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Multivalued almost contractions in metric space endowed with a graph

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ABSTRACT. The main goal of this paper is to introduce a multivalued almost contraction on a metric space with a graph. In terms of this new contraction, we establish some fixed point results on graph.

1. INTRODUCTION

Fixed point theory for multivalued map is started by Nadler [13]. Then many authors have improved this theory for many type contractions on metric space. Also, in 2013 Divenari and Frigon [11] presented some fixed point result for multivalued maps combining fixed point theory and graph theory. Then Beg and Butt [3] obtained sufficient conditions for the existence of a common fixed point for a multivalued mappings in metric space X endowed with a graph. Recently, in [10, 14] the authors proved some fixed point theorems in graph theory. Using F-contraction Acar and Altun [1] gave some results for multivalued mappings on a metric space involving a graph. Here, we shall investigate further fixed point theorems for multivalued mappings using almost contraction in the light of the paper [8] in a metric space endowed with a graph.

2. PRELIMINARIES

2.1. **Graph theory.** Let X be a nonempty set and Δ denotes the diagonal of Cartesian product $X \times X$. A graph on X is an object $G = (V(G), E(G))$, where $V(G)$ is vertex set, whose elements are called vertices and $E(G)$ is edge set. We assume that G has no parallel edges and $\Delta \subset E(G)$.

If x and y are vertices of G, then a path in G from x to y of length $k \in \mathbb{N}$ is a finite sequence $\{x_n\}_{n\in\{0,1,2,\cdots,k\}}$ of vertices such that

$$x_0 = x, x_k = y \text{ and } (x_{i-1}, x_i) \in E(G) \text{ for } i \in \{1, 2, \cdots, k\}.$$

Notice that a graph G is connected if there is a path between any two vertices and it is weakly connected if $\tilde{G}$ is connected, where $\tilde{G}$ denotes the undirected graph obtained from G by ignoring the direction of edges.

Denote by G^{-1} the graph obtained from G by reversing the direction of edges. Thus

$$E(G^{-1}) = \{(x, y) \in X \times X : (y, x) \in E(G)\}.$$

Since it is more convenient to treat $\tilde{G}$ as a directed graph for which the set of its edges is symmetric, under this convention, we have that

$$E(\tilde{G}) = E(G) \cup E(G^{-1}).$$

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We call (V', E') a subgraph of G if $V' \subseteq V(G)$ and $E' \subseteq E(G)$ and for any edge $(x, y) \in E'$, $x, y \in V'$.

If G is symmetric and x is a vertex in G, then the subgraph G_x consisting of all edges and vertices which are contained in some path beginning at x is called the component of G containing x. In this case $V(G_x) = [x]_G$, where $[x]_G$ is the equivalence class of the following relation $\mathcal{R}$ defined on $V(G)$ by the rule:

$$y\mathcal{R}z \text{ if there is path in } G \text{ from } y \text{ to } z.$$

We can find more information about graph theory in [12].

Let (X, d) be a metric space. $P(X)$ denotes the family of all nonempty subsets of X, $CB(X)$ denotes the family of all nonempty, closed and bounded subsets of X and $K(X)$ denotes the family of all nonempty compact subsets of X. It is clear that, $K(X) \subseteq CB(X) \subseteq P(X)$. For $A, B \in CB(X)$, let

$$H(A, B) = \max\left\{\sup_{x\in A} D(x, B), \sup_{y\in B} D(y, A)\right\},$$

where $D(x, B) = \inf\{d(x, y) : y \in B\}$. Then H is a metric on $CB(X)$, which is called Pompeiu-Hausdorff metric induced by d. We can find detailed information about Pompeiu-Hausdorff metric in [2, 9].

Definition 2.1. Let (X, d) be a metric space, $G = (V(G), E(G))$ be a graph such that $V(G) = X$ and let $T : X \to CB(X)$. Then T is said to be graph-preserving if

$$(x, y) \in E(G) \Rightarrow (u, v) \in E(G) \text{ for all } u \in Tx \text{ and } v \in Ty.$$

Definition 2.2. Let (X, d) be a metric space, $G = (V(G), E(G))$ be a graph such that $V(G) = X$ and let $T : X \to CB(X)$. Then we say that T has weakly graph-preserving property whenever for each $x \in X$ and $y \in Tx$ with $(x, y) \in E(G)$ implies $(y, z) \in E(G)$ for all $z \in Ty$.

Remark 2.1. It is clear that every graph preserving map is a weakly graph preserving. But the converge may not be true. For example, let $X = [-1, 1]$ with the usual metric. Consider a graph given by $V(G) = X$ and $E(G) = X \times X \setminus \Delta$. Define $T : X \to CB(X)$ by

$$Tx = \begin{cases} \{-x\} & , \quad x \notin \{-1, 0\} \\ \{0, 1\} & , \quad x = -1 \\ \{1\} & , \quad x = 0 \end{cases}.$$

Then it can be seen that T is weakly graph-preserving map but not graph-preserving.

Lemma 2.1. *Let (X, d) be a metric space and $T : X \to P(X)$ be an upper semicontinuous mapping such that Tx is closed for all $x \in X$. If $x_n \to x_0$, $y_n \to y_0$ and $y_n \in Tx_n$, then $y_0 \in Tx_0$.*

2.2. **Almost contraction.** Berinde [5, 6, 7] defined almost contraction (or (δ, L)-weak contraction) mappings in a metric space as follow:

Definition 2.3. Let (X, d) be a metric space and $T : X \to X$ is a self operator. T is said to be a almost contraction (or (δ, L)-weak contraction) if there exists a constant $\delta \in (0, 1)$ and some $L \geq 0$ such that

$$d(Tx, Ty) \leq \delta d(x, y) + Ld(y, Tx) \tag{2.1}$$

for all $x, y \in X$.

Note that, by the symmetry property of the distance, the almost contraction condition implicitly includes the following dual one

$$d(Tx, Ty) \leq \delta d(x, y) + Ld(x, Ty) \tag{2.2}$$

for all $x, y \in X$. So, in order to check the almost contractiveness of a mapping T, it is necessary to check both (2.1) and (2.2).

In [6] and [7], Berinde shows that any Banach, Kannan, Chatterjea and Zamfirescu mappings are almost contraction. Using the concept of almost contraction mappings, Berinde [6] proved that if T is a almost contraction (or (δ, L)-weak contraction) self mapping of a complete metric space X, then T has a fixed point. Also, Berinde shows that any almost contraction (or (δ, L)-weak contraction) mapping is a Picard operator. Then, Berinde [4] introduced the nonlinear type almost contraction using a comparison function and proved the following fixed point theorem. A map $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$, where $\mathbb{R}^+ = [0, \infty)$, is called comparison function if it satisfies:

(i) φ is monotone increasing,
(ii) $\lim_{n\to\infty} \varphi^n(t) = 0$ for all $t \in \mathbb{R}^+$.

If φ satisfies (i) and

(iii) $\sum_{n=0}^{\infty} \varphi^n(t)$ converges for all $t \in \mathbb{R}^+$,

then φ is said to be (c)-comparison function.

We can find some properties and some examples of comparison and (c)-comparison functions in [7].

Definition 2.4. Let (X, d) be a metric space and $T : X \to X$ is a self operator. T is said to be a almost φ-contraction (or (φ, L)-weak contraction) if there exists a comparison function φ and some $L \geq 0$ such that

$$d(Tx, Ty) \leq \varphi(d(x, y)) + Ld(y, Tx) \tag{2.3}$$

for all $x, y \in X$.

Similar to the case of almost contraction, in order to check the almost φ-contractiveness of a mapping T, it is necessary to check both (2.3) and

$$d(Tx, Ty) \leq \varphi(d(x, y)) + Ld(x, Ty) \tag{2.4}$$

for all $x, y \in X$.

Clearly any almost contraction is a almost φ-contraction, but the converse may not be true. Also the class of almost φ-contractions includes Matkowski type nonlinear contractions.

Theorem 2.1. *Let (X, d) be a complete metric space and $T : X \to X$ a almost φ-contraction (or (φ, L)-weak contraction with φ) is (c)-comparison function, then T has a fixed point.*

Moreover, in [8] the authors extend the concept of weak contraction from the case of single-valued mappings to multivalued mappings. Hence they obtain general fixed point theorems which extend, improve and unify a multitude of corresponding results in literature for both single-valued and multivalued maps as well.

Definition 2.5. Let (X, d) be a metric space and $T : X \to P(X)$ be a multivalued operator. T is said to be a multivalued weak contraction or a multivalued (θ, L)-weak contraction if there exist two constants $\theta \in (0, 1)$ and $L \geq 0$ such that

$$H(Tx, Ty) \leq \theta d(x, y) + LD(y, Tx) \tag{2.5}$$

for all$x, y \in X$.

Due to the symmetry of d and H, in order to check that T is a multivalued weak contraction, we have also to check the dual of (2.5) , that is to check that T verifies

$$H(Tx, Ty) \leq \theta d(x, y) + LD(x, Ty), \text{for all} x, y \in X.$$

The next theorem is the main result of that paper.

Theorem 2.2. *Let (X, d) be a complete metric space and $T : X \to CB(X)$ a multivalued (θ, L)-weak contraction. Then*

(i) the set of fixed points of T is nonempty;

(ii) for any $x_0 \in X$, there exists an orbit $\{x_n\}_{n=0}^{\infty}$ of T at the point x_0 that converges to a fixed point u of T for which the following estimates hold:

$$\begin{aligned} d(x_n, u) &\leq \frac{h^n}{1-h} d(x_0, x_1), n = 0, 1, 2, \cdots \\ d(x_n, u) &\leq \frac{h}{1-h} d(x_{n-1}, x_n), n = 0, 1, 2, \cdots \end{aligned}$$

for a certain constant $h < 1$.

Also, in this paper they extend this result by considering instead of the term $\theta d(x, y)$ in (2.5) the expression

$$k(d(x, y))d(x, y) \text{ where } k : [0, \infty) \to [0, 1)$$

is a function satisfying certain conditions.

Theorem 2.3. *Let (X, d) be a complete metric space and $T : X \to CB(X)$ a generalized multivalued (α, L)-weak contraction, i.e., a mapping for which there exists a function $\alpha : [0, \infty) \to [0, 1)$ satisfying $\limsup_{r \to t^+} \alpha(r) < 1$, for every $t \in [0, \infty)$, such that*

$$H(Tx, Ty) \leq \alpha(d(x, y))d(x, y) + LD(y, Tx)$$

for all$x, y \in X$. Then T has at least one fixed point.

3. MAIN RESULTS

In this section we prove some fixed point theorems for multivalued almost contraction using graph theory.

Theorem 3.4. *Let (X, d) be a complete metric space, G be a directed graph on X and $T : X \to CB(X)$ be a multivalued mapping. Assume that T is upper semicontinuous and weakly graph-preserving map. Suppose that the following assertions hold:*

(i)There exists a strictly increasing (c)-comparison function $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ and $L \geq 0$ such that

$$H(Tx, Ty) \leq \varphi(M(x, y)) + L \min\{D(x, Ty), D(y, Tx)\} \tag{3.6}$$

for all $(x, y) \in E(G)$, where

$$M(x, y) = \max\left\{d(x, y), D(x, Tx), D(y, Ty), \frac{1}{2}\left[D(x, Ty) + D(y, Tx)\right]\right\}$$

(ii)$X_T = \{x \in X : (x, u) \in E(G)$ for some $u \in Tx\}$is nonempty.

Then T has a fixed point.

Proof. Let $x_0 \in X_T$. There exists $x_1 \in Tx_0$ such that $(x_0, x_1) \in E(G)$. So we can use the condition (i) for x_0 and x_1. Then we have

$$\begin{aligned} D(x_1, Tx_1) &\leq H(Tx_0, Tx_1) \\ &\leq \varphi(M(x_0, x_1)) + L\min\{D(x_0, Tx_1), D(x_1, Tx_0)\} \\ &= \varphi(M(x_0, x_1)) \\ &= \varphi\left(\max\left\{\begin{array}{c} d(x_0, x_1), D(x_0, Tx_0), D(x_1, Tx_1), \\ \frac{1}{2}\left[D(x_0, Tx_1) + D(x_1, Tx_0)\right] \end{array}\right\}\right) \\ &\leq \varphi(d(x_0, x_1)) \\ &< q\varphi(d(x_0, x_1)) \end{aligned}$$

where $q > 1$ is a constant. Therefore, there exists $x_2 \in Tx_1$ such that

$$d(x_1, x_2) < q\varphi(d(x_0, x_1)).$$

Since φ is strictly increasing, we have

$$0 < \varphi(d(x_1, x_2)) < \varphi(q\varphi(d(x_0, x_1)))$$

Get $q_1 = \frac{\varphi(q\varphi(d(x_0,x_1)))}{\varphi(d(x_1,x_2))}$. Then $q_1 > 1$. Since $(x_0, x_1) \in E(G)$, $x_1 \in Tx_0$ and $x_2 \in Tx_1$, using weakly graph-preserving property we can write $(x_1, x_2) \in E(G)$. Then

$$\begin{aligned} D(x_2, Tx_2) &\leq H(Tx_1, Tx_2) \\ &\leq \varphi(M(x_1, x_2)) + L\min\{D(x_1, Tx_2), D(x_2, Tx_1)\} \\ &= \varphi(M(x_1, x_2)) \\ &\leq \varphi(d(x_1, x_2)) \\ &< q_1\varphi(d(x_1, x_2)) \end{aligned}$$

Therefore there exists $x_3 \in Tx_2$ such that

$$d(x_2, x_3) \leq q_1\varphi(d(x_2, x_1)) = \varphi(q\varphi(d(x_0, x_1))).$$

Since φ is strictly increasing, we have

$$0 < \varphi(d(x_2, x_3)) < \varphi^2(q\varphi(d(x_0, x_1))).$$

Get $q_2 = \frac{\varphi^2(q\varphi(d(x_0,x_1)))}{\varphi(d(x_2,x_3))}$. Then $q_2 > 1$. By induction, we can find a sequence $\{x_n\}$ in X such that $x_{n+1} \in Tx_n$, $(x_n, x_{n+1}) \in E(G)$ and

$$d(x_n, x_{n+1}) \leq \varphi^n(q\varphi(d(x_0, x_1))).$$

In order to show that $\{x_n\}$ is a Cauchy sequence, consider $m, n \in \mathbb{N}$ such that $m > n$. Using the triangular inequality for the metric, we have

$$\begin{aligned} d(x_n, x_m) &\leq \sum_{k=n}^{m-1} d(x_k, x_{k+1}) \\ &\leq \sum_{k=n}^{m-1} \varphi^k(q\varphi(d(x_0, x_1))). \end{aligned}$$

Since φ is a (c)-comparision function, by the convergence of the series, we get $d(x_n, x_m) \to 0$ as $n \to \infty$. This yields that $\{x_n\}$ is a Cauchy sequence in (X, d). Since (X, d) is a complete metric space, the sequence $\{x_n\}$ converges to some point $z \in X$, that is, $\lim_{n\to\infty} x_n = z$. Since T is upper semicontinuous, then by Lemma 2.1 we have $z \in Tz$. □

Theorem 3.5. *Let (X, d) be a complete metric space, G be a directed graph on X satisfies the following property:*

$$(P)-property\left\{\begin{array}{c} \text{for any } \{x_n\} \text{ in } X, \text{ if } x_n \to x \text{ and } (x_n, x_{n+1}) \in E(G), \\ \text{then there is a subsequence } \{x_{n_k}\} \text{ with } (x_{n_k}, x) \in E(G). \end{array}\right.$$

Let $T : X \to CB(X)$ be a multivalued mapping. Assume that T is weakly graph-preserving map. Suppose that the following assertions hold:

(i)There exists a strictly increasing (c)-comparison function $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ and $L \geq 0$ such that

$$H(Tx, Ty) \leq \varphi(M(x, y)) + L \min\{D(x, Ty), D(y, Tx)\}$$

for all $(x, y) \in E(G)$.

(ii)X_T is nonempty.

Then T has a fixed point.

Proof. By similar way of proof of Theorem 3.4, we can construct a sequence $\{x_n\}$ such that $x_n \to z$ for some $z \in X$. By the $(P)-$property, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $(x_{n_k}, z) \in E(G)$ for each $k \in \mathbb{N}$. Now assume that $D(z, Tz) > 0$. Since $\lim_{n\to\infty} D(x_{n_k}, x_{n_k+1}) = 0$, $\lim_{n\to\infty} D(x_{n_k}, z) = 0$, there exists $n_0 \in \mathbb{N}$ such that for $n_k > n_0$

$$D(x_{n_k}, x_{n_k+1}) < \frac{1}{3} D(z, Tz) \tag{3.7}$$

and there exists $n_1 \in \mathbb{N}$ such that for $n_k > n_1$

$$D(x_{n_k}, z) < \frac{1}{3} D(z, Tz). \tag{3.8}$$

If we take $n_k > \max\{n_0, n_1\}$, then by 3.7 and 3.8, we have

$$\begin{aligned} D(x_{n_k+1}, Tz)) &\leq H(Tx_{n_k}, Tz) \\ &\leq \varphi(M(x_{n_k}, z)) + L \min\{D(x_{n_k}, Tz), D(z, Tx_{n_k})\} \\ &\leq \varphi\left(\max\left\{\begin{array}{c} d(x_{n_k}, z), D(x_{n_k}, Tx_{n_k}), D(z, Tz), \\ \frac{1}{2}[D(x_{n_k}, Tz) + D(z, Tx_{n_k})] \end{array}\right\}\right) \\ &\quad + L \min\{D(x_{n_k}, Tz), D(z, x_{n_k+1})\} \\ &< \varphi(D(z, Tz)) + L \min\{D(x_{n_k}, Tz), d(z, x_{n_k+1})\} \end{aligned}$$

Taking the limit $k \to \infty$, we have $D(z, Tz) \leq \varphi(D(z, Tz)) < D(z, Tz)$, which is a contradiction. Thus T has a fixed point. □

If we assume that $T : X \to K(X)$ in the previous theorems, we don't need to necessity of being strictly increasing of φ. So we can state the proof of theorems as following:

Corollary 3.1. *Let (X, d) be a complete metric space, G be a directed graph on X and $T : X \to K(X)$ be a multivalued mapping. Assume that T is upper semicontinuous and weakly graph-preserving map. Suppose that the following assertions hold:*
(i)There exists a (c)-comparison function $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ and $L > 0$ such that

$$H(Tx, Ty) \leq \varphi(M(x, y)) + L \min\{D(x, Ty), D(y, Tx)\}$$

for all $(x, y) \in E(G)$
(ii)X_T is nonempty.
Then T has a fixed point.

Proof. Let $x_0 \in X_T$. There exists $x_1 \in Tx_0$ such that $(x_0, x_1) \in E(G)$. So we can use the condition (i) for x_0 and x_1. Then we have

$$\begin{aligned} D(x_1, Tx_1) &\leq H(Tx_0, Tx_1) \\ &\leq \varphi(M(x_0, x_1)) + L\min\{D(x_0, Tx_1), D(x_1, Tx_0)\} \\ &= \varphi(M(x_0, x_1)) \\ &= \varphi\left(\max\left\{\begin{array}{c} d(x_0, x_1), D(x_0, Tx_0), D(x_1, Tx_1), \\ \frac{1}{2}\left[D(x_0, Tx_1) + D(x_1, Tx_0)\right] \end{array}\right\}\right) \\ &\leq \varphi(d(x_0, x_1)) \end{aligned}$$

Since Tx_1 is compact, there exists $x_2 \in Tx_1$ such that $d(x_1, x_2) = D(x_1, Tx_1)$. So

$$d(x_1, x_2)) \leq \varphi(d(x_0, x_1)).$$

Since $(x_0, x_1) \in E(G)$, $x_1 \in Tx_0$ and $x_2 \in Tx_1$, using weakly graph-preserving property, we can write $(x_1, x_2) \in E(G)$. Then

$$\begin{aligned} D(x_2, Tx_2) &\leq H(Tx_1, Tx_2) \\ &\leq \varphi(M(x_1, x_2)) + L\min\{D(x_1, Tx_2), D(x_2, Tx_1)\} \\ &= \varphi(M(x_1, x_2)) \\ &\leq \varphi(d(x_1, x_2)). \end{aligned}$$

Since Tx_2 is compact, there exists $x_3 \in Tx_2$ such that $d(x_2, x_3) = D(x_2, Tx_2)$. Therefore we have

$$d(x_2, x_3) \leq \varphi(d(x_2, x_1)).$$

By induction, we can find a sequence $\{x_n\}$ in X such that $x_{n+1} \in Tx_n$, $(x_n, x_{n+1}) \in E(G)$ and

$$\begin{aligned} d(x_n, x_{n+1}) &\leq \varphi(d(x_{n-1}, x_n)) \\ &\leq \varphi^2(d(x_{n-2}, x_{n-1})) \\ &\vdots \\ &\leq \varphi^n(d(x_0, x_1)). \end{aligned}$$

The rest of the proof can be completed as in the proof of Theorem3.4. □

Corollary 3.2. *Let (X, d) be a complete metric space, G be a directed graph on X satisfies the $(P)-$property. Let $T : X \to K(X)$ be a multivalued mapping. Assume that T is weakly graph-preserving map. Suppose that the following assertions hold:*
(i)There exists a (c)-comparison function $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ and $L > 0$ such that

$$H(Tx, Ty) \leq \varphi(M(x, y)) + L\min\{D(x, Ty), D(y, Tx)\}$$

for all $(x, y) \in E(G)$.
(ii)X_T is nonempty.
Then T has a fixed point.

Corollary 3.3. *Let (X, d) be a complete metric space, G be a directed graph on X and $T : X \to K(X)$ be a multivalued mapping. Assume that T is upper semicontinuous and weakly graph-preserving map. Also suppose that there exists a (c)-comparison function $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ and $L > 0$ such that*

$$H(Tx, Ty) \leq \varphi(d(x, y))$$

for all $(x, y) \in E(G)$ and X_T is nonempty. Then T has a fixed point.

Corollary 3.4. *Let (X, d) be a complete metric space, G be a directed graph on X satisfies the $(P)-$property. Let $T : X \to K(X)$ be a multivalued mapping . Assume that T is weakly graph-preserving map. Also suppose that there exists a (c)-comparison function $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ and $L > 0$ such that*

$$H(Tx, Ty) \leq \varphi(d(x, y))$$

for all $(x, y) \in E(G)$ and X_T is nonempty. Then T has a fixed point.

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