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A survey on the stability of mean value points and functional equations involving some special functions

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ABSTRACT. The aim of this work is to put together some of the recent and classical results in the theory of stability. In the first part, we recall the results regarding the intermediary points arising from various Mean Value Theorems, then we study the stability of some functional equations involving the gamma and beta functions.

1. INTRODUCTION

The next question posed by the Polish-American mathematician Stanislaw Ulam in 1960 (see [63]) could be considered as a starting point of the stability concept:

"Under what conditions a slightly different solution of an equation, must be close to some exact solution of the given equation?"

Hyers (see [19]) answered to the Ulam's question by proving the following:

Theorem 1.1. *Let $f : U \to V$ be a function between two Banach spaces and let $\varepsilon > 0$. If*

$$\| f(x+y) - f(x) - f(y) \| < \varepsilon \quad (x, y \in U),$$

then there exists $\delta > 0$ (depending only on ε) and an unique additive function $A : U \to V$ such that

$$\| A(x) - f(x) \| < \delta \quad (x \in U).$$

In other words, the Cauchy additive functional equation

$$f(x+y) = f(x) + f(y)$$

is Hyers–Ulam stable. The so defined stability concept attracted many mathematicians, who obtained important results in the recent past. For details see, *e.g.*, [2]-[3], [5]-[6], [9]-[10], [12]-[30], [32]-[44], [46]-[48], [53]-[54], [56], [59]-[62], [64]-[65].

Besides the stability of the functional equations, the Hyers–Ulam stability is now also considered in the case of further mathematical objects such as differential equations, linear recurrences, convexity, intermediary points, etc.

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2. Intermediary points and Mean Value Theorems

After the famous answer given to Ulam's problem, Hyers proved some other results on the stability of extremum, or stationary points. See [20] and [21]. Later, Ulam and Hyers [18] put their forces together to present the following more general result which includes those mentioned above.

Theorem 2.2 (Ulam-Hyers). *Let $f:\mathbb{R}\to\mathbb{R}$ be n-times differentiable in a neighborhood N of a point η. Suppose that $f^{(n)}(\eta)=0$ and $f^{(n)}(x)$ changes the sign at η. Then, for every $\varepsilon>0$, there exists $\delta>0$ such that for every n-times differentiable function $g:\mathbb{R}\to\mathbb{R}$ with the property:*

$$|f(x)-g(x)|<\delta \quad (x\in N),$$

there exists a point $\gamma\in N$ with $g^{(n)}(\gamma)=0$ and $|\gamma-\eta|<0$.

The classical Mean Value Theorem refers to any function $f:[a,b]\to\mathbb{R}$, continuous on $[a,b]$ and differentiable on (a,b). Such a function admits a point $c\in(a,b)$ (called *Lagrange's point*) such that

$$\frac{f(b)-f(a)}{b-a}=f'(c).$$

The *Lagrange's point* is stable in the sense of the following:

Theorem 2.3 (Pawlikowska [54]). *Let $f:[a,b]\to\mathbb{R}$ be a continuously differentiable function having an unique Lagrange's point $c\in(a,b)$. Then for every $\varepsilon>0$, there exists $\delta>0$ with the following property: Every continuously differentiable function $g:[a,b]\to\mathbb{R}$, with the property that:*

$$|g(x)-f(x)|<\delta \quad (a\le x\le b),$$

admits a Lagrange's point $d\in(a,b)$ such that $|d-c|<\varepsilon$.

The proof uses the continuously differentiable function $F:[a,b]\to\mathbb{R}$, defined by the formula

$$F(x)=f(x)-\frac{f(b)-f(a)}{b-a}\cdot(x-a) \quad (a\le x\le b),$$

with $F(c)=0$. Moreover, F changes its sign at c. The conclusion follows by applying the Ulam-Hyers stability theorem in the case of the inequality

$$|G(x)-F(x)|<3\delta \quad (a\le x\le b),$$

where

$$G(x)=g(x)-\frac{g(b)-g(a)}{b-a}\cdot(x-a) \quad (a\le x\le b).$$

Further contribution was brought by Găvruţă et al. [13], who obtained some similar results subject to minor changes in the hypotesis. Furthermore, they proved the following stability result in [13, Theorem 2.3].

Theorem 2.4 (Găvruţă et al. [13]). *Let $\varepsilon>0$ and let $f:[a,b]\to\mathbb{R}$ be a twice continuously differentiable function satisfying either $f''(x)>0$ or $f''(x)<0$, for every $x\in[a,b]$. Let $\gamma\in(a,b)$ be such that*

$$\left|f'(\gamma)-\frac{f(b)-f(a)}{b-a}\right|<\varepsilon.$$

Then there exists a Lagrange's point $c\in(a,b)$ of f such that

$$|\gamma-c|\le\frac{\varepsilon}{\min_{x\in[a,b]}|f''(x)|}.$$

It is of general knowledge that for every functions $f, g : [a, b] \to \mathbb{R}$, continuous on $[a, b]$ and differentiable on (a, b), there exists a point $c \in (a, b)$ (called *Cauchy's point of the functions f and g*) such that

$$(f(b) - f(a))\, g'(c) = (g(b) - g(a))\, f'(c).$$

Peter and Popa [56] investigated the stability of Cauchy's points and established the next result:

Theorem 2.5 (Peter and Popa [56]). *Let $f, g : [a, b] \to \mathbb{R}$ be twice continuously differentiable functions, that admits an unique Cauchy's point $c \in (a, b)$. Assume moreover that*

$$(f(b) - f(a))\, g''(c) - (g(b) - g(a))\, f''(c) \neq 0.$$

Then for every $\varepsilon > 0$, there exists $\delta > 0$ with the following property: Every continuously differentiable functions $F, G : [a, b] \to \mathbb{R}$, satisfying the conditions

$$|F(x) - f(x)| < \delta \text{ and } \quad |G(x) - g(x)| < \delta \quad (a \leq x \leq b),$$

admits a Cauchy's point $d \in (a, b)$ such that

$$|d - c| < \varepsilon.$$

In their proof, Peter and Popa [56] defined the twice continuously differentiable function $h : [a, b] \to \mathbb{R}$, by the formula:

$$h(x) = (f(b) - f(a))\,(g(x) - g(a)) - (g(b) - g(a))\,(f(x) - f(a))$$

and proved that $h'(c) = 0$. Moreover, as h'' keeps constant sign on $[a, b]$, the function h' changes its sign on $[a, b]$. The conclusion follows again by using the Ulam-Hyers stability theorem in the case of the inequality:

$$|H(x) - h(x)| < \delta_1,$$

where

$$\delta_1 = 4\delta \left(\delta + \max_{x,y \in [a,b]} |f(x) - f(y)| + \max_{x,y \in [a,b]} |g(x) - g(y)| \right)$$

and $H : [a, b] \to \mathbb{R}$ is the function defined by the law:

$$H(x) = (F(b) - F(a))\,(G(x) - G(a)) - (G(b) - G(a))\,(F(x) - F(a)).$$

In 1946, the Romanian mathematician Dimitrie D. Pompeiu (1873-1954) derived a variant of the Mean Value Theorem with an interesting geometric interpretation, now known as Pompeiu's Mean Value Theorem.

More exactly, for every function f continuous on $[a, b]$ and differentiable on (a, b) such that $0 \notin [a, b]$, there exists a point $c \in (a, b)$ (called *Pompeiu's point*) such that

$$\frac{af(b) - bf(a)}{a - b} = f(c) - cf'(c).$$

For proof and further details, see, *e.g.*, [62, p. 83].

Peter and Popa [56, Theorem 8] presented a stability result of Pompeiu's point for twice continuously differentiable functions as follows:

Theorem 2.6. *Let $a, b \in \mathbb{R}$ be such that $0 \notin [a, b]$ and let $f : [a, b] \to \mathbb{R}$ be a twice continuously differentiable function, having an unique Pompeiu's point $c \in (a, b)$. Assume moreover that $f''(c) \neq 0$. Then for every $\varepsilon > 0$, there exists $\delta > 0$ with the following property: Every differentiable function $g : [a, b] \to \mathbb{R}$, satisfying the condition*

$$|g(x) - f(x)| < \delta \quad (a \leq x \leq b),$$

admits a Pompeiu's point $d \in (a,b)$ *such that*

$$|d - c| < \varepsilon.$$

The history of the stability of intermediary points arising from various Mean Value Theorems continues to the paper of Das et al [9], who investigated the stability of Flett's point in some sense as we explain in the sequel.

In 1958, Thomas M. Flett [11] proved that for every differentiable function $f : [a,b] \to \mathbb{R}$ with $f'(a) = f'(b)$, there exists a point $c \in (a,b)$ (called *Flett's point*) such that

$$\frac{f(c) - f(a)}{c - a} = f'(c).$$

A nice survey on Flett's mean value theorem, its generalizations and extensions may be found in recent paper [17]. Das et al [9] defined the set

$$\mathcal{F} = \{f : [a,b] \to \mathbb{R} \mid f \text{ is continuously differentiable, } f(a) = 0, f'(a) = f'(b)\}$$

and proved the following

Theorem 2.7 (Das et al [9]). *Let* $f \in \mathcal{F}$ *be a function having an unique Flett's point* $c \in (a,b)$. *Then for every* $\varepsilon > 0$, *there exists* $\delta > 0$ *with the following property: Every function* $g \in \mathcal{F}$, *satisfying:*

$$|g(x) - f(x)| < \delta \quad (a \le x \le b),$$

admits a Flett's point $d \in (a,b)$ *such that* $|d - c| < \varepsilon$.

Lee et al. [44] have presented a more general result, after they have corrected a detail in the proof given in [9].

Sahoo and Riedel [62] proved that for every differentiable function $f : [a,b] \to \mathbb{R}$, there exists a point $c \in (a,b)$ (called *Sahoo-Riedel point*) such that

$$f(c) - f(a) = f'(c)(c - a) - \frac{1}{2} \cdot \frac{f'(c) - f'(a)}{c - a}(c - a)^2.$$

The stability of Sahoo-Riedel point was proved by Lee et al. [44], in the following form:

Theorem 2.8. *Let* $f : [a,b] \to \mathbb{R}$ *be a twice differentiable function having an unique Sahoo-Riedel point* $c \in (a,b)$ *such that*

$$f''(c)(c - a) - 2f'(c) + \frac{2(f(c) - f(a))}{c - a} \ne 0.$$

Then for every $\varepsilon > 0$, *there exists* $\delta > 0$ *with the following property: Every differentiable function* $g \in [a,b] \to \mathbb{R}$, *with*

$$g'(b) - g'(a) = f'(b) - f'(a)$$

such that:

$$|g(x) - g(a) - f(x) + f(a)| < \delta \quad (a \le x \le b)$$

admits a Sahoo-Riedel point $d \in (a,b)$ *such that* $|d - c| < \varepsilon$.

The proof is based on the functions $F, G : [a,b] \to \mathbb{R}$ defined by the following formulas:

$$F(x) = \frac{f(x) - f(a)}{x - a} - \frac{f'(b) - f'(a)}{2(b - a)}(x - a) \quad (a < x \le b)$$

and

$$G(x) = \frac{g(x) - g(a)}{x - a} - \frac{g'(b) - g'(a)}{2(b - a)}(x - a) \quad (a < x \le b),$$

with $F(a) = f'(a)$ and $G(a) = g'(a)$.

Lee et al. [44] proved that $F'(c) = 0$ and F' changes its sign at c. After some algebra they obtained

$$|G(x) - F(x)| < (b-a)\,\delta \quad (a \leq x \leq b)$$

and the conclusion follows by Ulam-Hyers stability theorem.

Pawlikowska [55] introduced the notion of *a Flett's point of n-th order* as being each point $c \in (a,b)$ associated to an n-times differentiable function $f : [a,b] \to \mathbb{R}$ with $f^{(n)}(b) = f^{(n)}(a)$, such that

$$f(c) - f(a) = \sum_{k=1}^{n} \frac{(-1)^k}{k!} f^{(k)}(c)\,(c-a)^k .$$

A different proof of the generalized Flett's mean value theorem due to Pawlikowska is provided in [45]. According to a result of Pawlikowska [54], the Flett's point of n-th order is stable in the following sense:

Theorem 2.9. *Let $f : [a,b] \to \mathbb{R}$ be a n-times continuously differentiable function with $f^{(n)}(a) = f^{(n)}(b)$, that admits an unique Flett's point of n-th order. Then for every $\varepsilon > 0$, there exists $\delta > 0$ with the following property: Every n-times continuously differentiable function $g : [a,b] \to \mathbb{R}$ satisfying the condition:*

$$|g(x) - f(x)| + |g'(x) - f'(x)| + ... + \left|g^{(n)}(x) - f^{(n)}(x)\right| < \delta \quad (a \leq x \leq b),$$

with $g^{(n)}(a) = g^{(n)}(b)$, admits a Flett's point of n-th order $d \in (a,b)$ such that

$$|d - c| < \varepsilon.$$

Pawlikowska [54] explored the idea from the classical proof of Flett's theorem, by considering the functions $F : [a,b] \to \mathbb{R}$, defined by the formula

$$F(x) = \frac{f(x) - f(a)}{x - a} \quad (a < x \leq b),$$

with $F(a) = f(a)$ and

$$H_f(x) = F^{(n-1)}(x) \quad (a < x \leq b),$$

with

$$H_f(a) = \frac{1}{n!} f^{(n)}(a).$$

Note that $H_f'(c) = 0$. The function H_g is defined similarly and the conclusion follows using the inequality

$$|H_f(x) - H_g(x)| < \delta_1 \quad (a \leq x \leq b)$$

where

$$\delta_1 = \frac{\delta}{(n-1)!\left(\frac{1}{0!c^n} + \frac{1}{1!c^{n-1}} + \cdots + \frac{1}{(n-2)!c^2} + \frac{1}{c}\right)}.$$

In the second part of her work, Pawlikowska [54, Theorem 7] extended this result in the spirit of Sahoo-Riedel Theorem, by removing the boundary condition. More exactly, she proved that every n-times differentiable function $f : [a,b] \to \mathbb{R}$ admits a point $c \in (a,b)$ (called *the Sahoo-Riedel point of n-th order*) such that

$$f(c) - f(a) = \sum_{k=1}^{n} \frac{(-1)^k}{k!} f^{(k)}(c)\,(c-a)^k + \frac{\left(f^{(n)}(b) - f^{(n)}(a)\right)}{(n+1)!}(c-a)^{n+1} .$$

According to Pawlikowska [54, Theorem 7], the Sahoo-Riedel point of n-th order is stable:

Theorem 2.10 (Pawlikowska [54]). *Let $f : [a, b] \to \mathbb{R}$ be a n-times continuously differentiable function that admits an unique Sahoo-Riedel point of n-th order $c \in (a, b)$. Then for every $\varepsilon > 0$, there exists $\delta_1, \delta_2, \delta_3 > 0$ with the following property: Every n-times continuously differentiable function $g : [a, b] \to \mathbb{R}$ satisfying the conditions:*

$$|g(x) - f(x)| + |g'(x) - f'(x)| + ... + \left| g^{(n)}(x) - f^{(n)}(x) \right| < \delta_1 \quad (a \le x \le b)$$

and

$$\left| g^{(n)}(a) - f^{(n)}(a) \right| < \delta_2 , \quad \left| g^{(n)}(b) - f^{(n)}(b) \right| < \delta_3,$$

admits a Sahoo-Riedel point of n-th order $d \in (a, b)$ such that

$$|d - c| < \varepsilon.$$

Now let us consider a function $f : I \to \mathbb{R}$ that is n-times differentiable in the open interval $I \subseteq \mathbb{R}$. For every $a \in I$,

$$T_n(f, a, x) = f(a) + \frac{(x-a)}{1!} f'(a) + \frac{(x-a)^2}{2!} f''(a) + ... + \frac{(x-a)^n}{n!} f^{(n)}(a)$$

is *the Taylor polynomial of n-th degree associated to the function f at the point a.*

If f is $(n+1)$-times differentiable and $a, b \in I$, $a < b$, then there exists a point $c \in (a, b)$ (called *Lagrange-Taylor point*) such that

$$f(b) = T_n(f, a, b) + \frac{(b-a)^{n+1}}{(n+1)!} f^{(n+1)}(c).$$

Note that for $n = 0$, the point c is a classical Lagrange's point of f.

The stability of the Lagrange-Taylor point was proved by Peter and Popa [56, Theorem 9]:

Theorem 2.11 (Peter and Popa [56]). *Let $f : I \to \mathbb{R}$ be a $(n+2)$-times differentiable function and let $a, b \in I$, $a < b$ such that f admits an unique Lagrange-Taylor point c of $(n+1)$-th order in (a, b). Assume moreover that $f^{(n+2)}(c) \neq 0$. Then for every $\varepsilon > 0$, there exists $\delta > 0$ with the following property: Every $(n+1)$-times differentiable function $g : I \to \mathbb{R}$ satisfying the condition*

$$\left| g^{(k)}(x) - f^{(k)}(x) \right| < \delta \quad (k \in \{0, 1, ..., n\}, \ a \le x \le b),$$

admits a Lagrange-Taylor point $d \in (a, b)$ such that

$$|d - c| < \varepsilon.$$

The original proof due to Peter and Popa [56, Theorem 9] is based on the functions $F, G : I \to \mathbb{R}$ defined by the following formulas, for every $x \in I$:

$$F(x) = T_n(f, x, b) + \frac{f(b) - T_n(f, a, b)}{(n+1)!} (b-x)^{n+1},$$

$$G(x) = T_n(g, x, b) + \frac{g(b) - T_n(g, a, b)}{(n+1)!} (b-x)^{n+1}.$$

As these functions are continuously differentiable, with $F(a) = F(b) = 0$, we deduce that c is a Rolle's point of F. Hence $F'(c) \neq 0$ and consequently, F changes its sign at c. The conclusion follows by using the relation:

$$|G(x) - F(x)| < \delta_1 \quad (x \in I),$$

where

$$\delta_1 = \delta \left(1 + 2 \sum_{k=0}^{n} \frac{(b-a)^k}{k!} \right).$$

3. Equations of gamma-beta type

The gamma function defined for every real number $x > 0$ by the formula

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt \tag{3.1}$$

is widely used in pure mathematics, as well in other branches such as theory of probabilities and statistics, or combinatorics, since it is part of various distribution functions.

As $\Gamma(n+1) = n!$, for every integer $n \geq 0$, the gamma function can be viewed as an extension of the factorial function to the positive real numbers.

Formula (3.1) remains true when x is any complex number with positive real part, and moreover, the definition of the gamma function can be extended to the set of all complex numbers excepting the non-positive integers. For further classical and recent results on the gamma and related functions, see, *e.g.*, [7]-[8], [49]-[52], or the basic monographs [1, Chapter 6], or [4].

As by definition,

$$\Gamma(x+1) = x\Gamma(x) \quad (x > 0)$$

the following functional equation

$$f(x+1) = xf(x) \quad (x > 0)$$

is called *the gamma functional equation*.

In 1997, S.-M. Jung proved one of the first stability results of the gamma functional equation, which in fact it is a property of asymptotic stability.

Theorem 3.12 (Jung [23])**.** *Let n_0 be any positive integer. If the mapping $f : (0, \infty) \to \mathbb{R}$ satisfies the following inequality*

$$|f(x+1) - xf(x)| \leq \delta \quad (x > n_0),$$

then there exists an unique function $F : (0, \infty) \to \mathbb{R}$ such that

$$F(x+1) = xF(x), \tag{3.2}$$

with

$$|F(x) - f(x)| \leq \frac{3\delta}{x} \quad (x > n_0).$$

The sequence of functions

$$P_n(x) = f(x+n) \prod_{i=0}^{n-1} (x+i)^{-1} \quad (n \geq 1) \tag{3.3}$$

is proven to be a Cauchy sequence, so the requested mapping F satisfying (3.2) is

$$F(x) = \lim_{n\to\infty} P_n(x) \quad (x > n_0).$$

This function F can be inductively prolonged to its domain $(0, \infty)$ by considering the functions

$$F_i : (n_0 - i, n_0 - i + 1] \to \mathbb{R}$$

defined by the recursive formula

$$F_i(x) = \frac{1}{x} F_{i-1}(x+1).$$

Finally, for every $i = 1, 2, ..., n_0$, define

$$F(x) = F_i(x) \quad (n_0 - i < x \leq n_0 - i + 1).$$

In 1993, R. Ger introduced a different type of stability for a functional equation in the general form

$$E_1(f) = E_2(f). \tag{3.4}$$

More exactly, we say that this equation is *stable in the sense of Ger* if for every function g which satisfies the relation

$$\left| \frac{E_1(g)(x)}{E_2(g)(x)} - 1 \right| \leq \psi(x),$$

(the function ψ is fixed), there exists a solution f of (3.4) such that

$$\alpha(x) \leq \frac{g(x)}{f(x)} \leq \beta(x),$$

for some fixed functions $\alpha(x)$ and $\beta(x)$ depending only on $\psi(x)$. For details, see [15]-[16].

Jung [23] proved that the gamma functional equation is stable in the sense of Ger:

Theorem 3.13 (Jung [23]). *Let ε and δ be fixed positive real numbers and let n_0 be a positive integer. If a mapping $f : (0, \infty) \to (0, \infty)$ satisfies the inequality*

$$\left| \frac{f(x+1)}{xf(x)} - 1 \right| \leq \frac{\delta}{x^{1+\varepsilon}} \quad (x > n_0),$$

then there exists an unique function $F : (0, \infty) \to (0, \infty)$ such that

$$F(x+1) = xF(x) \quad (x > 0),$$

with

$$\alpha(x) \leq \frac{F(x)}{f(x)} \leq \beta(x) \quad \left(x > \max\left\{n_0, \delta^{1/(1+\varepsilon)}\right\}\right),$$

where

$$\alpha(x) = \prod_{i=0}^{\infty} \left[1 - \delta(x+i)^{-(1+\varepsilon)}\right]$$

and

$$\beta(x) = \prod_{i=0}^{\infty} \left[1 + \delta(x+i)^{-(1+\varepsilon)}\right].$$

For the proof, Jung used again the sequence of functions $P_n(x)$ given in (3.3), then he proposed as the requested function

$$F(x) = \lim_{n\to\infty} P_n(x) \quad (x > 0).$$

An interesting result was presented by Kairies [31], who proved that if a mapping $f : (0, \infty) \to (0, \infty)$ is continuous and the sequence (3.3) converges to $T(x)$, uniformly on $(0, \infty)$, and satisfies

$$\prod_{k=0}^{p-1} T\left(\frac{x+k}{p}\right) = (2\pi)^{(1/2)(p-1)} p^{(1/2)-x} T(x) \quad (x > 0),$$

for all sufficiently large prime numbers p, then $T = \Gamma$.

Kim [32] and Kim and Lee [33] investigated the following extended form of the gamma functional equation:

$$f(x+p) = \varphi(x) f(x) \tag{3.5}$$

($p > 0$ and the function φ are given) to obtain three results on its stability.

Theorem 3.14 (Kim [32]). *Let δ, $p > 0$ and n_0 be a non-negative integer. Let $\varphi : (0,\infty) \to (0,\infty)$ be such that the function*

$$\gamma(x) := \sum_{j=0}^{\infty} \prod_{i=0}^{j} \frac{1}{\varphi(x+pi)} \quad (x > n_0)$$

is bounded. If a function $g : (0,\infty) \to \mathbb{R}$ satisfies the following inequality:

$$|g(x+p) - \varphi(x) g(x)| \leq \delta \quad (x > n_0),$$

then there exists an unique function $f : (0,\infty) \to \mathbb{R}$ such that the following conditions are fulfiled for every $x > n_0$:

$$f(x+p) = \varphi(x) f(x)$$

and

$$|g(x) - f(x)| \leq \gamma(x) \delta.$$

The requested function f is defined as the limit as $n \to \infty$ of the following sequence of functions:

$$P_n(x) = g(x+pn) \prod_{i=0}^{n-1} \frac{1}{\varphi(x+pi)} \quad (x > n_0).$$

For the uniqueness, for every function $h : (0,\infty) \to \mathbb{R}$ satisfying

$$|h(x+p) - \varphi(x) h(x)| \leq \delta$$

we have:

$$\begin{aligned} |f(x) - h(x)| &= \prod_{i=0}^{n-1} \frac{1}{\varphi(x+pi)} |f(x+pn) - h(x+pn)| \\ &\leq 2\delta\gamma(x+pn) \prod_{i=0}^{n-1} \frac{1}{\varphi(x+pi)}. \end{aligned}$$

The last expression tends to zero as $n \to \infty$, since the function γ is bounded. In consequence, $h = f$.

Kim [32] established also the Hyers-Ulam-Rassias stability of the functional equation (3.5) in the case of two functions $\varphi, \phi : (0,\infty) \to (0,\infty)$ satisfying the condition:

$$\Phi(x) = \sum_{j=0}^{\infty} \phi(x+pj) \prod_{i=0}^{j} \frac{1}{\varphi(x+pi)} < \infty \quad (x > 0).$$

He proved that if a function $g : (0,\infty) \to \mathbb{R}$ satisfies the following inequality:

$$|g(x+p) - \varphi(x) g(x)| \leq \phi(x) \quad (x > n_0),$$

then there exists an unique function $f : (0,\infty) \to \mathbb{R}$ such that

$$f(x+p) = \varphi(x) f(x) \quad (x > n_0)$$

and

$$|g(x) - f(x)| \leq \Phi(x) \quad (x > n_0).$$

Let $\psi : (0, \infty) \to (0, 1)$ be any function such that the following functions are bounded

$$\alpha(x) := \sum_{i=0}^{\infty} \ln(1 - \psi(x + pi)), \quad \beta(x) := \sum_{i=0}^{\infty} \ln(1 + \psi(x + pi)) \quad (x > n_0).$$

Then for every function $g : (0, \infty) \to (0, \infty)$ such that

$$\left| \frac{g(x + p)}{\varphi(x) g(x)} - 1 \right| \le \psi(x) \quad (x > n_0),$$

there exists an unique function $f : (0, \infty) \to (0, \infty)$ satisfying the following conditions:

$$f(x + p) = \varphi(x) f(x) \quad (x > n_0)$$

and

$$e^{\alpha(x)} \le \frac{f(x)}{g(x)} \le e^{\beta(x)} \quad (x > n_0).$$

For details, see [32, Theorem 3.2].

Kim [38] defined a more general functional background to finally obtain a stability result on the gamma functional equation. He considered the difference equation

$$f(x + p, y + q) - \varphi(x, y) f(x, y) - \psi(x, y) = 0 \tag{3.6}$$

to prove that it is Hyers-Ulam stable. As application (see [38, Corollary 3.1]), for every function $g : \mathbb{N} \to \mathbb{R}$ such that

$$\left| g(x + 1) + \frac{x}{\ln a} g(x) \right| \le \delta \quad (x \in \mathbb{N}),$$

for some $0 < a < 1$, there exists an unique function $f : \mathbb{N} \to \mathbb{R}$ such that the following relations hold true:

$$f(x + 1) + \frac{x}{\ln a} f(x) = 0 \quad (x \in \mathbb{N}) \tag{3.7}$$

and

$$|g(x) - f(x)| \le \left(\frac{1}{a} - 1 \right) \delta \quad (x \in \mathbb{N}).$$

Note that the gamma function is a solution of (3.7), when $a = 1/e$, but in its general form, the functional equation (3.7) admits the solution

$$f(x) = \int_0^{\infty} t^{x-1} a^t \, dt.$$

Further results on the stability of the functional equation (3.6) were presented in [34].

The close relationship between the gamma and the beta function

$$B(x, y) = \frac{\Gamma(x) \Gamma(y)}{\Gamma(x + y)}$$

motivated Lee and Choi [39] to investigate the (super)stability of Cauchy's gamma-beta functional equation

$$B(x, y) f(x + y) = f(x) f(y). \tag{3.8}$$

Firstly, note that this equation admits an unique continuous solution satisfying $f(1) = a > 0$, namely

$$f(x) = a^x \Gamma(x).$$

The functional equation (3.8) is superstable in the sense that if a function $f : (0, \infty) \to (0, \infty)$ satisfies the following conditions:

i) for an arbitrarily fixed $\delta > 0$, there exists $m \in (0, \infty) \cap \mathbb{N}$ such that

$$f(m) \geq \max\left\{2, 2\sqrt{\delta}\right\};$$

ii) the following inequality holds true:

$$|B(x, y) f(x + y) - f(x) f(y)| < \delta \quad (x > 0),$$

then:

$$B(x, y) f(x + y) = f(x) f(y) \quad (x > 0).$$

In the sense of Ger, if a function $f : (0, \infty) \to (0, \infty)$ satisfies the following inequality, for some $\delta > 0$ and every real number $x > 0$:

$$\left|\frac{B(x, y) f(x + y)}{f(x) f(y)} - 1\right| \leq \delta,$$

then there exists an unique function $F : (0, \infty) \to (0, \infty)$ such that

$$B(x, y) F(x + y) = F(x) F(y) \quad (x > 0).$$

For proofs, see [39, Theorem 2.2] and [39, Theorem 2.2].

Lee [43] extended the result to a larger class of the so-called *beta-type functions*, *i.e.*, any function $\beta : D \times D \to \mathbb{R}^*$ (D is an additive subset of $\mathbb{R}$, containing all positive integers) with the following five properties, for every $x, y, z \in D$ and positive integers m, n:

(a) $\beta(x, y) = \beta(y, x)$;
(b) $|\beta(m, n)| \leq 1$;
(c) $\beta(x, y) \beta(z, x + y) = \beta(x, y + z) \beta(y, z)$;
(d) $\lim_{k\to\infty} \prod_{i=1}^{k} |\beta(im, m)| = 0$;
(e) $|\beta(x, n)| < \infty$.

More concretely, he proved that under some conditions, the following functional equation involving beta-type functions

$$\beta(x, y) f(x + y) = f(x) f(y)$$

is superstable. As an example, let $\delta > 0$ and $\beta(x, y)$ be a beta-type function on $(0, \infty)$. Then for every function $g : (0, \infty) \to (0, \infty)$ with $g(m) \geq \max\left\{2, (12\delta)^{1/3}\right\}$, for some positive integer m and

$$|\beta(x, y) g(x + y) - g(x) g(y)| \leq \delta \quad (x > 0,\ y > 0),$$

we have:

$$\beta(x, y) g(x + y) = g(x) g(y) \quad (x > 0,\ y > 0).$$

The study of the beta-type functional equations was continued by Alimohammady and Sadeghi [3], Kim and Lee [35], Lee and Han [41], Lee and Kim [42], or Lee and Choi [40], who considered (3.8) in the following extended form:

$$f(\varphi(x), \phi(x)) = \psi(x, y) f(x, y) + \lambda(x, y),$$

possibly with $\lambda(x, y) \equiv 0$. Many applications can be found and we choose to recall the following, due to its simplicity.

Theorem 3.15 ([35]). *For every function $f : (0, \infty) \times (0, \infty) \to \mathbb{R}$ with the property*

$$|f(x + 1, y + 1) - (x + y) f(x, y)| \leq \delta \quad (x > n_0,\ y > n_0),$$

there exists an unique function $g : (0, \infty) \times (0, \infty) \to \mathbb{R}$ such that

$$g(x + 1, y + 1) - (x + y) g(x, y) = 0 \quad (x > 0,\ y > 0)$$

and

$$|g(x,y)-f(x,y)|<\frac{\delta}{x+y}\left(1+\frac{1}{2}+\frac{1}{2\cdot 4}+\frac{1}{2\cdot 4\cdot 6}+\cdots\right)<\frac{5}{3(x+y)}\cdot\delta.$$

The gamma functional equation and some of its extensions are particular cases of *linear functional equations* of the form

$$f(a(x))=b(x)f(x)+c(x), \tag{3.9}$$

where a,b,c are given continuous functions. In the case $c\equiv 0$, this equation is called a *homogeneous linear equation.* From this point of view, a large amount of classical and new results were presented in time, starting with the work of Baker [6].

As Kim mentioned in [34], general results on linear functional equations can be particularized to obtain different types of stability properties for the gamma, the G-function, beta, Schröder functional equations, as follows:

$$\begin{aligned}
f(\varphi(x)) &= \phi(x)f(x)-\psi(x)\\
f(\varphi(x)) &= xf(x)\\
f(\varphi(x)) &= cf(x), \quad (c\in\mathbb{R})\\
f(x-1) &= x(f(x)-1)\\
f(x-1) &= (x+1)f(x)\\
f(x-p) &= \phi(x)f(x)\\
f(x-1) &= \phi(x)f(x)\\
f(x-1) &= xf(x),
\end{aligned}$$

with appropriate choice of the ϕ function, such as

$$\begin{aligned}
\phi(x) &= c>1 \quad (c\in\mathbb{R})\\
\phi(x) &= \left(1+\frac{1}{x}\right)^{x}\\
\phi(x) &= x^{n} \quad (n=1,2,3,...)\\
\phi(x) &= \arctan x\\
\phi(x) &= \arcsin x\\
\phi(x) &= \Gamma(x).
\end{aligned}$$

See also [2], [34]. Moreover, the n-dimensional version of the functional equation (3.9) and its stability were investigated in [36].

The functional equation, also called *the G-function functional equation:*

$$f(x+1)=\Gamma(x)f(x) \quad (x>0)$$

admits the solution

$$G(x)=(2\pi)^{\frac{x-1}{2}}e^{-\frac{x(x-1)}{2}}e^{-\frac{(x-1)^2}{2}\gamma}\prod_{k=1}^{\infty}\left[\left(1+\frac{x-1}{k}\right)^{k}e^{1-x+\frac{(x-1)^2}{2k}}\right]$$

($\gamma=0.577215\cdots$ is the Euler-Mascheroni constant). This function, firstly introduced by E. W. Barnes [8], is known as *the G-function.* For further details, see also [7].

Let us consider the functions:

$$\varepsilon:(0,\infty)\to\mathbb{R}^{*}, \quad \varphi:(0,\infty)\to(0,\infty).$$

Assume that

$$\omega(x) := \sum_{k=0} \frac{\varepsilon(\varphi_k(x))}{\prod_{j=0}^{k} |\Gamma(\varphi_j(x))|} < \infty \quad (x > 0),$$

where φ_s denotes the s-th iterate of φ :

$$\varphi_0(x) = x, \quad \varphi_n(x) = \varphi_{n-1}(\varphi(x)) \quad (x > 0,\ n = 1, 2, 3, ...).$$

The following result was presented by Kim [37]:

Theorem 3.16. *For every function* $g : (0, \infty) \to (0, \infty)$ *with the property*

$$|g(\varphi(x)) - \Gamma(x) g(x)| \leq \varepsilon(x) \quad (x > 0),$$

there exists an unique function $f : (0, \infty) \to (0, \infty)$ *such that*

$$f(x+1) = \Gamma(x) f(x) \quad (x > 0)$$

and

$$|g(x) - f(x)| \leq \omega(x).$$

In other words, we say that the G-function functional equation is *stable in the sense of Găvruţă*, thanks to a new approach of stability problems using the contraction principle, firstly presented by Găvruţă in [14].

Kim [37] introduced the sequence of functions

$$f_n(x) = \frac{f(\varphi_n(x))}{\prod_{j=0}^{n-1} \Gamma(\varphi_j(x))},$$

and proved that $\{f_n(x)\}$ is a Cauchy sequence, for every real number $x > 0$. Finally, he defined the requested function by the formula

$$f(x) = \lim_{n\to\infty} f_n(x) \quad (x > 0).$$

In the second part of this work, Kim [37] presented the following result on the stability of the G-function functional in the sense of Ger, under the supplementary condition that

$$0 < \varepsilon(x) < 1 \quad (x > 0),$$

such that

$$\alpha(x) := \prod_{j=0}^{\infty} [1 - \varepsilon(\varphi_j(x))] < \infty \quad (x > 0)$$

and

$$\beta(x) := \prod_{j=0}^{\infty} [1 + \varepsilon(\varphi_j(x))] < \infty \quad (x > 0).$$

Theorem 3.17. *For every function* $g : (0, \infty) \to (0, \infty)$ *with the property*

$$\left| \frac{g(\varphi(x))}{\Gamma(x) g(x)} - 1 \right| < \varepsilon(x) \quad (x > n_0),$$

there exists an unique function $f : (0, \infty) \to (0, \infty)$ *such that*

$$f(x+1) = \Gamma(x) f(x) \quad (x > 0)$$

and

$$\alpha(x) \leq \frac{f(x)}{g(x)} \leq \beta(x) \quad (x > n_0).$$

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