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Existence and behaviour of parameter classes of solutions of a system of quasilinear differential equations

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ABSTRACT. The paper presents some results on the existence and behaviour of parameter classes of solutions for system of quasilinear differential equations. Behaviour of integral curves in neighborhood of an arbitrary or a certain curve is considered. The obtained results contain the answer to the question on stability as well as approximation of solutions whose existence is established. The errors of the approximation are defined by the functions that can be sufficiently small. To obtain our main results, the theory of qualitative analysis of differential equations and topological retraction method are used.

1. INTRODUCTION

Let us consider the system of quasilinear differential equations

$$\dot{x} = A(x,t)\,x + F(x,t), \tag{1.1}$$

where $x(t) = (x_1(t), ..., x_n(t))^T,\ n \geq 2\,;\ t \in I = \langle a, \infty\rangle,\ a \in \mathbb{R}\,; D \subset \mathbb{R}^n$ is open set, $\Omega = D \times I$, $A(x,t) = (a_{ij}(x,t))_{n\times n}$ is the matrix-function with elements $a_{ij} \in C(\Omega, \mathbb{R})$ $(i,j = 1,...,n)$. $F(x,t) = (f_1(x,t),...,f_n(x,t))^T$ is the vector-function with elements $f_i \in C(\Omega,\mathbb{R})$, functions $f_i(x,t)$, $i = 1,...,n$, satisfy the Lipschitz's condition with respect to the variable x.

The above conditions for the functions a_{ij}, f_i $(i,j = 1,...,n)$, are guaranting the existence and uniquenes of solution for every Cauchy's problem for the system (1.1), in Ω.

Let

$$\Gamma = \{(x,t) \in \Omega : x = \varphi(t),\ t \in I\}, \tag{1.2}$$

where $\varphi(t) = (\varphi_1(t),...,\varphi_n(t)),\ \varphi_j(t) \in C^1(I,\mathbb{R}),$ is a certain curve in Ω.

In this paper the behaviour of the solutions of the system (1.1) in the neighborhood of a curve Γ is considered. The qualitative analysis theory of differential equations and the topological retraction method of T. Ważewski [8], are used.

2. NOTATIONS AND PRELIMINARIES

We are interested in the study of the behaviour of integral curves $(x(t),t), t \in I$, of system (1.1) with respect to the set

$$\omega = \{(x,t) \in \Omega : |x_i - \varphi_i(t)| < r_i(t)\,,\ i = 1,...,n\}, \tag{2.3}$$

where $r_i \in C^1(I,\mathbb{R}^+),\ i = 1,...,n$. The boundary surfaces of the set ω with respect to the set Ω are

$$W_i^k = \{(x,t) \in Cl\omega \cap \Omega : B_i^k(x,t) := (-1)^k(x_i - \varphi_i(t)) - r_i(t) = 0\}, \tag{2.4}$$

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where $k = 1,2,\ i = 1,...,n$. Let us denote the tangent vector field to an integral curve $(x(t),t), t \in I$, of (1.1) by T. We have

$$T = (\sum_{j=1}^{n} a_{1j}x_j + f_1, ..., \sum_{j=1}^{n} a_{ij}x_j + f_i, ..., \sum_{j=1}^{n} a_{nj}x_j + f_n, 1).$$

The vectors ∇B_i^k are the external normal on surfaces W_i^k. We have

$$\nabla B_i^k = (\,\delta_{1i}\,, ..., \,\delta_{ni}\,,\, (-1)^{k+1}\varphi_i' - r_i'),$$

where δ_{ji} is the Kronecker delta symbol.

Considering the sign of the scalar products

$$P_i^k(x,t) = (\nabla B_i^k, T) \ \text{ on } \ W_i^k,\ k = 1,2\,,\ i = 1,...,n\,,$$

we shall establish the behavior of integral curve of (1.1), with respect to the set ω.

Let us denote by $S^p(I),\ p \in \{0,1,...,n\}$, a class of solutions of the system (1.1), defined on I, which depends on p parameters. We shall simply say that the class of solutions $S^p(I)$ belongs to the set ω if graphs of function in $S^p(I)$ are contained in ω. In that case we shall write $S^p(I) \subset \omega$.

The results of this paper are based on the following Lemma in (see [1], [3]). In the following $(n_1, \cdots, n_n)$ denote a permutation of the indices $(1, \cdots, n)$.

Lemma 2.1. *If, for the system* (1.1), *the scalar products*

$$P_i^k = (\nabla B_i^k, T) < 0 \quad \text{on } W_i^k, k = 1,2,\ i = n_1, \cdots, n_p, \tag{2.5}$$

and

$$P_i^k = (\nabla B_i^k, T) > 0 \quad \text{on } W_i^k,\ k = 1,2,\ i = n_{p+1}, \cdots, n_n, \tag{2.6}$$

where $p \in \{0,1,\cdots,n\}$, then the system (1.1) *has a $p-$parameter class of solutions which belongs to the set ω (graphs of solutions belong to ω) for all $t \in I$.*

Notice that, according to this Lemma, the case $p = 0$ means that the system (1.1) has at least one solution belonging to the set ω for all $t \in I$.

The conditions (2.5) and (2.6) imply that the set $U = \bigcup_{i=n_1}^{n_p} (W_i^1 \cup W_i^2)$ has no point of exit and $V = \bigcup_{i=n_{p+1}}^{n_n} (W_i^1 \cup W_i^2)$ is the set of points of strict exit from set ω with respect to the set Ω, for integral curves of system (1.1), which according to the retraction method [8], makes the statement of Lemma valid (see [1], [2], [7]). In the case $p = n$ this Lemma gives the statement of Lemma 1, and for $p = 0$ the statement of Lemma 2 in [6].

3. Main results

Let

$$\begin{aligned} X_i(x,t) \quad &: \quad = \sum_{j=1}^{n} a_{ij}(x,t)x_j + f_i(x,t) - \varphi_i'(t), \\ \Phi_i(x,t) \quad &: \quad = \sum_{j=1}^{n} a_{ij}(x,t)\varphi_j + f_i(x,t) - \varphi_i'(t),\ i = 1,...,n. \end{aligned}$$

Theorem 3.1. *Let $r_i \in C^1(I, \mathbb{R}^+)$. If, on $W_i^k,\ k = 1, 2,$*

$$|X_i(x,t)| < r_i'(t), \quad i = n_1, ..., n_p, \tag{3.7}$$

and

$$|X_i(x,t)| < -r_i'(t), \quad i = n_{p+1}, ..., n_n, \tag{3.8}$$

$(p \in \{1, ..., n\})$, *then the system* (1.1) *has a $p-$parameter class of solutions $S^p(I)$ which belongs to the set ω for all $t \in I$, i.e. $S^p(I) \subset \omega$.*

Proof. Let us consider the behavior of the integral curves of system (1.1) with respect the set ω, which are defined by (2.3). For the scalar products $P_i^k(x,t)$ on $W_i^k,\ k = 1, 2$, $i = 1, ..., n$, we have

$$P_i^k(x,t) = (-1)^k(\sum_{j=1}^{n} a_{ij}x_j + f_i) - (-1)^k\varphi_i' - r_i'(t) = (-1)^k X_i(x,t) - r_i'(t)\,.$$

According to (3.7) and (3.8) we have

$$P_i^k(x,t) \le |X_i(x,t)| - r_i'(t) < 0\,, \ \text{ on } W_i^k,\ k = 1, 2\,,\ i = n_1, ..., n_p,$$

and

$$P_i^k(x,t) \ge -|X_i(x,t)| - r_i'(t) > 0\,, \ \text{ on } W_i^k,\ k = 1, 2\,,\ i = n_{p+1}, ..., n_n\,.$$

Hence, in direction of p axis we have $P_i^k(x,t) < 0$ on W_i^k, and in the direction of other $n-p$ axis $P_i^k(x,t) > 0$ on $W_i^k,\ k = 1, 2$. These estimates, according to Lemma 2.1., confirm the statements of this Theorem. □

Theorem 3.2. *Let $r_i \in C^1(I, \mathbb{R}^+)$. If*

$$|X_i(x,t) + a_{ii}(x,t)(\varphi_i(t) - x_i(t))| < -a_{ii}(x,t)\, r_i(t) + r_i'(t) \tag{3.9}$$

on $W_i^k,\ k = 1, 2\,,\ i = n_1, ..., n_p$, and

$$|X_i(x,t) + a_{ii}(x,t)(\varphi_i(t) - x_i(t))| < a_{ii}(x,t)\, r_i(t) - r_i'(t) \tag{3.10}$$

on $W_i^k,\ k = 1, 2\,,\ i = n_{p+1}, ..., n_n$, then the system (1.1) *has the $p-$parameter class of solutions $S^p(I) \subset \omega$.*

Proof. Let us consider the behavior the integral curves of system (1.1) with respect the set ω, which defined by (2.3). For the scalar products $P_i^k(x,t)$ on $W_i^k,\ k = 1, 2$, $i = 1, ..., n$, we have

$$\begin{aligned} P_i^k(x,t) &= (-1)^k[\sum_{j=1}^{n} a_{ij}(x,t)\, x_j + f_i(x,t) - \varphi_i'(t)] - r_i'(t) \\ &= (-1)^k a_{ii}(x,t)(x_i - \varphi_i(t)) + (-1)^k a_{ii}(x,t)\varphi_i(t) + \\ &\quad +(-1)^k(\sum_{j=1 (j\neq i)}^{n} a_{ij}(x,t)x_j + f_i(x,t) - \varphi_i'(t)) - r_i'(t) \qquad (3.11) \\ &= a_{ii}(x,t) r_i(t) + (-1)^k[X_i + a_{ii}(x,t)(\varphi_i(t) - x_i)] - r_i'(t) \end{aligned}$$

According to (3.9) and (3.10), we have on $W_i^k,\ k=1,2$,

$$P_i^k(x,t) \le a_{ii}\, r_i(t) + [X_i + a_{ii}(\varphi_i - x_i)] - r_i'(t) < 0,\ \ i = n_1, ..., n_p,$$

$$P_i^k(x,t) \ge a_{ii}\, r_i(t) - [X_i + a_{ii}(\varphi_i - x_i)] - r_i'(t) > 0,\ \ i = n_{p+1}, ..., n_n.$$

These estimates for $P_i^k(x,t)$ on W_i^k imply the statement of the Theorem. □

Theorem 3.3. *Let $r_i \in C^1(I, \mathbb{R}^+)$. If*

$$\sum_{j=1 (j\neq i)}^{n} |a_{ij}(x,t)|\, r_j(t) + |\Phi_i(x,t)| < -a_{ii}(x,t)\, r_i(t) + r_i'(t) \tag{3.12}$$

on $W_i^k,\ k=1,2$, $i = n_1, ..., n_p$, and

$$\sum_{j=1 (j\neq i)}^{n} |a_{ij}(x,t)|\, r_j(t) + |\Phi_i(x,t)| < a_{ii}(x,t)\, r_i(t) - r_i'(t) \tag{3.13}$$

on $W_i^k,\ k=1,2$, $i = n_{p+1}, ..., n_n$, then the system (1.1) *has a $p-$ parameter class of solutions $S^p(I)$ which belongs the set ω for all $t \in I$, i.e. $S^p(I) \subset \omega$.*

Proof. Using (3.11) for the scalar products $P_i^k(x,t)$ on $W_i^k,\ k=1,2$,we have

$$\begin{aligned} P_i^k(x,t) &= (-1)^k [\sum_{j=1}^{n} a_{ij}\, x_j + f_i - \varphi_i'] - r_i' \\ &= (-1)^k a_{ii}(x_i - \varphi_i) + (-1)^k a_{ii}\varphi_i + \\ &\quad + (-1)^k (\sum_{j=1 (j\neq i)}^{n} a_{ij}(x_j - \varphi_j) + \sum_{j=1 (j\neq i)}^{n} a_{ij}\varphi_j + f_i - \varphi_i' t) - r_i' \\ &= a_{ii} r_i + (-1)^k \sum_{j=1 (j\neq i)}^{n} a_{ij}(x_j - \varphi_j) + (-1)^k \Phi_i - r_i' \end{aligned}$$

Now, it is enough to note that, on $W_i^k,\ k=1,2$, according to (3.12) and (3.13)

$$P_i^k(x,t) \le a_{ii}\, r_i + \sum_{j=1 (j\neq i)}^{n} |a_{ij}|\, r_j + |\Phi_i| - r_i' < 0,\ \ i = n_1, ..., n_p,$$

$$P_i^k(x,t) \ge a_{ii}\, r_i - \sum_{j=1 (j\neq i)}^{n} |a_{ij}|\ r_j - |\Phi_i| - r_i' > 0,\ \ i = n_{p+1}, ..., n_n.$$

□

Using Theorem 3.1. and Theorem 3.2 we can give special results. For example,

Corollary 3.1. *Let* $r_i \in C^1(I, \mathbb{R}^+)$ *and let the curve* Γ *be t-axis* $(\varphi(t) = 0)$. *If*

$$\left| \sum_{j=1(j\neq i)}^{n} a_{ij}\,(x,t)\,x_j + f_i(x,t) \right| < r_i'(t)\,, \tag{3.14}$$

on $W_i^k,\ k = 1,2\ ,\ i = n_1, ..., n_p$, *and*

$$\left| \sum_{j=1(j\neq i)}^{n} a_{ij}(x,t)x_j + f_i(x,t) \right| < -r_i'(t)\,, \tag{3.15}$$

on $W_i^k,\ k = 1,2\ ,\ i = n_{p+1}, ..., n_n$, *then the system* (1.1) *has a* $p-$*parameter class of solutions* $S^p(I)$ *which belongs the set* ω *for all* $t \in I$, *i.e.* $S^p(I) \subset \omega$.

Corollary 3.2. *Let* $r_i \in C^1(I, \mathbb{R}^+)$. *If*

$$\left| \sum_{j=1(j\neq i)}^{n} a_{ij}(x,t)x_j + f_i(x,t) \right| < -a_{ii}(x,t)\,r_i(t) + r_i'(t)\,, \tag{3.16}$$

on $W_i^k,\ k = 1,2\ ,\ i = n_1, ..., n_p$, *and*

$$\left| \sum_{j=1(j\neq i)}^{n} a_{ij}(x,t)x_j + f_i(x,t) \right| < a_{ii}(x,t)\,r_i(t) - r_i'(t)\,, \tag{3.17}$$

on $W_i^k,\ k = 1,2\ ,\ i = n_{p+1}, ..., n_n$, *then the system* (1.1) *has a* $p-$*parameter class of solutions* $S^p(I)$ *which belongs the set* ω *for all* $t \in I$, *i.e.* $S^p(I) \subset \omega$.

4. Applications

Let us consider the system of differential equations of Volterra type

$$\dot{x}_i = (q_i + g_i\,(x))\ x_i\ ,\ i = 1, ..., n\,, \tag{4.18}$$

where $g_i \in C\,(D, \mathbb{R})\,,\ i = 1, ..., n$. Functions g_i satisfy the Lipschitz's condition with respect to the variable x on D.

We consider the behavior of the integral curves of the system (4.18) in neighborhoods of the curve $\left(x^0, t\right)\,,\ t \in I$, where $x^0 = \left(x_1^0, \ldots, x_n^0\right) \in \mathbb{R}^n$.

Theorem 4.4. *Let* $r_i \in C^1\,(I, \mathbb{R}^+)$ *and let*

$$r\,(t) = \sqrt{r_1^2\,(t) + \cdots + r_n^2\,(t)}$$

and

$$|g_i\,(x) - g_i\,(y)| \leqslant L_i\,\|x - y\| \quad for\ all\ x, y \in D.$$

If

$$L_i\left|x_i^0\right|\,r\,(t) < -\,(q_i + g_i\,(x))\ r_i\,(t) + r_i'\,(t)$$

on $W_i^k,\ k = 1,2$, *for* $i = n_1, ..., n_p$, *and*

$$L_i\left|x_i^0\right|\,r\,(t) < (q_i + g_i\,(x))\ r_i\,(t) - r_i'\,(t)$$

on $W_i^k,\ k=1,2$ *, for* $i=n_{p+1},...,n_n$*, then the system* (4.18) *has a p-parameter class of solutions* $x(t)$ *satisfying the conditions*

$$\left|x_i(t)-x_i^0\right|<r_i(t), \quad i=1,\ldots,n, \ \ t\in I.$$

Proof. For the scalar product $P_i^k(x,t)$ we have

$$\begin{aligned} P_i^k(x,t) &= (q_i+g_i(x))\,r_i(t)+(-1)^k(q_i+g_i(x))\,x_i^0-r_i'(t)\\ &= (q_i+g_i(x))\,r_i+(-1)^k\left[q_i+g_i\left(x^0\right)+g_i-g_i\left(x^0\right)\right]\,x_i^0-r_i'(t)\\ &= (q_i+g_i(x))\,r_i+(-1)^k\left[g_i(x)+g_i\left(x^0\right)\right]\,x_i^0-r_i'(t)\end{aligned}$$

In view of assumptions of the theorem, we have

$$\begin{aligned} P_i^k(x,t) &\leqslant (q_i+g_i(x))\,r_i(t)+\left|g_i(x)+g_i\left(x^0\right)\right|\ \left|x_i^0\right|-r_i'(t)\\ &\leqslant (q_i+g_i(x))\,r_i(t)+L_i\ \left|x_i^0\right|\ r(t)-r_i'(t)<0\end{aligned}$$

on $W_i^k,\ k=1,2$, for $i=n_1,...,n_p$, and

$$P_i^k(x,t)\geqslant(q_i+g_i(x))\,r_i-L_i\ \left|x_i^0\right|\ r(t)-r_i'(t)>0$$

on $W_i^k,\ k=1,2$, for $i=n_{p+1},...,n_n$.

The conclusion of the theorem now follows according to Lemma 2.1. □

As an example, we can can consider the following system which is known as predator-prey ecosystem model, (see [4]):

$$\begin{aligned}\dot{x}_1 &= (1-x_1-ax_2)\,x_1,\\ \dot{x}_2 &= (-b+ax_1)\,x_2, \quad a,b\in\mathbb{R}^+.\end{aligned} \tag{4.19}$$

For the system (4.19) we may state the following result:

Corollary 4.3. *Let* $\alpha,\beta\in\mathbb{R}^+$.

a) If $1+\beta>\alpha(1+a)$ *and* $b>\alpha a+\beta$ *then system* (4.19) *has a one-parameter class of solutions* $x(t)$ *which satisfying conditions*

$$|x_1(t)|<\alpha e^{-\beta t}, \ \ |x_2(t)|<\alpha e^{-\beta t} \ \ for\ t>0.$$

b) If $\beta>(1+a)(1+\alpha)$, $b>a(1+\alpha)+\beta$ *then system* (4.19) *has a one-parameter class of solutions* $x(t)$ *wich satisfying conditions*

$$|x_1(t)-1|<\alpha e^{-\beta t}, \ \ |x_2(t)|<\alpha e^{-\beta t} \ \ for\ t>0.$$

This *Corollary* follows from *Theorem 4.4.* In this example we consider $x^0=(0,0)$ in case a) and $x^0=(1,0)$ in case b). Here we have on $W_i^k(i,k=1,2)$:

a)

$$P_2^k(x_1,x_2,t)=(-b+ax_1)\alpha e^{-\beta t}+\beta\alpha e^{-\beta t}<(-b+a\alpha+\beta)\alpha e^{-\beta t}\le 0,$$
$$P_1^k(x_1,x_2,t)=(1-x_1-ax_2)\alpha e^{-\beta t}+\beta\alpha e^{-\beta t}>(1-\alpha-a\alpha+\beta)\alpha e^{-\beta t}\ge 0;$$

b)

$$P_2^k(x_1,x_2,t)=(-b+ax_1)\alpha e^{-\beta t}+\beta\alpha e^{-\beta t}<(-b+a(1+\alpha)+\beta)\alpha e^{-\beta t}\le 0,$$

$$\begin{aligned} P_1^k(x_1,x_2,t) &= (1-x_1-ax_2)\alpha e^{-\beta t} + (-1)^k(1-x_1-ax_2) + \beta\alpha e^{-\beta t} \\ &\geq (-\alpha e^{-\beta t} - a\alpha e^{-\beta t} + \beta)\alpha e^{-\beta t} - \alpha e^{-\beta t} - a\alpha e^{-\beta t} + \beta\alpha e^{-\beta t} \\ &> (-\alpha - a\alpha - 1 - a + \beta)\alpha e^{-\beta t} \geq 0. \end{aligned}$$

Remark 4.1. The obtained results also contain answers to questions on approximation and stability or instability of solutions $x(t)$ whose existence is established. The errors of the approximation and the function of stability or instability (including autostability and stability along the coordinates) are defined by functions $r_i(t), i = 1, ..., n$.

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