma,\gamma')}$-$p\,\mathrm{Cl}(V)\subset U$.*
(iii) *For each (γ,γ')-preclosed set F of X, $\cap\{\tau_{(\gamma,\gamma')}$-$p\,\mathrm{Cl}(V) : F\subset V, V\in(\gamma,\gamma')$-$PO(X)\} = F$*
(iv) *For each A subset of X and each $U\in(\gamma,\gamma')$-$PO(X)$ with $A\cap U\neq\emptyset$, there exists a $V\in(\gamma,\gamma')$-$PO(X)$ such that $A\cap U\neq\varnothing$ and $\tau_{(\gamma,\gamma')}$-$p\,\mathrm{Cl}(V)\subset U$.*
(v) *For each nonempty subset A of X and each (γ,γ')-preclosed subset F of X with $A\cap F=\emptyset$, there exists $V,W\in(\gamma,\gamma')$-$PO(X)$ such that $A\cap V\neq\emptyset$, $F\subset W$ and $W\cap V=\emptyset$*
(vi) *For each (γ,γ')-preclosed set F and $x\notin F$, there exists $U\in(\gamma,\gamma')$-$PO(X)$ and a (γ,γ')-pg.open set V such that $x\in U$, $F\subset V$ and $U\cap V=\emptyset$.*

(vii) *For each $A \subset X$ and each (γ,γ')-preclosed set F with $A\cap F=\emptyset$, there exists $U\in(\gamma,\gamma')$-$PO(X)$ and a (γ,γ')-pg.open set V such that $A\cap U\neq\emptyset$, $F\subset V$ and $U\cap V=\emptyset$.*

(viii) *For each (γ,γ')-preclosed set F of X, $F=\cap\{\tau_{(\gamma,\gamma')}\text{-}p\operatorname{Cl}(V) : F\subset V, V$ is (γ,γ')-pg.open$\}$*

Proof. (i)⇒ (ii) Let $x\notin X\backslash U$, where $U\in(\gamma,\gamma')$-$PO(X,x)$. Then there exists $G,V\in(\gamma,\gamma')$-$PO(X)$ such that $(X\backslash U)\subset G$, $x\in V$ and $G\cap V=\emptyset$. Therefore $V\subset(X\backslash G)$ and so $x\in V\subset\tau_{(\gamma,\gamma')}\text{-}p\operatorname{Cl}(V)\subset(X\backslash G)\subset U$.

(ii)⇒ (iii) Let $(X\backslash F)\in(\gamma,\gamma')$-$PO(X,x)$. Then by (2) there exists an $U\in(\gamma,\gamma')$-$PO(X,x)$ such that $x\in U\subset\tau_{(\gamma,\gamma')}\text{-}p\operatorname{Cl}(U)\subset(X\backslash F)$. So, $F\subset X\backslash\tau_{(\gamma,\gamma')}\text{-}p\operatorname{Cl}(U)=V$, $V\in(\gamma,\gamma')$-$PO(X)$ and $V\cap U=\emptyset$. Then by Theorem 2.1, $x\notin\tau_{(\gamma,\gamma')}\text{-}p\operatorname{Cl}(V)$. Thus $F\supset\cap\{\tau_{(\gamma,\gamma')}\text{-}p\operatorname{Cl}(V):F\subset V, V\in(\gamma,\gamma')\text{-}PO(X)\}$.

(iii)⇒ (iv) Let $U\in(\gamma,\gamma')$-$PO(X)$ with $x\in U\cap A$. Then $x\notin(X\backslash U)$ and hence by (iii) there exists a (γ,γ')-preopen set W such that $(X\backslash U)\subset W$ and $x\notin\tau_{(\gamma,\gamma')}\text{-}p\operatorname{Cl}(W)$. We put $V=X\backslash\tau_{(\gamma,\gamma')}\text{-}p\operatorname{Cl}(W)$, which is a (γ,γ')-preopen set containing x and hence $V\cap U\neq\emptyset$. Now $V\subset(X\backslash W)$ and so $\tau_{(\gamma,\gamma')}\text{-}p\operatorname{Cl}(V)\subset(X\backslash W)\subset U$

(iv)⇒ (v) Let F be a set as in hypothesis of (v). Then $(X\backslash F)$ is (γ,γ')-preopen and $(X\backslash F)\cap A\neq\emptyset$. Then there exists $V\in(\gamma,\gamma')$-$PO(X)$ such that $A\cap V\neq\emptyset$ and $\tau_{(\gamma,\gamma')}\text{-}p\operatorname{Cl}(V)\subset(X\backslash F)$. If we put $W=X\backslash\tau_{(\gamma,\gamma')}\text{-}p\operatorname{Cl}(V)$, then $F\subset W$ and $W\cap V=\emptyset$. (v)⇒ (i) Let F be a (γ,γ')-preclosed set not containing x. Then by (v), there exist $W,V\in(\gamma,\gamma')$-$PO(X)$ such that$F\subset W$ and $x\in V$ and $W\cap V=\emptyset$.

(i)⇒ (vi) Obvious.

(vi)⇒ (vii) For $a\in A$, $a\notin F$ and hence by (vi) there exists $U\in(\gamma,\gamma')$-$PO(X)$ and a (γ,γ')-pg.open set V such that $a\in U$, $F\subset V$ and $U\cap V=\emptyset$. So, $A\cap U\neq\emptyset$.

(vii)⇒ (i) Let $x\notin F$, where F is (γ,γ')-pg.closed. Since $\{x\}\cap F=\emptyset$, by (vii) there exists $U\in(\gamma,\gamma')$-$PO(X)$ and (γ,γ')-pg.open set W such that $x\in U$, $F\subset W$ and $U\cap W=\emptyset$. Now put $V=\tau_{(\gamma,\gamma')}\text{-}p\operatorname{Int}(W)$. Using definition of (γ,γ')-pg.open sets we get $F\subset V$ and $V\cap U=\emptyset$.

(iii)⇒ (viii) We have $F\subset\cap\{\tau_{(\gamma,\gamma')}\text{-}p\operatorname{Cl}(V):F\subset V$ and V is (γ,γ')-pg.open$\}\subset\cap\{\tau_{(\gamma,\gamma')}\text{-}p\operatorname{Cl}(V):F\subset V$ and V is (γ,γ')-preopen$\}=F$. (viii)⇒ (i) Let F be a (γ,γ')-preclosed set in$X$ not containing x. Then by (viii) there exists a (γ,γ')-pg.open set W such that $F\subset W$ and $x\in X\backslash\tau_{(\gamma,\gamma')}\text{-}p\operatorname{Cl}(W)$. Since F is (γ,γ')-preclosed and W is (γ,γ')-pg.open , $F\subset\tau_{(\gamma,\gamma')}\text{-}p\operatorname{Int}(W)$. Take $V=\tau_{(\gamma,\gamma')}\text{-}p\operatorname{Int}(W)$. Then $F\subset V$, $x\in U=X\backslash\tau_{(\gamma,\gamma')}\text{-}p\operatorname{Cl}(V)$ and $U\cap V=\emptyset$. □

Definition 3.9. A topological space (X,τ) with operations γ and γ' on τ is called (γ,γ')-prenormal if for any pair of disjoint (γ,γ')-preclosed sets A and B of X, there exist disjoint (γ,γ')-preopen sets U and V such that $A\subset U$ and $F\subset V$.

The following examples show that normality and (γ,γ')-prenormality are independent concepts.

Example 3.3. Let $X=\{a,b,c\}$ and $\tau=\{\emptyset,X,\{a\},\{a,b\},\{a,c\}\}$. Let $\gamma:\tau\to\mathcal{P}(X)$ and $\gamma':\tau\to\mathcal{P}(X)$ be operations defined as follows: for every $A\in\tau$
$A^\gamma=\operatorname{Cl}(A)$ and

$$A^{\gamma'}=\begin{cases} A & \text{if c} \notin \text{A},\\ \operatorname{Cl}(A) & \text{if c} \in \text{A}.\end{cases}$$

This space is (γ,γ')-prenormal but not normal.

Example 3.4. Let $X=\{a,b,c\}$ and τ be the discrete topology on X. Let $\gamma:\tau\to\mathcal{P}(X)$ and $\gamma':\tau\to\mathcal{P}(X)$ be operations defined as follows: for every $A\in\tau$

$$A^{\gamma} = \begin{cases} \{b\} & \text{if A = \{b\}}, \\ A \cup \{b\} & \text{if A} \neq \text{\{b\}} \end{cases}$$

and

$$A^{\gamma} = \begin{cases} \mathrm{Cl}(A) & \text{if A = \{b\}}, \\ A & \text{if A} \neq \text{\{b\}} \end{cases}$$

Then this space is normal but not (γ, γ')-prenormal.

Theorem 3.18. *For a topological space (X, τ) with operations γ and γ' on τ, the following are equivalent:*

(i) X is (γ, γ')-prenormal.
(ii) For each pair of disjoint (γ, γ')-preclosed sets A and B of X, there exist disjoint (γ, γ')-pg.open sets U and V such that $A \subset U$ and $B \subset V$.
(iii) For each (γ, γ')-preclosed set A and any (γ, γ')-preopen set V containing A, there exists a (γ, γ')-pg.open set U such that $A \subset U \subset \tau_{(\gamma,\gamma')}\text{-}p\,\mathrm{Cl}(U) \subset V$.
(iv) For each (γ, γ')-preclosed set A and any (γ, γ')-pg.open set B containing A, there exists a (γ, γ')-pg.open set U such that $A \subset U \subset \tau_{(\gamma,\gamma')}\text{-}p\,\mathrm{Cl}(U) \subset \tau_{(\gamma,\gamma')}\text{-}p\,\mathrm{Int}(B)$.
(v) For each (γ, γ')-preclosed set A and any (γ, γ')-pg.open set B containing A, there exists a (γ, γ')-preopen set G such that $A \subset G \subset \tau_{(\gamma,\gamma')}\text{-}p\,\mathrm{Cl}(G) \subset \tau_{(\gamma,\gamma')}\text{-}p\,\mathrm{Int}(B)$.
(vi) For each (γ, γ')-pg.closed set A and any (γ, γ')-preopen set B containing A, there exists a (γ, γ')-preopen set U such that $\tau_{(\gamma,\gamma')}\text{-}p\,\mathrm{Cl}(A) \subset U \subset \tau_{(\gamma,\gamma')}\text{-}p\,\mathrm{Cl}(U) \subset B$.
(vii) For each (γ, γ')-pg.closed set A and any (γ, γ')-preopen set B containing A, there exists a (γ, γ')-pg.open set G such that $\tau_{(\gamma,\gamma')}\text{-}p\,\mathrm{Cl}(A) \subset G \subset \tau_{(\gamma,\gamma')}\text{-}p\,\mathrm{Cl}(G) \subset B$.

Proof. (i) $\Rightarrow$ (ii): Follows from the fact that every (γ, γ')-preopen set is (γ, γ')-pg.open.

(ii) $\Rightarrow$ (iii): let A be a (γ, γ')-closed set and V any (γ, γ')-preopen set containing A. Since A and $(X \backslash V)$ are disjoint (γ, γ')-preclosed sets, there exist (γ, γ')-pg.open sets U and W such that $\subset U$, $(X \backslash V) \subset W$ and $U \cap W = \emptyset$. By Theorem 3.13, we get $(X \backslash V) \subset \tau_{(\gamma,\gamma')}\text{-}p\,\mathrm{Int}(W)$. Since $U \cap \tau_{(\gamma,\gamma')}\text{-}p\,\mathrm{Int}(W) = \emptyset$, we have $\tau_{(\gamma,\gamma')}\text{-}p\,\mathrm{Cl}(U) \cap \tau_{(\gamma,\gamma')}\text{-}p\,\mathrm{Int}(W) = \emptyset$, and hence $\tau_{(\gamma,\gamma')}\text{-}p\,\mathrm{Cl}(U) \subset X \backslash \tau_{(\gamma,\gamma')}\text{-}p\,\mathrm{Int}(W) \subset V$. Therefore, $A \subset U \subset \tau_{(\gamma,\gamma')}\text{-}p\,\mathrm{Cl}(U) \subset V$.

(iii) $\Rightarrow$ (i): Let A and B be any disjoint (γ, γ')-preclosed sets of X. Since $(X \backslash B)$ is an (γ, γ')-preopen set containing A, there exists a (γ, γ')-pg.open set G such that $A \subset G \subset \tau_{(\gamma,\gamma')}\text{-}p\,\mathrm{Cl}(G) \subset X \backslash B$. Since G is a (γ, γ')-pg.open set, using Theorem 3.13, we have $A \subset \tau_{(\gamma,\gamma')}\text{-}p\,\mathrm{Int}(G)$. Taking $U = \tau_{(\gamma,\gamma')}\text{-}p\,\mathrm{Int}(G)$ and $V = X \backslash \tau_{(\gamma,\gamma')}\text{-}p\,\mathrm{Cl}(G)$, we have two disjoint (γ, γ')-preopen sets U and V such that $A \subset U$ and $B \subset V$. Hence X is (γ, γ')-prenormal.

(v) $\Rightarrow$ (iii): let A be a (γ, γ')-closed set and V any (γ, γ')-preopen set containing A. Since every (γ, γ')-preopen set is (γ, γ')-pg.open, there exists a (γ, γ')-preopen set G such that $A \subset G \subset \tau_{(\gamma,\gamma')}\text{-}p\,\mathrm{Cl}(G) \subset \tau_{(\gamma,\gamma')}\text{-}p\,\mathrm{Int}(V)$. Also. we have a (γ, γ')-pg.open set G such that $A \subset G \subset \tau_{(\gamma,\gamma')}\text{-}p\,\mathrm{Cl}(G) \subset \tau_{(\gamma,\gamma')}\text{-}p\,\mathrm{Int}(V) \subset V$.

(iii) $\Rightarrow$ (v): Let A be a (γ, γ')-closed set and B by a (γ, γ')-pg.open set containing A. Using Theorem 3.13 of a (γ, γ')-pg.open set we get $A \subset \tau_{(\gamma,\gamma')}\text{-}p\,\mathrm{Int}(B) = V$, say. Then applying (iii), we get a (γ, γ')-pg.open set U such that $A = \tau_{(\gamma,\gamma')}\text{-}p\,\mathrm{Cl}(A) \subset U \subset \tau_{(\gamma,\gamma')}\text{-}p\,\mathrm{Cl}(U) \subset V$. Again, using the same Theorem 3.13 we get $A \subset \tau_{(\gamma,\gamma')}\text{-}p\,\mathrm{Int}(U)$, and hence $A \subset \tau_{(\gamma,\gamma')}\text{-}p\,\mathrm{Int}(U) \subset U \subset \tau_{(\gamma,\gamma')}\text{-}p\,\mathrm{Cl}(U) \subset V$; which implies $A \subset \tau_{(\gamma,\gamma')}\text{-}p\,\mathrm{Int}(U) \subset \tau_{(\gamma,\gamma')}\text{-}p\,\mathrm{Cl}(\tau_{(\gamma,\gamma')}\text{-}p\,\mathrm{Int}(U)) \subset \tau_{(\gamma,\gamma')}\text{-}p\,\mathrm{Cl}(U) \subset V$, that is, $A \subset G \subset \tau_{(\gamma,\gamma')}\text{-}p\,\mathrm{Cl}(G) \subset \tau_{(\gamma,\gamma')}\text{-}p\,\mathrm{Int}(B)$, where $G = \tau_{(\gamma,\gamma')}\text{-}p\,\mathrm{Int}(U)$.

(iii) ⇒ (vii): Let A be a (γ,γ')-pg.closed set and B any (γ,γ')-preopen set containing A. Since A is a (γ,γ')-pg.closed set, we have $\tau_{(\gamma,\gamma')}$-$p\,\mathrm{Cl}(A) \subset B$, therefore, we can find a (γ,γ')-pg.open set U such that $\tau_{(\gamma,\gamma')}$-$p\,\mathrm{Cl}(A) \subset U \subset \tau_{(\gamma,\gamma')}$-$p\,\mathrm{Cl}(U) \subset B$.

(vii) ⇒ (vi): Let A be a (γ,γ')-pg.closed set and B any (γ,γ')-preopen set containing A, then by (vii) there exists a (γ,γ')-pg.open set G such that τ_β-$\mathrm{Cl}(A) \subset G \subset \tau_{(\gamma,\gamma')}$-$p\,\mathrm{Cl}(G) \subset B$. Since G is a (γ,γ')-pg.open set, then by Theorem 3.13, we get $\tau_{(\gamma,\gamma')}$-$p\,\mathrm{Cl}(A) \subset \tau_{(\gamma,\gamma')}$-$p\,\mathrm{Int}(G)$. If we take $U = \tau_{(\gamma,\gamma')}$-$p\,\mathrm{Int}(G)$, the proof follows. □

Definition 3.10. Let (X,τ) be a topological space and $A \subset X$.

(i) The set denoted by $\tau_{(\gamma,\gamma')}$-$pD(A)$ and defined by $\{x :$ for every (γ,γ')-preopen set U containing x, $U \cap (A\backslash\{x\}) \neq \emptyset\}$ is called the $\tau_{(\gamma,\gamma')}$-prederived set of A.

(ii) The $\tau_{(\gamma,\gamma')}$-prefrontier of A, denoted by $\tau_{(\gamma,\gamma')}$-$pFr(A)$ is defined as $\tau_{(\gamma,\gamma')}$-$p\,\mathrm{Cl}(A)\cap \tau_{(\gamma,\gamma')}$-$p\,\mathrm{Cl}(X\backslash A)$.

Proposition 3.3. *Let (X,τ) be a topological space and $A \subset X$. Then the following properties hold:*

(i) $\tau_{(\gamma,\gamma')}$-$p\,\mathrm{Int}(A) = A\backslash(\tau_{(\gamma,\gamma')}$-$pD(X\backslash A)$.

(ii) $\tau_{(\gamma,\gamma')}$-$p\,\mathrm{Cl}(A) = A \cup \tau_{(\gamma,\gamma')}$-$pD(A)$.

Proof. Clear. □

4. (γ,β)-PRECONTINUOUS FUNCTIONS

Definition 4.11. A function $f : (X,\tau) \to (Y,\sigma)$ is said to be (γ,β)-precontinuous [1] at a point $x \in X$ if for each (β,β')-preopen subset V in Y containing $f(x)$, there exists a (γ,γ')-preopen subset U of X containing x such that $f(U) \subset V$.

Theorem 4.19. *For a function $f : (X,\tau) \to (Y,\sigma)$, the following statements are equivalent:*

(i) f is (γ,β)-precontinuous;

(ii) The inverse image of each (β,β')-preopen set in Y is (γ,γ')-preopen in X;

(iii) For each subset B of Y, $\tau_{(\gamma,\gamma')}$-$p\,\mathrm{Cl}(f^{-1}(B)) \subset f^{-1}(\sigma_{(\beta,\beta')}$-$p\,\mathrm{Cl}(B))$;

(iv) For each subset A of X, $f(\tau_{(\gamma,\gamma')}$-$p\,\mathrm{Cl}(A)) \subset \sigma_{(\beta,\beta')}$-$p\,\mathrm{Cl}(f(A))$;

(v) For each subset A of X, $f(\tau_{(\gamma,\gamma')}$-$pD(A)) \subset \sigma_{(\beta,\beta')}$-$p\,\mathrm{Cl}(f(A))$;

(vi) For any subset B of Y, $f^{-1}(\sigma_{(\beta,\beta')}$-$p\,\mathrm{Int}(B)) \subset \tau_{(\gamma,\gamma')}$-$p\,\mathrm{Int}(f^{-1}(A))$;

(vii) For each subset C of Y, $\tau_{(\gamma,\gamma')}$-$pFr(f^{-1}(C)) \subset f^{-1}(\sigma_{(\beta,\beta')}$-$pFr(B))$.

Proof. Easy proof and hence omitted. □

Theorem 4.20. *Let $f : (X,\tau) \to (Y,\sigma)$ be a (γ,β)-precontinuous function. Then for each subset V of Y, $f^{-1}(\sigma_{(\beta,\beta')}$-$p\,\mathrm{Int}(V)) \subset \tau_{(\gamma,\gamma')}$-$p\,\mathrm{Cl}(\tau_{(\gamma,\gamma')}$-$p\,\mathrm{Int}(f^{-1}(V)))$.*

Proof. Let V be any subset of Y. Then $\sigma_{(\beta,\beta')}$-$p\,\mathrm{Int}(V)$ is (β,β')-preopen in Y and so $f^{-1}(\sigma_{(\beta,\beta')}$-$p\,\mathrm{Int}(V))$ is (γ,γ')-preopen in X.

Hence $f^{-1}(\sigma_{(\beta,\beta')}$-$p\,\mathrm{Int}(V)) \subset \tau_{(\gamma,\gamma')}$-$p\,\mathrm{Cl}(\tau_{(\gamma,\gamma')}$-$p\,\mathrm{Int}(f^{-1}(\sigma_{(\beta,\beta')}$-$p\,\mathrm{Int}(V)))) \subset \tau_{(\gamma,\gamma')}$-$p\,\mathrm{Cl}(\tau_{(\gamma,\gamma')}$-$p\,\mathrm{Int}(f^{-1}(V)))$. □

Corollary 4.1. *Let $f : (X,\tau) \to (Y,\sigma)$ be a (γ,β)-precontinuous function. Then for each subset V of Y, $\tau_{(\gamma,\gamma')}$-$p\,\mathrm{Int}(\tau_{(\gamma,\gamma')}$-$p\,\mathrm{Cl}(f^{-1}(V))) \subset f^{-1}(\sigma_{(\beta,\beta')}$-$p\,\mathrm{Cl}(V))$.*

Theorem 4.21. *Let $f : (X,\tau) \to (Y,\sigma)$ be a bijection. Then f is (γ,β)-precontinuous if and only if $\sigma_{(\beta,\beta')}$-$p\,\mathrm{Int}(f(U)) \subset f(\tau_{(\gamma,\gamma')}$-$p\,\mathrm{Int}(U))$ for each subset U of X.*

Proof. Let U be any subset of X. Then by Theorem 4.19, $f^{-1}(\sigma_{(\beta,\beta')}\text{-}p\operatorname{Int}(f(U))) \subset \tau_{(\gamma,\gamma')}\text{-}p\operatorname{Int}(f^{-1}(f(U)))$. Since f is bijection, $\sigma_{(\beta,\beta')}\text{-}p\operatorname{Int}(f(U)) = f(f^{-1}(\tau_{(\gamma,\gamma')}\text{-}p\operatorname{Int}(f(U))) \subset f(\tau_{(\gamma,\gamma')}\text{-}p\operatorname{Int}(U))$. Conversely, let V be any subset of Y. Then $\sigma_{(\beta,\beta')}\text{-}p\operatorname{Int}(f(f^{-1}(V))) \subset f(\tau_{(\gamma,\gamma')}\text{-}p\operatorname{Int}(f^{-1}(V)))$. Since f is bijection, $\sigma_{(\beta,\beta')}\text{-}p\operatorname{Int}(V) = \sigma_{(\beta,\beta')}\text{-}p\operatorname{Int}(f(f^{-1}(V))) \subset f(\tau_{(\gamma,\gamma')}\text{-}p\operatorname{Int}(f^{-1}(V)))$; hence $f^{-1}(\sigma_{(\beta,\beta')}\text{-}p\operatorname{Int}(V) \subset \tau_{(\gamma,\gamma')}\text{-}p\operatorname{Int}(f^{-1}(V))$. Therefore, by Theorem 4.19, f is (γ,β)-precontinuous. □

Theorem 4.22. *Let $f : (X,\tau) \to (Y,\sigma)$ be a function. Then $X \backslash \tau_{(\gamma,\gamma')}\text{-}pC(f) = \cup\{\tau_{(\gamma,\gamma')}\text{-}pFr(f^{-1}(V)) : V \in (\beta,\beta') - PO(Y), f(x) \in V, x \in X\}$, where $\tau_{(\gamma,\gamma')}\text{-}pC(f)$ denotes the set of points at which f is (γ,β)-precontinuous.*

Proof. Let $x \in X\backslash\tau_{(\gamma,\gamma')}\text{-}pC(f)$. Then there exists $V \in (\beta,\beta') - PO(Y)$ containing $f(x)$ such that $f(U)$ is not a subset of V, for every (γ,γ')-preopen set U containing x. Hence $U \cap (X\backslash f^{-1}(V)) \neq \emptyset$ for every (γ,γ')-preopen set U containing x. Then, $x \in \tau_{(\gamma,\gamma')}\text{-}p\operatorname{Cl}(X\backslash f^{-1}(V))$. Then $x \in f^{-1}(V) \cap \tau_{(\gamma,\gamma')}\text{-}p\operatorname{Cl}(X\backslash f^{-1}(V)) \subset \tau_{(\gamma,\gamma')}\text{-}pFr(f^{-1}(V))$. So,

$$X\backslash\tau_{(\gamma,\gamma')} - pC(f) \subset \cup\{\tau_{(\gamma,\gamma')} - pFr(f^{-1}(V)) : V \in (\beta,\beta') - PO(Y), f(x) \in V, x \in X\}.$$

Conversely, let $x \notin X\backslash\tau_{(\gamma,\gamma')}\text{-}pC(f)$. Then for each $V \in (\beta,\beta') - PO(Y)$ containing $f(x)$, $f^{-1}(V)$ is a (γ,γ')-preopen set U containing x. Thus, $x \in \tau_{(\gamma,\gamma')}\text{-}p\operatorname{Int}(f^{-1}(V))$ and hence $x \notin \tau_{(\gamma,\gamma')}\text{-}pFr(f^{-1}(V))$, for every $V \in \beta SO(Y)$ containing $f(x)$. Therefore,

$$X\backslash\tau_{(\gamma,\gamma')} - pC(f) \supset \cup\{\tau_{(\gamma,\gamma')} - pFr(f^{-1}(V)) : V \in (\beta,\beta')PO(Y), f(x) \in V, x \in X\}.$$

□

Theorem 4.23. *Let $f : (X,\tau) \to (Y,\sigma)$ be a (γ,β)-precontinuous, (γ,β)-preclosed, (γ,β)-preopen surjective function on (X,τ). If X is (γ,γ')-preregular, then Y is (β,β')-preregular.*

Proof. Let K be a (β,β')-preclosed in Y and $y \in K$. Since f is (γ,β)-precontinuous and X is (γ,γ')-preregular for each point $x \in f^{-1}(y)$, there exist disjoint $V, W \in (\gamma,\gamma')\text{-}PO(X)$ such that $x \in V$ and $f^{-1}(K) \subset W$. Now, since f is (γ,β)-preclosed, there exists a (β,β')-preopen set U containing K such that $f^{-1}(U) \subset W$. As f is a (γ,β)-preopen function, we have $y = f(x) \in f(V)$ and $f(V)$ is (β,β')-preopen in Y. Now, $f^{-1}(U) \cap V = \emptyset$; hence $U \cap f(V) = \emptyset$. Therefore, Y is (β,β')-preregular. □

Theorem 4.24. *If $f : (X,\tau) \to (Y,\sigma)$ is a (γ,β)-precontinuous, (γ,β)-preclosed surjective function and X is (γ,γ')-prenormal, then Y is (β,β')-prenormal.*

Proof. Let A and B be two disjoint (β,β')-preclosed sets in Y. Since f is (γ,β)-precontinuous, $f^{-1}(A)$ and $f^{-1}(B)$ are disjoint (γ,γ')-preclosed sets in X. Now as X is (γ,γ')-prenormal, there exist disjoint (γ,γ')-preopen sets V and W such that $f^{-1}(A) \subset V$ and $f^{-1}(B) \subset W$. Since f is (γ,β)-preclosed, there exist (β,β')-preopen sets M and N such that $A \subset M$, $B \subset N$, $f^{-1}(M) \subset V$ and $f^{-1}(N) \subset W$. Since $V \cap W = \emptyset$, we have $M \cap N = \emptyset$; hence Y is (β,β')-prenormal. □

Definition 4.12. A function $f : (X,\tau) \to (Y,\sigma)$ is said to be:

(i) (γ,β)-preopen if $f(U)$ is a (β,β')-preopen set of Y for every (γ,γ')-preopen set U of X.

(ii) (γ,β)-preclosed [1] if $f(U)$ is a (β,β')-preclosed set of Y for every (γ,γ')-preclosed set U of X.

Theorem 4.25. *For a bijective function $f : (X,\tau) \to (Y,\sigma)$, the following statements are equivalent:*

(i) f *is* (γ,β)*-preopen;*
(ii) f *is* (γ,β)*-preclosed;*
(iii) f *is* (γ,β)*-precontinuous.*

Proof. The proof is clear. □

Theorem 4.26. *For a function* $f:(X,\tau)\to(Y,\sigma)$*, the following statements are equivalent:*
(i) f *is* (γ,β)*-preopen;*
(ii) $f(\tau_{(\gamma,\gamma')}\text{-}p\operatorname{Int}(U))\subset\sigma_{(\beta,\beta')}\text{-}p\operatorname{Int}(f(U))$ *for each subset* U *of* X*;*
(iii) $\tau_{(\gamma,\gamma')}\text{-}p\operatorname{Int}(f^{-1}(V))\subset f^{-1}(\sigma_{(\beta,\beta')}\text{-}p\operatorname{Int}(V))$ *for each subset* V *of* Y*.*

Proof. $(i)\Rightarrow(ii)$: Let U be any subset of X. Then $\tau_{(\gamma,\gamma')}\text{-}p\operatorname{Int}(U)$ is a (γ,γ')-preopen set of X. Then $f(\tau_{(\gamma,\gamma')}\text{-}p\operatorname{Int}(U))$ is a (β,β')-preopen set of Y. Since $f(\tau_{(\gamma,\gamma')}\text{-}p\operatorname{Int}(U))\subset f(U)$, $f(\tau_{(\gamma,\gamma')}\text{-}p\operatorname{Int}(U))=\sigma_{(\beta,\beta')}\text{-}p\operatorname{Int}(f(\tau_{(\gamma,\gamma')}\text{-}p\operatorname{Int}(U)))\subset\sigma_{(\beta,\beta')}\text{-}p\operatorname{Int}(f(U))$.

$(ii)\Rightarrow(iii)$: Let V be any subset of Y. Then $f^{-1}(V)$ is a subset of X. Hence $f(\tau_{(\gamma,\gamma')}\text{-}p\operatorname{Int}(f^{-1}(V)))\subset\sigma_{(\beta,\beta')}\text{-}p\operatorname{Int}(f(f^{-1}(V)))\subset\sigma_{(\beta,\beta')}\text{-}p\operatorname{Int}(V))$. Then $\tau_{(\gamma,\gamma')}\text{-}p\operatorname{Int}(f^{-1}(V))\subset f^{-1}(f(\tau_{(\gamma,\gamma')}\text{-}p\operatorname{Int}(f^{-1}(V))))\subset f^{-1}(\sigma_{(\beta,\beta')}\text{-}p\operatorname{Int}(V))$.

$(iii)\Rightarrow(i)$: Let U be any (γ,γ')-preopen set of X. Then $\tau_{(\gamma,\gamma')}\text{-}p\operatorname{Int}(U)=U$ and $f(U)$ is a subset of Y. Now, $V=\tau_{(\gamma,\gamma')}\text{-}p\operatorname{Int}(V)\subset\tau_{(\gamma,\gamma')}\text{-}p\operatorname{Int}(f^{-1}(f(V)))\subset f^{-1}(\sigma_{(\beta,\beta')}\text{-}p\operatorname{Int}(f(V)))$. Then $f(V)\subset f(f^{-1}(\sigma_{(\beta,\beta')}\text{-}p\operatorname{Int}(f(V))))\subset\sigma_{(\beta,\beta')}\text{-}p\operatorname{Int}(f(V))$ and $\sigma_{(\beta,\beta')}\text{-}p\operatorname{Int}(f(V))\subset f(V)$. Hence $f(V)$ is a (β,β')-preopen set of Y; hence f is (γ,β)-preopen. □

Corollary 4.2. *Let* $f:(X,\tau)\to(Y,\sigma)$ *be a function. Then* f *is* (γ,β)*-preclosed and* (γ,β)*-precontinuous if and only if* $f(\tau_{(\gamma,\gamma')}\text{-}p\operatorname{Cl}(V))=\sigma_{(\beta,\beta')}\text{-}p\operatorname{Cl}(f(V)))$ *for every subset* V *of* X*.*

Corollary 4.3. *Let* $f:(X,\tau)\to(Y,\sigma)$ *be a function. Then* f *is* (γ,β)*-preopen and* (γ,β)*-precontinuous if and only if* $f^{-1}(\sigma_{(\beta,\beta')}\text{-}p\operatorname{Cl}(V)))=\tau_{(\gamma,\gamma')}\text{-}p\operatorname{Cl}(f^{-1}((V)))$ *for every subset* V *of* Y*.*

Theorem 4.27. *Let* $f:(X,\tau)\to(Y,\sigma)$ *be a function. Then* f *is a* (γ,β)*-preclosed function if and only if for each subset* V *of* X*,* $\sigma_{(\beta,\beta')}\text{-}p\operatorname{Cl}(f(V))\subset f(\tau_{(\gamma,\gamma')}\text{-}p\operatorname{Cl}(V))$*.*

Proof. Let f be a (γ,β)-preclosed function and V any subset of X. Then $f(V)\subset f(\tau_{(\gamma,\gamma')}\text{-}p\operatorname{Cl}(V))$ and $f(\tau_{(\gamma,\gamma')}\text{-}p\operatorname{Cl}(V))$ is a (β,β')-preclosed set of Y. We have $\sigma_{(\beta,\beta')}\text{-}p\operatorname{Cl}(f(V))\subset\sigma_{(\beta,\beta')}\text{-}p\operatorname{Cl}(f(\tau_{(\gamma,\gamma')}\text{-}p\operatorname{Cl}(V)))=f(\tau_{(\gamma,\gamma')}\text{-}p\operatorname{Cl}(V))$. Conversely, let V be a (γ,γ')-preopen set of X. Then $f(V)\subset\sigma_{(\beta,\beta')}\text{-}p\operatorname{Cl}(f(V))\subset f(\tau_{(\gamma,\gamma')}\text{-}p\operatorname{Cl}(V))=f(V)$; hence $f(V)$ is a (β,β')-preclosed subset of Y. Therefore, f is a (γ,β)-preclosed function. □

Theorem 4.28. *Let* $f:(X,\tau)\to(Y,\sigma)$ *be a function. Then* f *is a* (γ,β)*-preclosed function if and only if for each subset* V *of* Y*,* $f^{-1}(\sigma_{(\beta,\beta')}\text{-}p\operatorname{Cl}(V))\subset\tau_{(\gamma,\gamma')}\text{-}p\operatorname{Cl}(f^{-1}(V))$*.*

Proof. Let V be any subset of Y. Then by Theorem 4.27, $\sigma_{(\beta,\beta')}\text{-}p\operatorname{Cl}(V)\subset f(\tau_{(\gamma,\gamma')}\text{-}p\operatorname{Cl}(f^{-1}(V)))$. Since f is bijection, $f^{-1}(\sigma_{(\beta,\beta')}\text{-}p\operatorname{Cl}(V))=f^{-1}(\sigma_{(\beta,\beta')}\text{-}p\operatorname{Cl}(f(f^{-1}(V))))\subset f^{-1}(f(\tau_{(\gamma,\gamma')}\text{-}p\operatorname{Cl}(f^{-1}(V))))=\sigma_{(\beta,\beta')}\text{-}p\operatorname{Cl}(f^{-1}(V))$. Conversely, let U be any subset of X. Since f is bijection,
$\sigma_{(\beta,\beta')}\text{-}p\operatorname{Cl}(f(U))=f(f^{-1}(\sigma_{(\beta,\beta')}\text{-}p\operatorname{Cl}(f(U)))\subset f(\tau_{(\gamma,\gamma')}\text{-}p\operatorname{Cl}(f^{-1}(f(U))))=f(\tau_{(\gamma,\gamma')}\text{-}p\operatorname{Cl}(U))$.

Therefore, by Theorem 4.27, f is a (γ,β)-preclosed function. □

Theorem 4.29. *Let* $f:(X,\tau)\to(Y,\sigma)$ *be a* (γ,β)*-preopen function. If* V *is a subset of* Y *and* U *is a* (γ,γ')*-preclosed subset of* X *containing* $f^{-1}(V)$*, then there exists a* (β,β')*-preclosed set* F *of* Y *containing* V *such that* $f^{-1}(F)\subset U$*.*

Proof. Let V be any subset of Y and U a (γ, γ')-preclosed subset of X containing $f^{-1}(V)$, and let $F = Y\backslash(f(X\backslash V))$. Then $f(X\backslash V) \subset f(f^{-1}(X\backslash V)) \subset X\backslash V$ and $X\backslash U$ is a (γ, γ')-preopen set of X. Since f is (γ, β)-preopen, $f(X\backslash U)$ is a (β, β')-preopen set of Y. Hence F is a (β, β')-preclosed set of Y and $f^{-1}(F) = f^{-1}(Y\backslash(f(X\backslash U)) \subset U$. □

Theorem 4.30. *Let $f : (X, \tau) \to (Y, \sigma)$ be a (γ, β)-preclosed function. If V is a subset of Y and U is a (γ, γ')-preopen subset of X containing $f^{-1}(V)$, then there exists (β, β')-preopen set F of Y containing V such that $f^{-1}(F) \subset U$.*

Proof. The proof is similar to the Theorem 4.29. □

Theorem 4.31. *Let $f : (X, \tau) \to (Y, \sigma)$ be a (γ, β)-preopen function. Then for each subset V of Y, $f^{-1}(\sigma_{(\beta,\beta')}\text{-}p\,\mathrm{Int}(\sigma_{(\beta,\beta')}\text{-}p\,\mathrm{Cl}(V)) \subset \tau_{(\gamma,\gamma')}\text{-}p\,\mathrm{Cl}(f^{-1}(V)$.*

Proof. Let V be any subset of Y. Then $\tau_{(\gamma,\gamma')}\text{-}p\,\mathrm{Cl}(f^{-1}(V)$ is a (γ, γ')-preclosed set of X containing $f^{-1}(V)$. Since f is (γ, β)-preopen, by Theorem 4.29, there is a (β, β')-preopen set F of Y containing V such that $f^{-1}(\sigma_{(\beta,\beta')}\text{-}p\,\mathrm{Int}(\sigma_{(\beta,\beta')}\text{-}p\,\mathrm{Cl}(V)) \subset \tau_{(\gamma,\gamma')}\text{-}p\,\mathrm{Int}(\tau_{(\gamma,\gamma')}\text{-}p\,\mathrm{Cl}(F)) \subset f^{-1}(F) \subset \tau_{(\gamma,\gamma')}\text{-}p\,\mathrm{Cl}(f^{-1}(V))$. □

Theorem 4.32. *Let $f : (X, \tau) \to (Y, \sigma)$ be a bijection such that for each subset V of Y, $f^{-1}(\sigma_{(\beta,\beta')}\text{-}p\,\mathrm{Int}(\sigma_{(\beta,\beta')}\text{-}p\,\mathrm{Cl}(V)) \subset \tau_{(\gamma,\gamma')}\text{-}p\,\mathrm{Cl}(f^{-1}(V))$. Then f is a (γ, β)-preopen function.*

Proof. Let U be a (γ, γ')-preopen subset of X. Then $f(X\backslash U)$ is a subset of Y and $f^{-1}(\sigma_{(\beta,\beta')}\text{-}p\,\mathrm{Int}(\sigma_{(\beta,\beta')}\text{-}p\,\mathrm{Cl}(f(X\backslash U)))) \subset \tau_{(\gamma,\gamma')}\text{-}p\,\mathrm{Cl}(f^{-1}(f(X\backslash U))) = \tau_{(\gamma,\gamma')}\text{-}p\,\mathrm{Cl}(X\backslash U) = X\backslash U$, and so $\sigma_{(\beta,\beta')}\text{-}p\,\mathrm{Int}(\sigma_{(\beta,\beta')}\text{-}p\,\mathrm{Cl}(f(X\backslash U))) \subset f(X\backslash U)$. Hence $f(X\backslash U)$ is a (β, β')-preclosed set of Y and $f(U) = X\backslash(f(X\backslash U))$ is a (β, β')-preopen set of Y. Therefore, f is a (γ, β)-preopen function. □

Definition 4.13. [1] *A function $f : (X, \tau) \to (Y, \sigma)$ is said to be (γ, β)-prehomeomorphism if f and f^{-1} are (γ, β)-precontinuous.*

Theorem 4.33. *Let $f : (X, \tau) \to (Y, \sigma)$ be a bijection. Then the following statements are equivalent:*

(i) *f is (γ, β)-prehomeomorphism;*
(ii) *f^{-1} is (γ, β)-prehomeomorphism;*
(iii) *f and f^{-1} are (γ, β)-preopen ((γ, β)-preclosed);*
(iv) *f is (γ, β)-precontinuous and (γ, β)-preopen ((γ, β)-preclosed);*
(v) *$f(\tau_{(\gamma,\gamma')}\text{-}p\,\mathrm{Cl}(V)) = \sigma_{(\beta,\beta')}\text{-}p\,\mathrm{Cl}(f(V))$ for each subset V of X;*
(vi) *$f(\tau_{(\gamma,\gamma')}\text{-}p\,\mathrm{Int}(V)) = \sigma_{(\beta,\beta')}\text{-}p\,\mathrm{Int}(f(V))$ for each subset V of X;*
(vii) *$f^{-1}(\sigma_{(\beta,\beta')}\text{-}p\,\mathrm{Int}(V)) = \tau_{(\gamma,\gamma')}\text{-}p\,\mathrm{Int}(f^{-1}(V))$ for each subset V of Y;*
(viii) *$\tau_{(\gamma,\gamma')}\text{-}p\,\mathrm{Cl}(f^{-1}(V)) = f^{-1}(\sigma_{(\beta,\beta')}\text{-}p\,\mathrm{Cl}(V))$ for each subset V of Y.*

Proof. $(i)\Rightarrow(ii)$: It follows immediately from the definition of a (γ, β)-prehomeomorphism.

$(ii) \Rightarrow (iii) \Rightarrow (iv)$: It follows from Theorem 4.25.

$(iv) \Rightarrow (v)$: It follows from Theorem 4.28 and Corollary 4.2.

$(v)\Rightarrow(vi)$: Let U be a subset of X. Then by Theorem 2.1, $f(\tau_{(\gamma,\gamma')}\text{-}p\,\mathrm{Int}(U)) = X\backslash f(\tau_{(\gamma,\gamma')}\text{-}p\,\mathrm{Cl}(X\backslash U)) = X\backslash\sigma_{(\beta,\beta')}\text{-}p\,\mathrm{Cl}(f(X\backslash U)) = \sigma_{(\beta,\beta')}\text{-}p\,\mathrm{Int}(f(U))$.

$(vi) \Rightarrow (vii)$: Let V be a subset of Y.

Then $f(\tau_{(\gamma,\gamma')}\text{-}p\,\mathrm{Int}(f^{-1}(V))) = \sigma_{(\beta,\beta')}\text{-}p\,\mathrm{Int}(f(f^{-1}(V))) = \sigma_{(\beta,\beta')}\text{-}p\,\mathrm{Int}(f(V))$. Hence $f^{-1}(f(\tau_{(\gamma,\gamma')}\text{-}p\,\mathrm{Int}(f^{-1}(V)))) = f^{-1}(\sigma_{(\beta,\beta')}\text{-}p\,\mathrm{Int}(V))$. Therefore, $f^{-1}(\sigma_{(\beta,\beta')}\text{-}p\,\mathrm{Int}(V)) = \tau_{(\gamma,\gamma')}\text{-}p\,\mathrm{Int}(f^{-1}(V))$.

$(vii) \Rightarrow (viii)$: Let V be a subset of Y.
Then by Theorem 2.1, $\tau_{(\gamma,\gamma')}\text{-}p\,\mathrm{Cl}(f^{-1}(V)) = X\backslash(f^{-1}(\sigma_{(\beta,\beta')}\text{-}p\,\mathrm{Int}(Y\backslash V))) = X\backslash(\tau_{(\gamma,\gamma')}\text{-}p\,\mathrm{Int}(f^{-1}((X\backslash V)))) = f^{-1}(\sigma_{(\beta,\beta')}\text{-}p\,\mathrm{Cl}(V))$.

$(viii) \Rightarrow (i)$: It follows from Theorem 4.28 and Corollary 4.3. □

Theorem 4.34. *Every topological space (X, τ) with operations γ and γ' on τ is (γ, γ')-pre-$T_{1/2}$.*

Proof. Let $x \in X$. We prove (X, τ) is (γ, γ')-pre-$T_{1/2}$, it is sufficient to show that $\{x\}$ is (γ, γ')-preopen or (γ, γ')-preclosed. Now, if $\{x\}$ is (γ, γ')-open, then it is obviously (γ, γ')-preopen. If $\{x\}$ is not (γ, γ')-open, then $\tau_{(\gamma,\gamma')}\text{-Int}(\{x\}) = \emptyset$; hence $\tau_{(\gamma,\gamma')}\text{-Cl}(\tau_{(\gamma,\gamma')}\text{-Int}(\{x\})) = \emptyset \subset \{x\}$. Therefore, $\{x\}$ is (γ, γ')-preclosed. □

References

[1] Carpintero, C., Rajesh, N. and Rosas, E., *On a class of (γ, γ')-preopen sets in a topological space*, Fasc. Math. **46** (2011), 25–36

[2] Kasahara, S., *Operation-compact spaces*, Math. Japan. **24** (1979), 97–105

[3] Rajesh, N. and Vijayabharathi, V., *Bioperation-via Regular open sets* (submitted)

[4] Umehara, J., Maki, H. and Noiri, T., *Bioperation on topological spaces and some separation axioms*, Mem. Fac. Sci. Kochi Univ. (A) **13** (1992), 45–59

DEPARTMENT OF MATHEMATICS
RAJAH SERFOJI GOVT. COLLEGE
THANJAVUR-613005, TAMILNADU, INDIA
E-mail address: nrajesh_topology@yahoo.co.in

DEPARTMENT OF MATHEMATICS
NATIONAL INSTITUTE OF TECHNOLOGY, TIRUCHIRAPPALLI
TAMILNADU, INDIA
E-mail address: vijayabharathi_v@yahoo.com