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On a functional equation arising in mathematical biology and theory of learning

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ABSTRACT. V. Istrăţescu [Istrăţescu, V. I., *On a functional equation*, J. Math. Anal. Appl., **56** (1976), No. 1, 133–136] used the Banach contraction mapping principle to establish an existence and approximation result for the solution of the functional equation

$$\varphi(x) = x\varphi((1-\alpha)x+\alpha) + (1-x)\varphi((1-\beta)x),\ x \in [0,1],\ (0 < \alpha \leq \beta < 1),$$

which is important for some mathematical models arising in biology and theory of learning.

This equation has been studied by Lyubich and Shapiro [A. P. Lyubich, Yu. I. and Shapiro, A. P., *On a functional equation* (Russian), Teor. Funkts., Funkts. Anal. Prilozh. **17** (1973), 81–84] and subsequently, by Dmitriev and Shapiro [Dmitriev, A. A. and Shapiro, A. P., *On a certain functional equation of the theory of learning* (Russian), Usp. Mat. Nauk **37** (1982), No. 4 (226), 155–156].

The main aim of this note is to solve this functional equation with more general arguments for φ on the right hand side, by using appropriate fixed point tools.

1. INTRODUCTION

Lyubich and Shapiro [22] studied the existence and uniqueness of a continuous solution $\varphi : [0,1] \to [0,1]$ of the functional equation

$$\varphi(x) = x\varphi((1-\alpha)x+\alpha) + (1-x)\varphi((1-\beta)x),\ x \in [0,1], \tag{1.1}$$

where $0 < \alpha \leq \beta < 1$.

The functional equation (1.1) arises in mathematical models of biology to study behaviour of predator animals that prey two categories of preys. This behaviour is mathematically described by a Markov process having the state space $[0,1]$ with the probabilities of transition from the state x to the state $(1-\alpha)x+\alpha$ and from the state x to the state $(1-\beta)x$, given by

$$P(x \longrightarrow (1-\alpha)x+\alpha) = x,\ P(x \longrightarrow (1-\beta)x) = 1-x,$$

respectively.

In this mathematical model, the solution φ is the final probability of the event when the predator is fixed on one category of prey, knowing that the initial probability for this category to be chosen is equal to x.

Lyubich and Shapiro [22] used the Schauder fixed point theorem to prove the existence of a solution of (1.1) of the following form

$$\varphi(x) = \sum_{i=1}^{\infty} a_i x^i,\ a_i \geq 0,$$

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and satisfying the conditions

$$\varphi(0) = 0,\ \varphi(1) = 1. \tag{1.2}$$

A few years later, Istrăţescu [20] established an existence result for the solution of the functional equation (1.1) by using an alternate fixed point tool, namely, the Banach contraction mapping principle, to find a solution satisfying merely conditions (1.2).

Note that equation (1.1) is also important in the theory of learning (see [11], [12], [13], [14], [15], [16]). In this context, Dmitriev and Shapiro [14] (see also [15], in Russian), found a solution of (1.1) by a direct method. They denoted by $h = 1 - \alpha$ and $k = 1 - \beta$ and used the substitution

$$\varphi(x) := x + (k - h)x(1 - x)\psi(x), \tag{1.3}$$

to reduce the functional equation (1.1) with the unknown function φ to the following functional equation

$$\psi(x) = h(1 - h(1 - x))\psi(1 - h(1 - x)) + k(1 - kx)\psi(kx) + 1, \tag{1.4}$$

with the unknown function ψ. In this way, they proved that the solution of (1.4) may be represented as the sum of the series

$$\psi(x) = \sum_{m=0}^{+\infty} \varphi_m(x), \tag{1.5}$$

where

$$\varphi_0(x) = \sum_{n=0}^{+\infty} k^n \alpha_n(x, k);\ \varphi_{m+1}(x) = h \sum_{n=0}^{+\infty} k^n \alpha_n(x, k)(1 - (1 - k^n x))\varphi_m(1 - (1 - k^n x)),$$

and

$$\alpha_0(x, k) = 1 \text{ and } \alpha_n(x, k) = (1 - kx)\dots(1 - k^n x),\ n \geq 1.$$

Now, in order to obtain the solution φ of (1.1) in closed form, we shall use (1.5) and (1.3).

We note that to obtain the solution φ of the equation (1.1), we must use some complicated computations. Contrary to that, Istrăţescu's approach [20] allows us to construct the solution φ of (1.1) iteratively, which is not in closed form, and not for all values of α.

A careful examination of the arguments in [20], reveals that the functions $f, g : [0, 1] \to [0, 1]$, are given by

$$f(x) = (1 - \alpha)x + \alpha;\ g(x) = (1 - \beta)x,\ x \in [0, 1],$$

and corresponding to the two states of the Markov process, are in fact (linear) contractions satisfying the boundary conditions

$$f(1) = 1 \text{ and } g(0) = 0. \tag{1.6}$$

With this observation in mind, the main aim of this note is to solve the following more general functional equation

$$\varphi(x) = x\varphi(f(x)) + (1 - x)\varphi(g(x)),\ x \in [0, 1], \tag{1.7}$$

by considering $f, g : [0, 1] \to [0, 1]$ as contractions (generally nonlinear) and satisfying (1.6).

An iterative method for constructing the solution of (1.7), which solves in particular the initial functional equation (1.1), is also provided.

Illustrative examples are given to show that our results are signifycantly more general than the ones existing in literature and mentioned before.

For a similar fixed point approach for other classes of functional equations; see [2], [4] and [21].

As we shall need Banach contraction mapping principle in proving our main result, we state it completely.

Theorem 1.1. *Let (X, d) be a complete metric space and $T : X \longrightarrow X$ a map satisfying*

$$d(Tx, Ty) \le a\, d(x, y)\,, \quad \text{for all } x, y \in X\,, \tag{1.8}$$

where $0 \le a < 1$ is constant. Then:

$(p1)$ T has a unique fixed point x^* in X;

$(p2)$ The Picard iteration $\{x_n\}_{n=0}^{\infty}$ defined by

$$x_{n+1} = Tx_n\,, \quad n = 0, 1, 2, \ldots \tag{1.9}$$

converges to x^*, for any $x_0 \in X$.

$(p3)$ The following estimate holds:

$$d(x_{n+i-1}, x^*) \le \frac{a^i}{1-a}\, d(x_n, x_{n-1})\,, \quad n = 1, 2, \ldots;\ i = 1, 2, \ldots \tag{1.10}$$

$(p4)$ The rate of convergence of Picard iteration is given by

$$d(x_n, x^*) \le a\, d(x_{n-1}, x^*)\,, \quad n = 1, 2, \ldots \tag{1.11}$$

Remarks.

Theorem 1.1 is a powerful tool and has many applications in solving nonlinear equations. It not only guarantees the existence and uniqueness of the fixed point x^* of the contraction T but also approximates x^* by means of Picard iteration (1.9). Moreover, for this iterative method both *a priori*

$$d(x_i, x^*) \le \frac{a^i}{1-a}\, d(x_0, x_1)\,, \quad i = 1, 2, \ldots$$

and *a posteriori*

$$d(x_n, x^*) \le \frac{a}{1-a}\, d(x_n, x_{n-1})\,, \quad n = 1, 2, \ldots$$

error estimates are available, which are both obtained from (1.10), by taking $n = 1$ and $i = 1$, respectively.

Note that the inequality (1.11) shows that the rate of convergence of Picard iteration (1.9) is only *linear*, hence we cannot expect $\{x_n\}$ to be a fast iterative scheme (see Examples 2.1-2.3 below to illustrate this fact).

2. Main result

Let $E = \{\varphi \in C[0, 1] : \varphi(0) = 0,\ \varphi(1) = 1\}$. Then the mapping

$$\|\varphi\| = \sup_{t \ne s} \frac{|\varphi(t) - \varphi(s)|}{|t - s|},\ \varphi \in E, \tag{2.12}$$

is a norm on E and $(E, \|\cdot\|)$ is a Banach space (see [22]).

Our main result in this paper is the following:

Theorem 2.2. *If f and g are contractions on $[0, 1]$ (endowed with usual norm), with contraction coefficients α and β, respectively, satisfying $\alpha, \beta \in (0, 1)$, $\alpha \le \beta$ and $2\alpha < 1$, then*

1) Equation (1.7) has a unique solution $\overline{\varphi}$ in E;

2) The sequence of successive approximations $\{\varphi_n\}$, defined by

$$\varphi_{n+1}(x) = x\varphi_n(f(x)) + (1 - x)\varphi_n(g(x)),\ x \in [0, 1],\ n \ge 0 \tag{2.13}$$

converges to $\overline{\varphi}$, as $n \to \infty$, for any $\varphi_0 \in E$.

3) The error estimate of $\{\varphi_n\}$ is given by

$$\|\varphi_{n+i-1} - \overline{\varphi}\| \leq \frac{(2\alpha)^i}{1-2\alpha} \|\varphi_n - \varphi_{n-1}\|, \quad n = 1, 2, \ldots; \, i = 1, 2, \ldots \tag{2.14}$$

4) The rate of convergence of the iterative method $\{\varphi_n\}$ is linear, i.e.,

$$\|\varphi_n - \overline{\varphi}\| \leq \alpha \|\varphi_{n-1} - \overline{\varphi}\|, \quad n = 1, 2, \ldots \tag{2.15}$$

Proof. We consider the operator $T : E \to C[0,1]$, defined by

$$(T\varphi)(x) = x\varphi(f(x)) + (1-x)\varphi(g(x)), \, x \in [0,1], \tag{2.16}$$

where $f, g : [0,1] \to [0,1]$ are contractions with the contraction coefficients α and β satisfying the "one side" boundary conditions (1.6). Obviously, T maps E into itself and the functional equation (1.7) is equivalent to the fixed point operator equation

$$T\varphi = \varphi. \tag{2.17}$$

In order to apply the contraction mapping principle to (2.17), we shall prove that T is a contraction on $(E, \|\cdot\|)$.

We observe that T is a linear operator and so, in order to evaluate

$$\|T\varphi - T\psi\| = \|T(\varphi - \psi)\|,$$

we evaluate the quantity

$$|\Delta_{t,s}| = \frac{|T(\varphi-\psi)(t) - T(\varphi-\psi)(s)|}{|t-s|}, t, s \in [0,1], \, t \neq s.$$

We have

$$\Delta_{t,s} = \frac{1}{t-s} \left[t(\varphi-\psi)(f(t)) + (1-t)(\varphi-\psi)(s) - s(\varphi-\psi)(f(s)) - (1-s)(\varphi-\psi)(g(s))\right]$$

$$= \frac{1}{t-s}[t(\varphi-\psi)(f(t)) - t(\varphi-\psi)(f(s)) + (1-t)(\varphi-\psi)(g(t)) - (1-t)(\varphi-\psi)(g(s))$$
$$+t(\varphi-\psi)(f(s)) - s(\varphi-\psi)(f(s)) + (1-t)(\varphi-\psi)(g(s)) - (1-s)(\varphi-\psi)(g(s))].$$

Hence

$$\begin{aligned}
|\Delta_{t,s}| \leq t \cdot \frac{|(\varphi-\psi)(f(t)) - (\varphi-\psi)(f(s))|}{|f(t)-f(s)|} \cdot \frac{|f(t)-f(s)|}{|t-s|} \\
+(1-t) \cdot \frac{|(\varphi-\psi)(g(t)) - (\varphi-\psi)(g(s))|}{|g(t)-g(s)|} \cdot \frac{|g(t)-g(s)|}{|t-s|} \\
+\left|\frac{t-s}{t-s}\right| \cdot \frac{|(\varphi-\psi)(f(s)) - (\varphi-\psi)(f(1))|}{|f(s)-f(1)|} \cdot \frac{|f(s)-f(1)|}{|s-1|} \cdot |(s-1)| \\
+ \left|\frac{t-s}{t-s}\right| \cdot \frac{|(\varphi-\psi)(g(s)) - (\varphi-\psi)(g(0))|}{|g(s)-g(0)|} \cdot \frac{|g(s)-g(0)|}{s-0} \cdot |(s-0)|, \, t \neq s.
\end{aligned} \tag{2.18}$$

Since f and g are contractions with the contraction coefficients α and β, respectively, we have

$$|f(t) - f(s)| \leq \alpha|t-s|; \, |g(t) - g(s)| \leq \beta|t-s|,$$

and

$$|f(t) - f(1)| \leq \alpha|s-1| = \alpha \cdot (1-s); \, |g(s) - g(0)| \leq \beta|s-0| = \beta \cdot s.$$

Therefore, by (2.18) we get

$$|\Delta_{t,s}| \leq \alpha t\|\varphi-\psi\| + \beta(1-t)\|\varphi-\psi\| + \alpha(1-s)\|\varphi-\psi\| + \beta s\|\varphi-\psi\|, \, t \neq s,$$

and thus we eventually obtain that T satisfies the Lipschitzian type condition

$$\|T\varphi - T\psi\| \leq \sup_{t\neq s}[\alpha t + \beta(1-t) + \alpha(1-s) + \beta s]\|\varphi - \psi\|, \ \forall \varphi, \psi \in E.$$

Now, since $\alpha, \beta \in (0,1)$, $\alpha \leq \beta$ and $2\alpha < 1$, we have

$$c = \alpha t + \beta(1-t) + \alpha(1-s) + \beta s = \alpha + \beta + (\alpha - \beta)t - (\alpha-\beta)s \leq \alpha + \beta + \alpha - \beta = 2\alpha < 1,$$

which shows that T is a contraction on the Banach space $(E, \|\cdot\|)$.

Hence, by Theorem 1.1, part (p_1), we get the needed solution $\overline{\varphi}$.

2) By Theorem 1.1, part (p_2), and (2.16) we get the estimate (2.13);

Conclusions 3)-4) are obtained by Theorem 1.1, part $(p_3) - (p_4)$, as the contraction coefficient of T is 2α. □

Remark 2.1. If f and g are given by

$$f(x) = (1-\alpha)x + \alpha; \ g(x) = (1-\beta)x, \ x \in [0,1],$$

then, by Theorem 2.2, we get the main result in [20], on the basis of conclusions 3) and 4) related to the error estimates and rate of convergence of the iterative process, respectively.

Example 2.1. If, in the above setting, f and g are given by

$$f(x) = (1-\alpha)x + \alpha; \ g(x) = (1-\beta)x, \ x \in [0,1],$$

then, by (1.7), we get the functional equation (1.1) studied in [14], [15] and [20].

Take as initial approximation $\varphi_0(x) = x$. Then the next two iterates are

$$\varphi_1(x) = (\beta - \alpha)x^2 + (1 + \alpha - \beta)x, \ x \in [0,1],$$

$$\varphi_2(x) = (\beta-\alpha)[(1-\alpha)^2 - (1-\beta)^2]x^3 + (2\alpha^3 + \beta^3 - 2\alpha^2\beta - \alpha\beta^2 - 3\alpha^2 - 3\beta^2 - 6\alpha\beta - 2\alpha + 2\beta)x^2$$
$$+(-\alpha^3 + \alpha^2\beta + \alpha^2 - \beta^2 - 2\alpha\beta + 2\alpha - 2\beta + 1)x, , \ x \in [0,1].$$

As $\varphi_0 - \varphi_1 = (\beta - \alpha)(x^2 - x)$, so we have

$$\|\varphi_0 - \varphi_1\| = (\beta - \alpha)\sup_{t\neq s}\frac{|t^2 - t - s^2 + s|}{|t-s|} = (\beta-\alpha)\sup_{t\neq s}|t + s - 1| = \beta - \alpha.$$

Hence, by the error estimate (2.14), we are able to get *a priori* to know how many iterates are needed to achieve a certain precision in approximating the solution $\overline{\varphi}$ by means of the iterate φ_n.

If we consider, for example, the particular case $\alpha = \dfrac{1}{3}$ and $\beta = \dfrac{1}{2}$, and fix the desired computational precision to $\epsilon = 10^{-p}$, then by

$$\|\varphi_n - \overline{\varphi}\| \leq \frac{(2\alpha)^n}{1-2\alpha}\|\varphi_1 - \varphi_0\|, \quad n = 1, 2, \ldots \tag{2.19}$$

we obtain that n must be the smallest number such that

$$\left(\frac{2}{3}\right)^n \leq 2\epsilon.$$

In case $p = 3$, for example, this shows that, we must compute $n \geq 15$ iterates or more (which requires a rather huge computational skill), to have the solution computed with 3 exact digits.

This slow convergence is mainly due to the *linear* rate of convergence of the Picard iteration, expressed by (2.15).

In such situations, it makes sense to consider an appropriate acceleration method for the iterative sequence; see [5]-[10].

Example 2.2. Let now f and g be given by

$$f(x) = \frac{x^2+4}{5};\ g(x) = \frac{x^2}{3},\ x \in [0,1].$$

Here f is a nonlinear contraction with the contraction coefficient $\alpha = \dfrac{1}{5}$, while g is a nonlinear contraction with the contraction coefficient $\beta = \dfrac{1}{4}$. Thus the conditions $\alpha, \beta \in (0,1)$, $\alpha \leq \beta$ and $2\alpha < 1$ are satisfied and hence we can apply Theorem 2.2.

If we take as initial approximation, the function as in Example 2.1, i.e., $\varphi_0(x) = x$, then the next iterate is given by

$$\varphi_1(x) = \frac{-2x^3+5x^2+12x}{15},\ x \in [0,1].$$

Remark 2.2. Note that the functions f and g in Examples 2.1 and 2.2 are strictly increasing. As a direct consequence of this fact, if the initial approximation φ_0 is increasing, then φ_n is increasing for all $n \geq 0$ and therefore the solution $\overline{\varphi}$ of (1.7) is also increasing. A similar result has been established in [14] for the solution of equation (1.1) when the monotonicity is considered with respect to the parameters α and β.

In the next example, we illustrate what happens if f and g are not increasing functions on $[0,1]$.

Example 2.3. Let $f : [0,1] \to [0,1]$ be given by

$$f(x) = -\frac{1}{3}x + 1,\ x \in \left[0, \frac{1}{2}\right) \text{ and } f(x) = \frac{1}{3}x + \frac{2}{3},\ x \in \left[\frac{1}{2}, 1\right].$$

Then f is a non-monotone contraction with the contraction coefficient $\alpha = \dfrac{1}{3}$.

Indeed, since the cases $x, y \in \left[0, \frac{1}{2}\right)$ and $x, y \in \left[\frac{1}{2}, 1\right]$ are obvious, we must consider only the case $x \in \left[0, \frac{1}{2}\right)$ and $y \in \left[\frac{1}{2}, 1\right]$. In this case, the contraction condition (1.8) reduces to

$$|x+y-1| \leq |x-y| \Longleftrightarrow 2x \leq 1 \leq 2y, \tag{2.20}$$

which is obviously true.

Let now $g : [0,1] \to [0,1]$ be given by

$$g(x) = \frac{2}{3}x,\ x \in \left[0, \frac{1}{2}\right) \text{ and } g(x) = -\frac{2}{3}x + \frac{2}{3},\ x \in \left[\frac{1}{2}, 1\right].$$

Then g is a non-monotone contraction with the contraction coefficient $\beta = \dfrac{2}{3}$.

Similarly, since the cases $x, y \in \left[0, \frac{1}{2}\right)$ and $x, y \in \left[\frac{1}{2}, 1\right]$ are obvious, we must consider only the case $x \in \left[0, \frac{1}{2}\right)$ and $y \in \left[\frac{1}{2}, 1\right]$. In this case, the contraction condition (1.8) reduces to the inequality (2.20).

If we consider as initial approximation, the identity function, $\varphi_0(x) = x$, then the next iterate is given by

$$\varphi_1(x) = -x^2 + \frac{5}{3}x,\ x \in \left[0, \frac{1}{2}\right) \text{ and } \varphi_1(x) = x^2 - \frac{2}{3}x + \frac{2}{3},\ x \in \left[\frac{1}{2}, 1\right],$$

which show that, in this case, the solution $\overline{\varphi}$ of the functional equation is no more monotone on the whole interval $[0,1]$.

3. Open problem

The fundamental problem of stability of functional equations in relation to Ulam-Hyers stability and Ulam-Hyers-Rassias stability has been studied in [1], [17]-[19], [23], [24].The interested reader can search for many more developments on this topic in mathematics databases. We leave the stability problem of the following two functional equations as an open problem:

$$\varphi(x) = x\varphi((1-\alpha)x+\alpha) + (1-x)\varphi((1-\beta)x),\ x \in [0,1],\ (0 < \alpha \leq \beta < 1),$$

and

$$\varphi(x) = x\varphi(f(x)) + (1-x)\varphi(g(x)),\ x \in [0,1],$$

where f and g are contractions on $[0,1]$, with contraction coefficients α and β, respectively, satisfying $\alpha, \beta \in (0,1)$, $\alpha \leq \beta$ and $2\alpha < 1$.

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