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Inversions in 312–permutations

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ABSTRACT. We introduce an inversion diagram and use it to describe a simple bijection between 312–avoiding permutations and Dyck paths that sends inversion statistic to area statistic. We define Tamari and Dyck lattice using inversion tables. Our unified approach allows to understand the connections between these lattices better. We present a simple bijection between pairs of 312–avoiding permutations and perfect matchings of ordered graphs that avoid the pattern $abccab$.

1. INTRODUCTION

Pattern avoidance is a central problem in recent research in enumerative combinatorics. The first surprising result ([3], [9]), that the number of permutations of $[n]$ that avoid a permutation pattern of length 3 is equal to the n–th Catalan number.

$$C_n = \frac{1}{n+1}\binom{2n}{n}.$$

Catalan number is a unique integer sequence that enumerates more than 200 combinatorial objects [10]. In this work we consider permutations that avoid the pattern 312. We start with definitions and notations.

Let $\pi = \pi_1\pi_2\cdots\pi_n$ be a permutation of $[n]$ and $\sigma = \sigma_1\sigma_2\cdots\sigma_k$ be a permutation of $[k]$, $k < n$. We say that a permutation π *contains the pattern* σ, if there are indices $1 \leq i_1 < i_2 < \cdots < i_k \leq n$ such that $\pi_{i_1}\pi_{i_2}\cdots\pi_{i_k}$ has the same relative order as $\sigma_1\sigma_2\cdots\sigma_k$. Otherwise, π is said to *avoid the pattern* σ, or alternatively we say that π is σ*–avoiding*. We denote the set of permutations of $[n]$ that avoid the pattern $\sigma = 312$ by $S_n(312)$.

It is well known that a permutation is uniquely defined by its inversions. We can define two unique integer sequences to each permutation according to its inversions.

Let $(c_1, c_2, \ldots, c_n)$ be the integer sequence, where c_k is the number of elements to the right of k that are less than k:

$$c_k = |\{\pi_i : \pi_i < k = \pi_j \quad \text{and} \quad i > j\}|.$$

A c–vector $(c_1, c_2, \ldots, c_n)$ of a 312–avoiding permutation fulfils the properties

(c.1) $c_1 = 0$
(c.2) $0 \leq c_{k+1} \leq c_k + 1$ for $1 \leq k < n$.

and every vector with these conditions defines uniquely a 312–avoiding permutation.

We can turn this point of view and define another sequence $(s_1, \ldots, s_n)$, where s_k is the number of elements to the left of k that are greater than k:

$$s_k = |\{\pi_i | \pi_i > k = \pi_j \text{ and } i < j\}|.$$

The s–vector $(s_1, \ldots, s_n)$ of a 312–avoiding permutation fulfills the conditions

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(s.1) $0 \le s_k \le n-k$ for $1 \le k \le n$

(s.2) $s_{k+i} \le s_k - i$ for $1 \le k \le n-2$ and $1 \le i \le s_k$.

and every s–vector with these properties defines uniquely a 312-avoiding permutation.

The *inversion diagram* is a triangular shape with 1 cell in the top–most row, 2 cells in the row below, etc. and $n-1$ cells in the $(n-1)$th row. We mark every inversion (i,j) of the permutation by an X entry in the cell (i,j). Hence s_k is simply the number of the X entries in the k'th column and c_k is the number of the X entries in the k'th row.

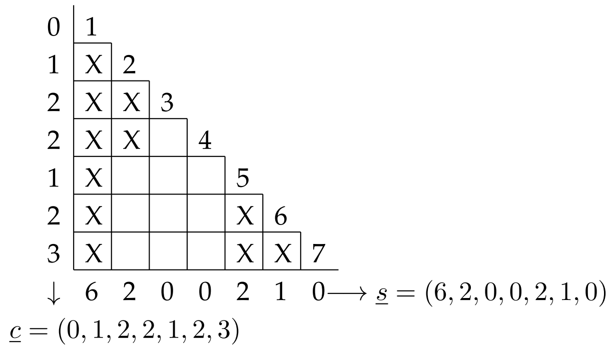

FIGURE 1. The sequences for $\pi = 3427651$

The diagram we obtained has a special form that rooted on the fact that in the permutation the pattern 312 can not appear. The X entries are arranged in each column in one block of the form $B_{i,l} = \{(i,j) : 1 \le i \le l\}$ and each such block has a „shadow" in that the occurrence of any X entry is forbidden. The block of length l in the i'th column (denoted by $B_{i,l}$) has the shadow:

$$\{i+l+1, i+l+2, \ldots, n\} \times \{i+1, i+2, \ldots, i+l\}.$$

We describe a bijection between 312–avoiding permutations and Dyck paths using the following trick on the diagram: slide all X entries to the right of the diagram. The boarder of the X entries determines the corresponding Dyck path. (See Figure 2.)

Our definition reveals the fact that the area of a Dyck path (the number of full squares „below" the path) coincides with the number of inversions in the corresponding 312–avoiding permutations.

2. LATTICES ON THE SET OF 312–AVOIDING PERMUTATIONS

The natural order on the set of s–, resp. c–vectors leads to the definition of posets on the set of 312–avoiding permutations. Both posets are well known lattices: the Tamari resp. the Dyck lattice. Various definitions of these lattices are known in the literature. We define them by the inversion tables of 312–avoiding permutations. This approach underlines the fact that these lattices are the restrictions of the weak resp. strong Bruhat order on permutations and that the Tamari lattice is a sublattice of the Dyck lattice.

Definition 2.1. [Tamari lattice] On the set of 312–avoiding permutations with consider the order relation

$$\pi \le_s \sigma \quad \text{iff} \quad s(\pi) \le s(\sigma),$$

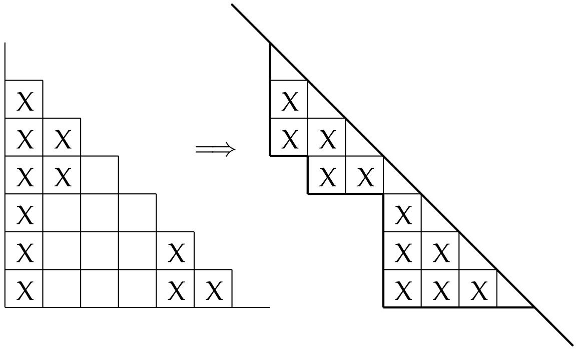

FIGURE 2. Inversion diagram and Dyck–path

where $s \leq s'$ is the usual relation on vectors: $(s_1, \ldots, s_n) \leq (s'_1, \ldots, s'_n)$ iff $s_j \leq s'_j$ for all $1 \leq j \leq n$.

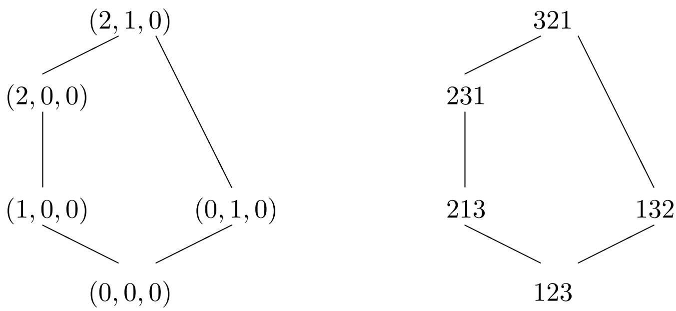

FIGURE 3. The Tamari lattice with five elements

Remark 2.1. In [[2], Section 9] Björner and Wachs studied the Tamari lattice. They discuss – as they remark – a surprisingly close connection that exists between Tamari lattices and weak order on the symmetric group. By defining a map from permutations to binary trees they showed that the sublattice of the weak order consisting of 312–avoiding permutations (and thus the Tamari lattice) is a quotient of the weak order in the order theoretic sense. It is interesting that though in [2] the Tamari lattice is actually defined on the set of inversion tables of 312–avoiding permutations, this simple fact is not mentioned there.

Definition 2.2. [Dyck lattice] On the set of 312–avoiding permutations with consider the order relation

$$\pi \leq_c \sigma \quad \text{iff} \quad c(\pi) \leq c(\sigma),$$

where $c \leq c'$ is defined as usual $(c_1, \ldots, c_n) \leq (c'_1, \ldots, c'_n)$ iff $c_j \leq c'_j$ for all $1 \leq j \leq n$.

These lattices are important in enumerative combinatorics since the numbers of intervals of these lattices arise as enumeration results. We mention some examples.

The intervals (pairs of comparable elements) of the Tamari lattice were enumerated by Chapoton [4] using generating function approach. The number of intervals in the Tamari lattice is

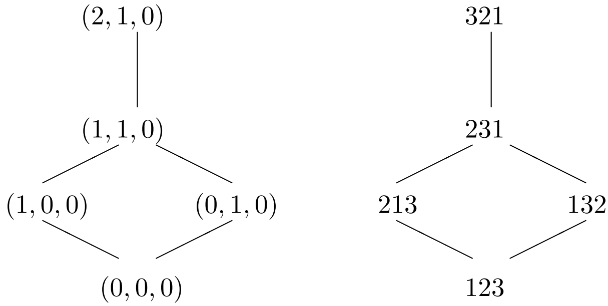

FIGURE 4. The Dyck lattice with five elements

$$|I_n^{\mathcal{T}}| = \frac{2(4n+1)!}{(n+1)!(3n+2)!}.$$

Chapoton noticed that this is the number of maximal planar maps and asked for an explanation. Bernardi and Bonichon [1] investigated realizers of maximal planar maps and established a bijection between special pairs of Dyck–paths and minimal realizers of size n.

The number of intervals of the Dyck lattice is given by a determinant. This result is a special case of the theory of noncrossing Dyck paths.

$$|I_n^{\mathcal{D}}| = \begin{vmatrix} C_n & C_{n+1} \\ C_{n+1} & C_{n+2} \end{vmatrix} = \frac{6(2n)!(2n+2)!}{n!(n+1)!(n+2)!(n+3)!}. \quad (2.1)$$

The list of combinatorial objects that are enumerated by the formula (2.1) is long and seems to grow. We give here a short (not complete) list with references: 2-triangulations [7][8], reduced pipe dreams of $\pi_{2,n}$ [11], 3-noncrossing and 3-nonnesting matchings, oscillating tableaux [5], realizers of triangulations [1].

3. BIJECTION BETWEEN PERFECT MATCHINGS AND PAIRS OF 312–PERMUTATIONS

Perfect matchings that avoid the pattern $abccab$ is a further example that are enumerated by the formula (2.1). Jelinek [6] constructed a bijection between pairs of noncrossing Dyck paths and these matchings.

We present in this section a bijection between these matchings and appropriate pairs of 312–avoiding matchings. Our bijection coincides with Jelinek's when we combine it with the bijection between the 312–avoiding permutations and Dyck paths that we described in section 1. Despite of this we think that it is worth to formulate our bijection because our approach is more natural and the description of our bijection is simpler.

We recall some basic definitions and notations. A *matching* M of size n is a graph on the vertex set $[2n]$ whose every vertex has degree one. For an arbitrary edge $e = \{i, j\}$ of M $i < j$ we say that i is an *l–vertex* and j is an *r–vertex* of M.

A matching $M = (V, E)$ *contains* a matching $M' = (V', E')$ if there is a monotone edge-preserving injection from V' to V; in other words, M contains M' if there is a function $f : V' \to V$ such that $u < v$ implies $f(u) \to f(v)$ and $\{u, v\} \in E'$ implies $\{f(u), f(v)\} \in E$.

Let $M_n(abccab)$ denote the matchings that avoid the matching pattern $\{\{1,5\},\{2,6\},\{3,4\}\}$. (See Figure 5.)

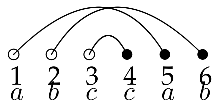

FIGURE 5. The matching pattern $\{\{1,5\},\{2,6\},\{3,4\}\}$

Theorem 3.1. [6]

$$|\mathcal{M}_n(abccab)| = |I_n^{\mathcal{D}}| = \begin{vmatrix} C_n & C_{n+1} \\ C_{n+1} & C_{n+2} \end{vmatrix}.$$

We prove this theorem by a bijection. Every matching $M \in \mathcal{M}_n(abccab)$ defines two permutations. Since the graph is ordered the nodes has a total order. We pick out the l vertices and label them by $1, \ldots, n$ according their positions in the total order.

First we ignore the edges and consider only the arrangement of l and r vertices. Let c_k be $k-1$ decreased by the number of r vertices to the left of the kth l vertex. Clearly $(c_1, \ldots, c_n)$ fulfils the conditions (c.1) and (c.2). Let σ be the 312–avoiding permutation with the so defined c–vector.

We obtain another permutation π regarding M as a perfect matching of the bipartite graph. We label the edges with the label of its l vertex. We visit the nodes of the graph in the total order and at each r vertex we record the label of the incident edge. Since M avoids the pattern $abccab$ the permutation π avoids the pattern 312.

Lemma 3.1. *Let σ and π be the permutations that we obtain by the procedure described above. It is true that*

$$c(\pi) \le c(\sigma).$$

Proof. Consider a perfect matching on an ordered graph. Let $M(c_k)$ be the number of arcs that are above the arc incident with the kth l vertex. To a given l-r vertex arrangement there is a unique matching that has a maximal associated $M(c_k)$ sequence. This matching is the noncrossing matching and the corresponding $M(c_k)$'s are the entries of the c-vector of σ. The kth entry of the c–vector of the permutation π is exactly the same as the associated $M(c_k)$ to a given $M \in M_n(abccab)$. □

Proof. The contant of the lemma is that the permutation pair that we obtain by the map we described is a pair that corresponds to an interval in the Dyck lattice. Hence the bijection proofs the Theorem 3.1. □

Example 3.1. We present an example for better understanding (see Figure 6.). Let the matching be

$$M = \{\{1,9\},\{2,3\},\{4,7\},\{5,6\},\{8,14\},\{10,12\},\{11,13\},\{15,16\}\}$$

First we label the l vertices from left to right. One permutation ($\pi = 24316758$) is easy to record. The first element of π is 2, because the first r vertex is connected to the l vertex labelled by 2. The second element of π is 4, because the second r vertex is connected to the l vertex labelled by 4 and so on.

The inversion table of the other permutation $(c_1, \ldots, c_7)$ can be recorded according to the arrangement of the l and r vertices. $c_1 = 0$ always, because the first vertex is always an l vertex. $c_2 = (2-1) = 1$, because there is no r vertex to the left of the second l vertex. $c_3 = (3-1) - 1$, because there is an r vertex to the left of the third l vertex, and so on. We record $(0,1,1,2,1,1,2)$. This inversion table defines uniquely

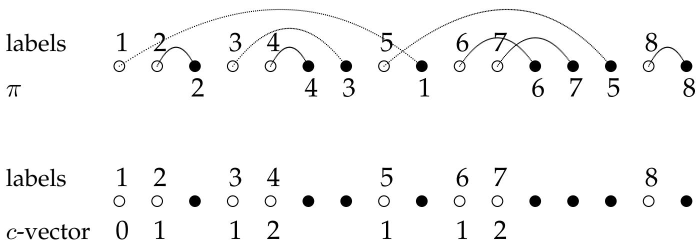

FIGURE 6. The bijection

the permutation 24357618. We obtained the following pair of 312–avoiding permutations $(\pi, \sigma) = (24316758, 24357618)$.

4. CONCLUSIONS

In this work we pointed out the crucial role of inversions in 312–avoiding permutations: we introduced the inversion diagram of 312–avoiding permutations and defined two basic lattices using inversion tables. The numbers of intervals of these lattices appear in many enumeration results. According to our observations these theorems can be proved bijectively by coding the combinatorial objects with appropriate pairs of 312–avoiding permutations. We underlined our idea by presenting one example: the case of perfect matchings that avoid the pattern $abccab$.

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