CREAT. MATH. INFORM. 24 (2015), No. 1, 17 - 26
Online version at http://creative-mathematics.ubm.ro/
Print Edition: ISSN 1584 - 286X Online Edition: ISSN 1843 - 441X

Convergence properties of Ibragimov-Gadjiev-Durrmeyer operators

EMRE DENIZ and ALI ARAL

ABSTRACT. The purpose of the present paper is to study the local and global direct approximation properties of the Durrmeyer type generalization of Ibragimov Gadjiev operators defined in [Aral, A. and Acar, T., *On Approximation Properties of Generalized Durrmeyer Operators*, (submitted)]. The results obtained in this study consist of Korovkin type theorem which enables us to approximate a function uniformly by new Durrmeyer operators, and estimate for approximation error of the operators in terms of weighted modulus of continuity. These results are obtained for the functions which belong to weighted space with polynomial weighted norm by new operators which act on functions defined on the non compact interval $[0.\infty)$. We finally present a direct approximation result.

1. INTRODUCTION

There are several integral modifications of the well known linear positive operators in the literature which include the most common modifications due to Kantorovich and Durrmeyer. In [2] authors defined general family of summation integral type (Durrmeyer type) operators which include some well-known operators as particular cases and studied asymptotic formula of Voronovskaya type and its quantitative version in terms of weighted modulus of continuity. The operators mentioned above are defined as follows:

Definition 1.1. Let $(\varphi_n(t))_{n\in\mathbb{N}}$ and $(\psi_n(t))_{n\in\mathbb{N}}$ are sequences of functions in $C(\mathbb{R}^+)$, which is the space of continuous function on $\mathbb{R}^+$, such that $\varphi_n(0) = 0$, $\psi_n(t) > 0$, for all t and $\lim_{n\to\infty} 1/n^2\psi_n(0) = 0$. Also let $(\alpha_n)_{n\in\mathbb{N}}$ denote a sequence of positive numbers satisfying the conditions

$$\lim_{n\to\infty}\frac{\alpha_n}{n} = 1 \text{ and } \lim_{n\to\infty}\alpha_n\psi_n(0) = l_1,\ l_1 > 0.$$

The Ibragimov-Gadjiev-Durrmeyer operators are defined by

$$\begin{aligned} M_n(f;x) &= (n-m)\,\alpha_n\psi_n(0)\sum_{\nu=0}^{\infty} K_n^{(\nu)}(x,0,\alpha_n\psi_n(0))\frac{[-\alpha_n\psi_n(0)]^{\nu}}{(\nu)!} \\ &\quad\times\int_0^{\infty} f(y)\,K_n^{(\nu)}(y,0,\alpha_n\psi_n(0))\frac{[-\alpha_n\psi_n(0)]^{\nu}}{(\nu)!}dy \qquad (1.1)\end{aligned}$$

where $K_n(x,t,u)$ ($x,t\in\mathbb{R}^+ = [0,\ \infty)$ and $-\infty < u < \infty$) is a sequence of functions of three variable and must meet several conditions under which the $\{M_n\}_{n\in\mathbb{N}}$ represents a method to approximate the function f. These conditions are:

(1) $K_n(x,0,0) = 1$ for $x\in\mathbb{R}^+$ and $n\in\mathbb{N}$,

Received: 10.11.2014. In revised form: 11.03.2015. Accepted: 31.03.2015
2010 *Mathematics Subject Classification.* 41A36; 41A25.
Key words and phrases. *Durrmeyer operators, Ibragimov-Gadjiev operators, Korovkin theorem, rate of convergence, modulus of continuity, weighted approximation.*
Corresponding author: Ali Aral; aliaral73@yahoo.com

(2) $\left[(-1)^{\nu}\ \frac{\partial^{\nu}}{\partial u^{\nu}} K_n(x,t,u)\big|_{u=u_1,t=0}\right] \geq 0$ for $\nu = 0,1,...,$ and $x \in \mathbb{R}^+$,

(3) $\frac{\partial^{\nu}}{\partial u^{\nu}} K_n(x,t,u)\big|_{u=u_1,t=0} = -nx\left[\frac{\partial^{\nu-1}}{\partial u^{\nu-1}} K_{m+n}(x,t,u)\Big|_{u=u_1,t=0}\right]$ for all $x \in \mathbb{R}^+$ and $n,\nu \in \mathbb{N}$, m is a fixed natural number.

Above three conditions were already assumed to define Ibragimov-Gadjiev operators. (For details see [12]). To define Durrmeyer modification of Ibragimov-Gadjiev operators, in [2] authors assume following two conditions as well.

(4) $K_n(0,0,u) = 1$ for any $u, x \in \mathbb{R}^+$, $p \in \mathbb{N}$ and $u = \varphi_n(t)\,,\ u_1 = \alpha_n \psi_n(t)$

$$\lim_{x\to\infty} x^p K_n^{(\nu)}(x,0,u_1) = 0,$$

(5) $\frac{d}{dx} K_n(x,0,u_1) = -nu_1 K_{m+n}(x,t,u)\,.$

The family of operators $M_n(f)$ is linear and positive. The condition (1) guarantees the positivity only on $\mathbb{R}^+$.

With the purpose of obtaining results for a wide class of linear positive operators, Ibragimov-Gadjiev operators and some generalizations were considered by several authors (see [6], [10], [5] and [1]).

Also as a continuation of [2], authors have studied L_p convergence and weighted L_p convergence properties of the operators M_n using $K-$functional and corresponding Ditzian-Totik modulus of smoothness (see [16]).

In the present paper, we first obtain pointwise estimate for the operators in terms of Petree $K-$functional. To determine the order of convergence in weighted space, we use weighted modulus of continuity given in [15]. Also we study uniform weighted approximation formula for the operators M_n.

2. Auxiliary results

Now we give some auxiliary results.

Lemma 2.1. *Let ν, $n \in \mathbb{N}$. For any natural r we have*

$$\begin{aligned} M_n(t^r;x) &= \frac{n^{2r}}{(n-2m)\dots(n-rm)(n-(r+1)m)(\alpha_n)^r(n^2\psi_n(0))^r} \\ &\quad \times \sum_{j=0}^{r} n(n+m)\dots(n+(j-1)m)\,C_{j,r}\left[\alpha_n\psi_n(0)\right]^j x^j, \end{aligned}$$

where $C_{j,r} = \frac{r!}{j!}\binom{r}{j}$. Also,

$$M_n(1;x) = 1,\ \ M_n(t;x) = \frac{n^2}{(n-2m)\alpha_n}\left(\frac{\alpha_n}{n}x + \frac{1}{n^2\psi_n(0)}\right)$$

$$M_n\left(t^2;x\right) = \frac{n^4}{(n-2m)(n-3m)\alpha_n^2}\left(\left(\frac{\alpha_n}{n}x\right)^2\frac{(m+n)}{n} + \frac{\alpha_n}{n}\frac{4}{n^2\psi_n(0)}x + \frac{2}{\left(n^2\psi_n(0)\right)^2}\right) \tag{2.2}$$

Proof. Using (1.1) we obtain

$$\begin{aligned} M_n\left(t^r;x\right) &= (n-m)\,\alpha_n\varphi_n(0)\sum_{\nu=0}^{\infty}K_n^{(\nu)}(x,t,u)\,\frac{\left[-\alpha_n\varphi_n(0)\right]^{\nu}}{(\nu)!} \\ &\quad\times\int_0^{\infty}y^rK_n^{(\nu)}(y,t,u)\,\frac{\left[-\alpha_n\varphi_n(0)\right]^{\nu}}{(\nu)!}dy. \end{aligned}$$

From (4) we get

$$\int_0^{\infty}x^rK_n^{\nu}(x,0,u_1)\,dx=\frac{(-1)^{\nu}(\nu+r)!}{(n-m)(n-2m)\ldots(n-rm)(n-(r+1)m)\,u_1^{\nu+r+1}}. \tag{2.3}$$

Using (2.3) we have

$$\begin{aligned} M_n\left(t^r;x\right) &= (n-m)\,\alpha_n\varphi_n(0)\sum_{\nu=0}^{\infty}K_n^{(\nu)}(x,t,u)\,\frac{\left[-\alpha_n\varphi_n(0)\right]^{\nu}}{(\nu)!} \\ &\quad\times\frac{(-1)^{\nu}(\nu+r)!}{(n-m)(n-2m)\ldots(n-rm)(n-(r+1)m)\left(\alpha_n\varphi_n(0)\right)^{\nu+r+1}}\,\frac{\left[-\alpha_n\varphi_n(0)\right]^{\nu}}{(\nu)!}. \\ &= \sum_{\nu=0}^{\infty}K_n^{(\nu)}(x,t,u)\,\frac{\left[-\alpha_n\varphi_n(0)\right]^{\nu}}{(\nu)!} \\ &\quad\times\frac{1}{(n-2m)\ldots(n-rm)(n-(r+1)m)\left(\alpha_n\varphi_n(0)\right)^{r}}(\nu+r)\ldots(\nu+1) \end{aligned}$$

Using the equality

$$(\nu+r)\ldots(\nu+1)=\sum_{j=0}^{r}C_{j,r}\prod_{l=0}^{j-1}(\nu-l)$$

we can write

$$\begin{aligned} M_n\left(t^r;x\right) &= \frac{1}{(n-2m)\ldots(n-rm)(n-(r+1)m)\left(\alpha_n\varphi_n(0)\right)^{r}} \\ &\quad\times\sum_{j=0}^{r}C_{j,r}\sum_{\nu=0}^{\infty}\prod_{l=0}^{j-1}(\nu-l)\,K_n^{(\nu)}(x,t,u)\,\frac{\left[-\alpha_n\varphi_n(0)\right]^{\nu}}{(\nu)!} \\ &= \frac{1}{(n-2m)\ldots(n-rm)(n-(r+1)m)\left(\alpha_n\varphi_n(0)\right)^{r}} \\ &\quad\times\sum_{j=0}^{r}C_{j,r}\sum_{\nu=j}^{\infty}K_n^{(\nu)}(x,t,u)\,\frac{\left[-\alpha_n\varphi_n(0)\right]^{\nu}}{(\nu-j)!} \\ &= \frac{1}{(n-2m)\ldots(n-rm)(n-(r+1)m)\left[-\alpha_n\varphi_n(0)\right]^{r+1}} \\ &\quad\times\sum_{j=0}^{r}C_{j,r}x^j\sum_{\nu=0}^{\infty}K_n^{(\nu)}(x,t,u)\,\frac{(-1)^j\left[-\alpha_n\varphi_n(0)\right]^{\nu+j}}{(\nu)!} \end{aligned}$$

$$= \frac{n^{2r}}{(n-2m)\ldots(n-rm)(n-(r+1)m)(\alpha_n)^r(n^2\varphi_n(0))^r} \times \sum_{j=0}^{r} n(n+m)\ldots(n+(j-1)m)C_{j,r}[\alpha_n\varphi_n(0)]^j x^j,$$

where $C_{j,r} = \frac{r!}{j!}\binom{r}{j}$. □

Lemma 2.2. *For each $x \geq 0$ and n large enough we have*

$$M_n\left((t-x)^2;x\right) \leq \frac{C}{(n-2m)\alpha_n\psi_n(0)}\left[\varphi^2(x) + \frac{1}{(n+3m)\alpha_n\psi_n(0)}\right], \tag{2.4}$$

where $\varphi(x) := \sqrt{x(1+xm\alpha_n\psi_n(0))}$ and C is positive constant.

Proof. Using Lemma 2.1 we obtain

$$\begin{aligned} & M_n\left((t-x)^2;x\right) \\ &= M_n\left(t^2;x\right) - 2xM_n(t;x) + x^2M_n(1;x) \\ &= \left[\frac{(2n+6m)}{(n-2m)(n-3m)\alpha_n\psi_n(0)}\right]\left[x(1+xm\alpha_n\psi_n(0)) + \frac{1}{(n+3m)\alpha_n\psi_n(0)}\right] \\ &\leq \frac{C}{(n-2m)\alpha_n\psi_n(0)}\left[\varphi^2(x) + \frac{1}{(n+3m)\alpha_n\psi_n(0)}\right] \end{aligned}$$

where C is a positive constant.. □

3. Direct estimate

Here we establish direct local approximation theorems for the operators $M_n(f;x)$ in ordinary approximation. Let the space $C_B[0,\infty)$ of all continuous and bounded functions be endowed with the norm $\|f\| = \sup\{|f| : f \in [0,\infty)\}$. We begin by considering the following Peetre's K-functional:

$$K_2(f;\delta) = \inf\left\{\|f-g\| + \delta\|g''\| : g \in W_\infty^2\right\}, \tag{3.5}$$

where $\delta > 0$ and $W_\infty^2 = \{g \in C_B[0,\infty) : g', g'' \in C_B[0,\infty)\}$. By the theorem as given in [4], there exists an absolute constant $C > 0$ such that

$$K_2(f,\delta) \leq C\omega_2\left(f,\sqrt{\delta}\right),$$

where

$$\omega_2\left(f,\sqrt{\delta}\right) = \sup_{0\leq h\leq\sqrt{\delta}} \sup_{x\in\mathbb{R}^+} |f(x+2h) - 2f(x+h) + f(x)|$$

is second order modulus of smoothness of $f \in C_B[0,\infty)$.

Also let

$$\omega(f,\delta) = \sup_{0\leq h\leq\delta} \sup_{x\in\mathbb{R}^+} |f(x+h) - f(x)|$$

be the usual modulus of continuity

Theorem 3.1. *Let $f \in C_B[0,\infty)$. Then for every $x \in \mathbb{R}^+$ and n large enough, we have*

$$\left|M_n(f;x) - f(x)\right| \leq C\omega_2\left(f, \sqrt{\frac{1}{(n-2m)\alpha_n\psi_n(0)}\left(\varphi^2(x) + \frac{1}{(n+3m)\alpha_n\psi_n(0)} + 1\right)}\right)$$
$$+\,\omega\left(f, \frac{1+2xm\alpha_n\psi_n(0)}{(n-2m)\alpha_n\psi_n(0)}\right)$$

where C is a positive constant which independent of n and f.

Proof. To give the proof of Theorem, we introduce the auxiliary operator as

$$\widetilde{M_n}(f;x) = M_n(f;x) - f\left(\frac{n^2}{(n-2m)\alpha_n}\left(\frac{\alpha_n}{n}x + \frac{1}{n^2\psi_n(0)}\right)\right) + f(x)$$

Let $g \in W_\infty^2$ and $t \in \mathbb{R}^+$. Applying the operator $\widetilde{M_n}$ toTaylor's expansion of g we have

$$\widetilde{M_n}(g;x) - g(x) = g'(x)\widetilde{M_n}((t-x);x) + \widetilde{M_n}\left(\int_x^t (t-u)g''(u)\,du; x\right).$$

Considering the definition of $\widetilde{M_n}$ and using Lemma 2.1, we get

$$\left|\widetilde{M_n}(g;x) - g(x)\right| \leq \left|g'(x)\widetilde{M_n}((t-x);x)\right| + \left|\widetilde{M_n}\left(\int_x^t (t-u)g''(u)\,du; x\right)\right|$$
$$= \widetilde{M_n}\left(\left|\int_x^t (t-u)g''(u)\,du\right|; x\right)$$

$$\leq M_n\left((t-x)^2; x\right)\|g''\|_\infty$$
$$+\left|\int_x^{\frac{n^2}{(n-2m)\alpha_n}\left[\frac{\alpha_n}{n}x+\frac{1}{n^2\psi_n(0)}\right]} \left(\frac{n^2}{(n-2m)\alpha_n}\left(\frac{\alpha_n}{n}x + \frac{1}{n^2\psi_n(0)}\right) - u\right)du\right|\|g''\|_\infty \tag{3.6}$$

Using Lemma 2.2, we get

$$\leq \frac{C}{(n-2m)\alpha_n\psi_n(0)}\left[\varphi^2(x) + \frac{1}{(n+3m)\alpha_n\psi_n(0)}\right]\|g''\|_\infty$$
$$+\left|\int_x^{\frac{n^2}{(n-2m)\alpha_n}\left[\frac{\alpha_n}{n}x+\frac{1}{n^2\psi_n(0)}\right]} \left(\frac{n^2}{(n-2m)\alpha_n}\left(\frac{\alpha_n}{n}x + \frac{1}{n^2\psi_n(0)}\right) - x\right)du\right|\|g''\|_\infty$$

$$= \frac{C}{(n-2m)\alpha_n\psi_n(0)}\left[\varphi^2(x) + \frac{1}{(n+3m)\alpha_n\psi_n(0)}\right]\|g''\|_\infty$$
$$+\left|\int_x^{\frac{n^2}{(n-2m)\alpha_n}\left[\frac{\alpha_n}{n}x+\frac{1}{n^2\psi_n(0)}\right]} \left(\frac{1+2xm\alpha_n\psi_n(0)}{(n-2m)\alpha_n\psi_n(0)}\right)du\right|\|g''\|_\infty$$

$$\begin{aligned}
&\leq \left[\frac{C}{(n-2m)\,\alpha_n\psi_n(0)}\left[\varphi^2(x)+\frac{1}{(n+3m)\,\alpha_n\psi_n(0)}\right]+\left(\frac{1+2xm\alpha_n\psi_n(0)}{(n-2m)\,\alpha_n\psi_n(0)}\right)^2\right]\|g''\|_\infty \\
&\leq \frac{C}{(n-2m)\,\alpha_n\psi_n(0)}\left(\varphi^2(x)+\frac{1}{(n+3m)\,\alpha_n\psi_n(0)}+1\right)\|g''\|_\infty .
\end{aligned}$$

Since

$$|M_n(f;x)|\leq\|f\|_\infty$$

we can write

$$\begin{aligned}
|M_n(f;x)-f(x)| &\leq \left|\widetilde{M_n}(f-g;x)\right|+|f(x)-g(x)|+\left|\widetilde{M_n}(g;x)-g(x)\right| \\
&\quad+\left|f\left(\frac{n^2}{(n-2m)\,\alpha_n}\left(\frac{\alpha_n}{n}x+\frac{1}{n^2\psi_n(0)}\right)\right)-f(x)\right| \\
&= C\left[\|f-g\|_\infty+\frac{1}{(n-2m)\,\alpha_n\psi_n(0)}\left(\varphi^2(x)+\frac{1}{(n+3m)\,\alpha_n\psi_n(0)}+1\right)\|g''\|_\infty\right] \\
&\quad+\omega\left(f,\frac{1+2xm\alpha_n\psi_n(0)}{(n-2m)\,\alpha_n\psi_n(0)}\right).
\end{aligned}$$

Taking infimum overall $g\in W_\infty^2$ and using (3.5) we get

$$\begin{aligned}
|M_n(f;x)-f(x)| &= C\mathcal{K}_2\left(f,\frac{1}{(n-2m)\,\alpha_n\psi_n(0)}\left(\varphi^2(x)+\frac{1}{(n+3m)\,\alpha_n\psi_n(0)}+1\right)\right) \\
&\quad+\omega\left(f,\frac{1+2xm\alpha_n\psi_n(0)}{(n-2m)\,\alpha_n\psi_n(0)}\right) \\
&\leq C\omega_2\left(f,\sqrt{\frac{1}{(n-2m)\,\alpha_n\psi_n(0)}\left(\varphi^2(x)+\frac{1}{(n+3m)\,\alpha_n\psi_n(0)}+1\right)}\right) \\
&\quad+\omega\left(f,\frac{1+2xm\alpha_n\psi_n(0)}{(n-2m)\,\alpha_n\psi_n(0)}\right)
\end{aligned}$$

which proves the theorem. □

4. Weighted approximation

The operators M_n act on functions defined on the non compact interval $\mathbb{R}^+$ and then the uniform norm is not valid to compute the rate of convergence for unbounded functions. For this we will establish the theorems for functions of polynomial growth.

Let $B_{x^2}[0,\infty)$ be the set of all functions f defined on $\mathbb{R}^+$ satisfying the condition $|f(x)|\leq M_f\left(1+x^2\right)$ with some constant M_f, depending only on f, but independent of x. $B_{x^2}[0,\infty)$ is called weighted space and it is a Banach space endowed with the norm

$$\|f\|_{x^2}=\sup_{x\in\mathbb{R}^+}\frac{f(x)}{1+x^2}.$$

Let $C_{x^2}[0,\infty)=C[0,\infty)\cap B_{x^2}[0,\infty)$ and by $C_{x^2}^k[0,\infty)$, we denote subspace of all continuous functions $f\in B_{x^2}[0,\infty)$ for which $\lim_{x\to\infty}\frac{f(x)}{1+x^2}$ is finite.

We know that usual first modulus of continuity $\omega(\delta)$ does not tend to zero, as $\delta\to0$, on infinite interval. Thus we use weighted modulus of continuity $\Omega(f;\delta)$ defined on infinite interval $\mathbb{R}^+$ (see [15]). Let

$$\mathbf{\Omega}(f;\delta)=\sup_{|h|<\delta,\, x\in\mathbb{R}^+}\frac{|f(x+h)-f(x)|}{(1+h^2)(1+x^2)} \text{ for each } f\in C_{x^2}[0,\infty).$$

Now some elementary properties of $\Omega(f;\delta)$ are collected in the following Lemma.

Lemma 4.3. *Let $f\in C_{x^2}^k[0,\infty)$. Then,*

i) $\Omega(f;\delta)$ is a monotonically increasing function of δ, $\delta\geq 0$.
ii) For every $f\in C_{x^2}^k[0,\infty)$, $\lim\limits_{\delta\to 0}\Omega(f;\delta)=0$.
iii) For each $\lambda>0$,

$$\Omega(f;\lambda\delta)\leq 2(1+\lambda)\left(1+\delta^2\right)\Omega(f;\delta). \tag{4.7}$$

From the inequality (4.7) and definition of $\Omega(f;\delta)$ we get

$$|f(t)-f(x)|\leq 2\left(1+x^2\right)\left(1+(t-x)^2\right)\left(1+\frac{|t-x|}{\delta}\right)\left(1+\delta^2\right)\Omega(f;\delta) \tag{4.8}$$

for every $f\in C_{x^2}[0,\infty)$ and $x,t\in\mathbb{R}^+$.

Theorem 4.2. *If $f\in C_{x^2}^k[0,\infty)$. then the inequality*

$$\sup_{x\geq 0}\frac{|M_n(f;x)-f(x)|}{(1+x^2)^{\frac{5}{2}}}\leq K\Omega\left(f;\frac{1}{\sqrt{n-2m}}\right)$$

is satisfied for a sufficiently large n, where K is a constant.

Proof. From (4.8) we can write

$$|M_n(f;x)-f(x)|\leq 2\left(1+x^2\right)\left(1+\delta^2\right)\Omega(f;\delta)\left\{1+M_n\left((t-x)^2;x\right)+\frac{1}{\delta}M_n(|t-x|;x)\right.$$
$$\left.+\frac{1}{\delta}M_n\left((t-x)^2|t-x|;x\right)\right\}.$$

Applying Cauchy-Schwartz inequality we obtain

$$|M_n(f;x)-f(x)|\leq 2\left(1+x^2\right)\left(1+\delta^2\right)\Omega(f;\delta)\left\{1+M_n\left((t-x)^2;x\right)+\frac{1}{\delta}M_n\left((t-x)^2;x\right)^{\frac{1}{2}}\right.$$
$$\left.+\frac{1}{\delta}M_n\left((t-x)^2;x\right)^{\frac{1}{2}}M_n\left((t-x)^4;x\right)^{\frac{1}{2}}\right\}$$

By simple calculation we obtain

$$M_n\left((t-x)^2;x\right)=x^2\left[\frac{m(2n+6m)}{(n-2m)(n-3m)}\right]+\frac{(2n+6m)\alpha_n\psi_n(0)x+2}{(n-2m)(n-3m)\alpha_n^2\psi_n^2(0)} \tag{4.9}$$

and

$$\begin{aligned}
&M_n\left((t-x)^4;x\right)\\
&=\left\{\frac{120m^4+252nm^3-96n^2m^2}{(n-2m)\ldots(n-5m)}\right\}x^4+\left\{\frac{240m^3-174n^2m+504nm^2}{(n-2m)\ldots(n-5m)\alpha_n\psi_n(0)}\right\}x^3\\
&+\left\{\frac{12n^2+432nm-108m+240m^2}{(n-2m)\ldots(n-5m)\alpha_n^2\psi_n^2(0)}\right\}x^2+\left\{\frac{120n+120m}{(n-2m)\ldots(n-5m)\alpha_n^3\psi_n^3(0)}\right\}x\\
&+\frac{24}{(n-2m)\ldots(n-5m)\alpha_n^4\psi_n^4(0)}.
\end{aligned} \tag{4.10}$$

Using (4.9) and (4.10)

$$\begin{aligned}\sup_{x\geq 0}\frac{\left|M_n\left(f;x\right)-f\left(x\right)\right|}{\left(1+x^2\right)^{\frac{5}{2}}} &\leq 2\left(1+\delta^2\right)\Omega\left(f;\delta\right)\left\{1+\frac{m\left(2n+6m\right)}{\left(n-2m\right)\left(n-3m\right)}+\frac{1}{\delta}\left[\frac{m\left(2n+6m\right)}{\left(n-2m\right)\left(n-3m\right)}\right]^{\frac{1}{2}}\right.\\ &\left.+\frac{1}{\delta}\left(\frac{m\left(2n+6m\right)}{\left(n-2m\right)\left(n-3m\right)}\right)^{\frac{1}{2}}\left(\frac{120m^4+252nm^3-96n^2m^2}{\left(n-2m\right)\ldots\left(n-5m\right)}\right)^{\frac{1}{2}}\right\}\\ &\leq 2\left(1+\delta^2\right)\Omega\left(f;\delta\right)\left\{1+\frac{C}{n-2m}+\frac{1}{\delta}\frac{C}{\sqrt{n-2m}}+\frac{1}{\delta}\frac{C}{\left(n-2m\right)^{\frac{3}{2}}}\right\}\end{aligned}$$

where we denote constants by C and point out that they are not the same at each occurrence. Choosing $\delta=\frac{1}{\sqrt{n-2m}}$, for sufficiently large n, we obtain

$$\sup_{x\geq 0}\frac{\left|M_n\left(f;x\right)-f\left(x\right)\right|}{\left(1+x^2\right)^{\frac{5}{2}}}\leq K\Omega\left(f;\frac{1}{\sqrt{n-2m}}\right)$$

where K is a constant. □

Theorem 4.3. *For each $f\in C_{x^2}^k\left[0,\infty\right)$, we have*

$$\lim_{n\to\infty}\left\|M_n\left(f\right)-f\right\|_{x^2}=0.$$

Proof. Using the theorem in [9] we see that it is sufficient to verify the following three conditions

$$\lim_{n\to\infty}\left\|M_n\left(t^\nu,x\right)-x^\nu\right\|_{x^2}=0,\ \ \nu=0,1,2. \tag{4.11}$$

Since $M_n\left(1;x\right)=1$, the first condition of (4.11) is fulfilled for $\nu=0$.

By Lemma 2.1 we have for $n>2m$

$$\begin{aligned}\left\|M_n\left(t,x\right)-x\right\|_{x^2} &= \sup_{x\in\mathbb{R}^+}\frac{\left|M_n\left(t,x\right)-x\right|}{1+x^2}\\ &\leq \left(\frac{n}{\left(n-2m\right)}-1\right)+\frac{1}{\left(n-2m\right)\alpha_n\psi_n\left(0\right)}\end{aligned}$$

and the second condition of (4.11) holds for $\nu=1$ as $n\to\infty$.

Similarly we can write for $n>3m$

$$\begin{aligned}&\left\|M_n\left(t^2,x\right)-x^2\right\|_{x^2}=\sup_{x\in\mathbb{R}^+}\frac{\left|M_n\left(t^2,x\right)-x^2\right|}{1+x^2}\\ &\leq\left\{\frac{n\left(m+n\right)}{\left(n-2m\right)\left(n-3m\right)}-1\right\}+\frac{4n}{\left(n-2m\right)\left(n-3m\right)\alpha_n\psi_n\left(0\right)}+\frac{2}{\left(n-2m\right)\left(n-3m\right)\alpha_n^2\psi_n^2\left(0\right)}\end{aligned}$$

and the third condition of (4.11) holds for $\nu=2$ as $n\to\infty$.

Thus the proof is completed. □

Next, we give the following theorem to approximate all functions in $C_{x^2}\left[0,\infty\right)$. This type of result is given in [7] for locally integrable functions.

Theorem 4.4. *For each $f\in C_{x^2}\left[0,\infty\right)$ and $\alpha>0$, we have*

$$\lim_{n\to\infty}\sup_{x\in\mathbb{R}^+}\frac{\left|M_n\left(f,x\right)-f\left(x\right)\right|}{\left(1+x^2\right)^{1+\alpha}}=0.$$

Proof. For each $x_0 > 0$,

$$\begin{aligned}\sup_{x\in\mathbb{R}^+}\frac{|M_n(f,x)-f(x)|}{(1+x^2)^{1+\alpha}} &\leq \sup_{x\leq x_0}\frac{|M_n(f,x)-f(x)|}{(1+x^2)^{1+\alpha}}+\sup_{x\geq x_0}\frac{|M_n(f,x)-f(x)|}{(1+x^2)^{1+\alpha}}\\ &\leq \|M_n(f)-f\|_{C[0,x_0]}+\|f\|_{x^2}\sup_{x\geq x_0}\frac{\left|M_n\left(1+t^2,x\right)\right|}{(1+x^2)^{1+\alpha}}\\ &\quad+\sup_{x\geq x_0}\frac{|f(x)|}{(1+x^2)^{1+\alpha}}.\end{aligned}$$

Obviously, the first term of the above inequality tends to zero, which is evident from Theorem 3.1. By Lemma 2.1 for any fixed $x_0 > 0$, it is easily seen that $\sup_{x\geq x_0}\frac{\left|M_n\left(1+t^2,x\right)\right|}{(1+x^2)^{1+\alpha}}$ tends to zero as $n\to\infty$. Finally, we can choose $x_0 > 0$ so large that the last part of above inequality can be made small enough. □

5. Examples

Also, the operators $M_n(f)$ reduce to following well-known operators in special case as shown in the following table:

$M_n(f;x)$	$K_n(x,t,u)$	α_n	$\psi_n(0)$	m
Baskakov-Durrmeyer	$[1+t+ux]^{-n}$	n	$1/n$	1
Szasz-Durrmeyer	$e^{-n(t+ux)}$	n	$1/n$	0
Generalized Baskakov-Durrmeyer	$K_n(t+ux)$	n	$1/n$	1

Other classical sequences of linear positive operators can be obtained with suitable selection of K_n.

Using the above table, the operators $M_n(f)$ reduce to Baskakov-Durrmeyer operators given in [14]

$$\mathcal{B}_n(f;x)=(n-1)\sum_{k=0}^{\infty}v_{n,k}(x)\int_0^{\infty}v_{n,k}(t)\,f(t)\,dt,$$

where $v_{n,k}(x)=\begin{bmatrix} n+k-1\\ k\end{bmatrix}\frac{x^k}{(1+x)^{n+k}}$ and Szasz-Durrmeyer operators given in [13]

$$S_n(f;x)=n\sum_{k=0}^{\infty}p_{n,k}(x)\int_0^{\infty}p_{n,k}(t)\,f(t)dt,$$

where $p_{n,k}(x)=e^{-nx}\frac{(nx)^k}{k!}$.

Under the assumptions of Theorem 3.1 and Theorem 4.2 we can give following results for Szasz-Durrmeyer and Baskakov-Durrmeyer operators.

Theorem 5.5. *Let $f\in C_B[0,\infty)$. Then for every $x\in\mathbb{R}^+$ and n large enough, we have*

$$|B_n(f;x)-f(x)|\leq C\omega_2\left(f,\sqrt{\frac{1}{(n-2)}\left(x(1+x)+\frac{1}{n+3}+1\right)}\right)+\omega\left(f,\frac{1+2x}{(n-2)}\right)$$

and

$$|S_n(f;x)-f(x)|\leq C\omega_2\left(f,\sqrt{\frac{1}{n}\left(x+\frac{1}{n}+1\right)}\right)+\omega\left(f,\frac{1}{n}\right)$$

where C is a positive constant which independent of n and f.

Theorem 5.6. *If $f \in C_{x^2}^k[0,\infty)$. then the inequalities*

$$\sup_{x\geq 0}\frac{|B_n(f;x)-f(x)|}{(1+x^2)^{\frac{5}{2}}}\leq K\Omega\left(f;\frac{1}{\sqrt{n-2}}\right)$$

and

$$\sup_{x\geq 0}\frac{|S_n(f;x)-f(x)|}{(1+x^2)^{\frac{5}{2}}}\leq K\Omega\left(f;\frac{1}{\sqrt{n}}\right)$$

are satisfied for a sufficiently large n, where K is a constant.

References

[1] Aral, A., *Approximation by Ibragimov-Gadjiev operators in polynomial weighted space*, Proc. Inst. Math. Mech. Natl. Acad. Sci. Azerb., **19** (2003), 35–44

[2] Aral, A. and Acar, T., *On Approximation Properties of Generalized Durrmeyer Operators*, (submitted)

[3] Aral, A. and Gupta, V., *On the Durrmeyer type modification of the q-Baskakov type operators*, (English summary) Nonlinear Anal., **72** (2010), No. 3-4, 1171–1180

[4] Agratini, O., *On the iterates of a class of summation type linear positive operators*, Comput. Math. Appl., **55** (2008) 3795–3801

[5] Coskun, T., *On a construction of positive linear operators for approximation of continuous functions in the weighted Spaces*, Journal of Comput. Anal. and Appl., Vol. **13**, No. 4, 756–770

[6] Doğru, O., *On a certain family of linear positive operators*, Turkish J. Math., **21** (1997)

[7] Gadjiev, A. D., Efendiyev, R. O. and Ibikli, E., *On Korovkin type theorem in the space of locally integrable functions*, Czech. Math. J., **53 (128)** (2003), No.1, 45–53

[8] Gadjiev, A. D. and Aral, A., *Weighted Lp -approximation with positive linear operators on unbounded sets*, Appl. Math. Lett., **20** (2007), No. 10, 1046–1051

[9] Gadzhiev, A. D., *Theorems of the of P. P Korovkin type theorems*, Math. Zametki, **20** (1976), No. 5, 781-786, Math. Notes, **20** (1976), No. (5-6) 996–998 (English Translation)

[10] Ghorbanalizadeh, Arash M., *On the order of weighted approximation of unbounded functions by some generalizations of Gadjiev-Ibragimov operators*, J. Appl. Anal., **20** (2014), No. 1, 87–92

[11] Heilmann, M., *Direct and converse results for operators of Baskakov-Durrmeyer type*, Approx. Theory and Its Appl., **5** (1988), No. 1, 105–127

[12] Ibragimov, I. I. and Gadjiev, A. D., *On a certain Family of linear positive operators*, Soviet Math. Dokl., English Trans., **11** (1970), 1092–1095

[13] Mazhar, S. M. and Totik, V., *Approximation by modified Szasz operators*, Acta Sci. Math., **49** (1985), 257–269

[14] Sahai, A. and Prasad, G., *On simultaneous approximation by modified Lupas operators*, J. Approx. Theory, **45** (1985), No. 12, 122–128

[15] Ispir, N., *On modified Baskakov operators on weighted spaces*, Turk. J. Math., **26** (2001), No. 3, 355–365

[16] Ulusoy, G. and Aral, A., *Approximation Properties of Ibragimov-Gadjiev Durrmeyer operators in $L_p[0,\infty)$*, (submitted)

Department of Mathematics
Faculty of Science and Arts
Kirikkale University
71450 Yahsihan, Kirikkale, Turkey
E-mail address: `emredeniz--@hotmail.com`

Department of Mathematics
Faculty of Science and Arts
Kirikkale University
71450 Yahsihan, Kirikkale, Turkey
E-mail address: `aliaral73@yahoo.com`