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Commuting Regularity degree of finite semigroups

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ABSTRACT. A pair (x, y) of elements x and y of a semigroup S is said to be a commuting regular pair, if there exists an element $z \in S$ such that $xy = (yx)z(yx)$. In a finite semigroup S, the probability that the pair (x, y) of elements of S is commuting regular will be denoted by $dcr(S)$ and will be called the Commuting Regularity degree of S. Obviously if S is a group, then $dcr(S) = 1$. However for a semigroup S, getting an upper bound for $dcr(S)$ will be of interest to study and to identify the different types of non-commutative semigroups. In this paper, we calculate this probability for certain classes of finite semigroups. In this study we managed to present an interesting class of semigroups where the probability is $\frac{1}{2}$. This helps us to estimate a condition on non-commutative semigroups such that the commuting regularity of (x, y) yields the commuting regularity of (y, x).

1. INTRODUCTION

For a semigroup S the ordered pair (x, y) of the elements $x, y \in S$ is called a commuting regular pair if there exists an element $z \in S$ such that $xy = (yx)z(yx)$. A semigroup S is called a commuting regular semigroup, if every pair of the elements $x, y \in S$ is a commuting regular pair. A natural question may be posed here that "for a non-commutative non-commuting regular semigroup, if (x, y) is a commuting regular pair, when (y, x) is also a commuting regular pair?" Investigating this question needs the following definition which was formerly used in [5].

Definition 1.1. The Commuting Regularity degree of a semigroup S denoted by $dcr(S)$, is the probability of choosing two elements x and y of S such that (x, y) is a commuting regular pair.

Obviously, $dcr(S) = 1$ if S is a commuting regular semigroup. The article [5] computes this number for certain sub-semigroups of transformation semigroups and here we use this probability to investigate the above question. Indeed, we consider two infinite classes of finite non-commutative non-commuting regular semigroups, and by means of their Commuting Regularity degrees we propose a conjecture related to the above question.

Our notations are merely standard, following [7] one may get the elementary concepts of semigroups. Also we need some remarks and descriptions on the presentation of semigroups.

For a set A (often called the alphabet), let A^* be the set of all finite words over A and A^+ be the set of all non-empty words in A^*. A semigroup presentation is an ordered pair $\langle A \mid R\rangle$ where, $R \subseteq A^+ \times A^+$. The semigroup defined by a presentation $\langle A \mid R\rangle$ is indeed A^+/ρ where ρ is the smallest congruence on A^+ containing R.

In fact, a semigroup S is defined by the presentation $\langle A \mid R\rangle$ if $S \cong A^+/\rho$. So each word of A^+ represents an element of S. For $\omega_1, \omega_2 \in A^+$, we write $\omega_1 \equiv \omega_2$ if ω_1 and

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ω_2 are identical words and $\omega_1 = \omega_2$ if they represent the same element of S. Thus, for an example if $A = \{a, b\}$ and R is $\{ab = ba\}$, then $aba = a^2b$ but $aba \not\equiv a^2b$. For a detailed information on the presentation of semigroups one may see [2, 3, 10, 11]. As usual, we denote a semigroup presented by a presentation $\pi = \langle A \mid R\rangle$ as $Sg(\pi)$ and a group presented by π as $Gp(\pi)$.

In this paper for every $n \geq 2$, we consider the semigroups $S_1 = Sg(\pi_1)$ and $S_2 = Sg(\pi_2)$ where,

$$\pi_1 = \langle a, b | a^3 = a, b^n a = a, abab^2 = b\rangle$$

and

$$\pi_2 = \langle a, b | a^3 = a, b^{n+1} = b, abab^2 = b\rangle.$$

These well-known semigroups are of orders $4n$ and $6n + 2$, respectively. They are indeed, the semigroups related to the dihedral groups.

Before giving our main results we would like to provide a short history of such probabilistic numbers of algebraic structures indicating their importance and interests in calculation.

For a given algebraic structure A, the commutativity degree of A, denoted by $Pr(A)$ is defined as the probability of choosing a pair (x, y) of the elements of A such that x and y commute, i.e; $Pr(A) = \frac{|\{(x,y)\in A^2|xy=yx\}|}{|A^2|}$. If A is a group it is proved that $Pr(A) = \dfrac{\kappa(A)}{|A|}$ where, $\kappa(A)$ is the number of conjugacy classes of A (see [4]). As an interest upper bound Gustafon [6] showed that $Pr(A) \leq \frac{5}{8}$, for a finite non-abelian group A and MacHale [9] proved that $Pr(A) \leq \frac{5}{8}$, for a finite non-abelian ring. Estimating $Pr(A)$ for a group A, when it is less that or greater than $\frac{1}{2}$ was of interest and Lescot [8] studied some groups satisfying $\frac{1}{2} \leq Pr(A) \leq \frac{5}{8}$ and Doostie [4] presented certain classes of finite groups where $Pr(A) \leq \frac{1}{2}$. In continuation of this research the article [1] proved that the number $\frac{5}{8}$ is not an upper bound for $Pr(A)$ when A is a semigroups.

Our main results on the semigroups S_1 and S_2 are as follows:

Proposition A. *Let $S_1 = Sg(\pi_1)$. For every positive integer $n \geq 2$, the number of ordered pairs of elements of S_1, which are commuting regular equals $8n^2$. Moreover, $dcr(S_1) = \frac{1}{2}$.*

Proposition B. *For the semigroup $S_2 = Sg(\pi_2)$, $dcr(S_2) = \dfrac{12n^2 + 8n + 4}{(6n + 2)^2}$, for all positive integers $n \geq 2$. Moreover, $dcr(S_2)$ is strictly less than $\frac{1}{2}$ for sufficiently large values of n.*

2. The proof of propositions A and B

We use the following key lemma concerning useful new relators in the semigroup S_1.

Lemma 2.1. *The following relators hold in S_1:*
(i) $b^{n+1} = b, b^n a^2 = a^2$,
(ii) $a^2 b^i = b^i$, $(1 \leq i \leq n)$,
(iii) $bab = ab^n$ and $b^i ab^i = bab$, $(i \geq 2)$,
(iv) $(ab)^2 = b^n$,
(v) $b^i ab = ab^{n+1-i}$, $(i \leq n - 1)$,
(vi) $ab^i a = b^{n-i} a^2$, $(i \leq n - 1)$.
Moreover, $|S_1| = 4n$.

Proof. Let $R_1 : a^3 = a$, $R_2 : b^n a = a$ and $R_3 : abab^2 = b$. Then, R_3 yields $b^n(abab^2) = b^{n+1}$ and by using R_2, we get the relator $abab^2 = b^{n+1}$. So $b^{n+1} = b$. The second part of (i) is a quick result of R_2.

The assertion (ii) is obvious for $i = 1$ (by multiplying both sides of R_3 by a from the left and using R_1.) For every $i \geq 2$ we multiply both sides of R_3 by a^2 (from the left) and by b^{i-1} (from the right). Then we get $a^3bab^2b^{i-1} = a^2b^i$ which yields $abab^2b^{i-1} = a^2b^i$. So, $a^2b^i = b^i$.

Proving (iii) needs the relator R_3. Multiplying R_3 by a from the left and using (ii), gives us $bab^2 = ab$. By multiplying this new relator by b^{n-1} from the left and using (i), we get the result. One may get the second part of (iii) by using $bab = ab^n$ and (i).

The assertion (iv) is a quick result of (ii) and (iii).

The relator (v) for $i = 1$ is exactly the first assertion of (iii). Let $i \geq 2$. Then, (iii) yields $b^iab = b^{i-1}(bab) = b^{i-1}ab^n$ or, $b^{i-1}ab^n = b^{i-1}ab^{i-1}b^{n-(i-1)} = ab^nb^{n-(i-1)}$. Since $i - 1 < n$ so, $n - (i - 1) \geq 1$ and then (i) yields $ab^nb^{n-(i-1)} = ab^{n-i+1}$. Finally, (vi) is a result of (v) and the relators R_1 and R_2.

These assertions estimate the lengths of the elements of S_1 and we get that $S_1 = X \cup Y \cup Z \cup T$ where, $X = \{a, a^2\}$, $Y = \{b, b^2, \ldots, b^n\}$, $Z = \{ab^i | i = 1, \ldots, n\}$ and $T = \{b^ia^j | 1 \leq i \leq n - 1, j = 1, 2\}$. Consequently, $|S_1| = 4n$. □

Proof of Proposition A. Using the results of above lemma we intend to find all of the pairs of elements which are commuting regular. In the following table we have summarized all the cases for commuting regular pair (x, y) together with the suitable z such that $xy = (yx)z(yx)$holds.

x	y	z	
a^i	a^j	a^{i+j}	
b^i	b^j	b^t	$t \equiv n - (i + j) \pmod n$
ab^i	ab^j	b^t	$t \equiv 3(i - j) \pmod n$
b^ia	b^ja	b^t	$t \equiv 3(i - j) \pmod n$
b^ia	b^ja^2	$ab^{4n-(i+3j)}$	
b^ia^2	b^ja	ab^{3i-j}	
b^ia^2	b^ja^2	$b^{2n-(i+j)}$	
a	b^ia	b^{n-3i}	
a	b^ia^2	$b^{3i}a$	
b^ia	a	b^{3i}	
b^ia^2	a	$b^{3n-3i}a$	
a^2	b^ia	ab^{n-i}	
b^ia	a^2	ab^{n-i}	
a^2	b^ia^2	b^{n-i}	
b^ia^2	a^2	b^{n-i}	
b^i	ab^j	$b^{4n-(3i+j)}$	
ab^j	b^i	b^{n+3j-i}	

All of the remained pairs are not commuting regular, for example, if $(x, y) = (a, b^j)$ then there is no element $z \in S_1$ satisfying $xy = (yx)z(yx)$. To prove this, we have to examine all possible values of z. The possible values for z are $z = a$, $z = a^2$, $z = b^i$, $z = ab^i$, $z = b^ia$ and $z = b^ia^2$. Considering each case we get the contradictions as follows:

$$z = a \Rightarrow \quad b^jazb^ja \quad = b^{2j}a \neq ab^j,$$

$$z = a^2 \Rightarrow \quad b^jazb^ja \quad = b^jab^ja = ab^na = a^2 \neq ab^j,$$

$$z = b^i \Rightarrow \quad b^j a z b^j a = b^j a b^i b^j a = b^j a b^j b^i a = a b^n b^i a = a b^i a = b^{n-i} a^2 \neq a b^j,$$

$$z = a b^i \Rightarrow \quad b^j a z b^j a = b^j a^2 b^i b^j a = b^{2j+i} a \neq a b^j,$$

$$z = b^i a \Rightarrow \quad b^j a z b^j a = b^j a b^i a b^j a = b^j b^{n-i} a^2 b^j a = b^{n-i+2j} a \neq a b^j,$$

$$z = b^i a^2 \Rightarrow \quad b^j a z b^j a = b^j a b^i a^2 b^j a = b^j a b^i b^j a = b^j a b j b^i a = a b^n b^i a = a b^i a = b^{n-i} a^2 \neq a b^j.$$

So, the pair (a, b^j) is not a commuting regular pair. A similar proof may be used for other pairs which are absent in the above table. Counting all of the commuting regular pairs appeared in the table gives us:

$|\{(x,y)|x,y \in S_1, xy = yxzyx\}| = 4 + 2n^2 + 4(n-1)^2 + 8(n-1) + 2n^2 = 8n^2.$

So, $dcr(S_1) = \dfrac{8n^2}{(4n)^2} = \frac{1}{2}$, as required.

Proving the Proposition B needs certain new information about S_2. In the semigroup S_2, the relators $b^i a b^i = a b^n$ and $a^2 b^i = b^i$ hold for every i ($1 \leq i \leq n$). Moreover, for every i and j where $i \neq j$, we get $b^i a b^j = \begin{cases} a b^{j-i}, & j > i \\ a b^{n-(i-j)}, & j < i. \end{cases}$

Summarizing these information, we easily get $|S_2| = 6n + 2$.

Proof of Proposition B. An almost similar proof to that of Proposition A may be used here. A table of commuting regular pairs, similar to that of S_1 is as follows:

x	y	z	
a^i	a^j	a^{i+j}	
b^i	b^j	b^t	$t \equiv n - (i+j) \pmod n$
a^2	$b^i a$	$b^i a$	
a^2	$b^i a^2$	$a b^i a$	
$b^i a$	a^2	$b^i a$	
$b^i a^2$	a^2	$a b^i a$	
a^2	$a b^i a$	b^i	
a^2	$a b^i a^2$	$a b^i$	
$a b^i a$	a^2	b^i	
$a b^i a^2$	a^2	$a b^i$	
b^i	$a b^j$	$a b^{j+3i}$	
$a b^i$	$a b^j$	b^t	$t \equiv 3(i-j) \pmod n$
$a b^i a$	$a b^j a$	b^{i+j}	
$a b^i a^2$	$a b^j a^2$	b^t	$t \equiv 3(i-j) \pmod n$
$b^i a$	$b^j a$	a	
$a b^i$	b^j	$b^t a$	$t \equiv 3j - i \pmod n$
$b^i a^2$	$b^j a^2$	$a b^{i+j} a$	
$b^j a$	$a b^i a$	$a b^t$	$t \equiv 3i - j \pmod n$
$a b^i a$	$b^j a$	$a b^{3i+j}$	
$b^j a^2$	$a b^i a^2$	$a b^{4n-(i+3j)}$	
$a b^i a^2$	$b^j a^2$	$a b^t$	$t \equiv 3j - i \pmod n$

By considering the relators of S_2 we conclude that the ordered pairs of the forms $(a, b^j a)$, $(a, b^j a^2)$, $(a b^i a, a b^j a)$,$\ldots$ are not commuting regular pairs. Enumerating the pairs which have been appeared in this table shows that:

$$|\{(x,y)|x,y\in S_2,\exists z\in S,\ xy=(yx)z(yx)\}|=12n^2+8n+4$$

Hence $dcr(S_2)=\dfrac{12n^2+8n+4}{(6n+2)^2}$. A simple hand calculation shows that $dcr(S_2)$ is strictly less than $\frac{1}{2}$.

3. Conclusion

One may see easily that $\dfrac{12n^2+8n+4}{(6n+2)^2}$ is a decreasing sequence and also

$$\lim_{n\to\infty}\frac{12n^2+8n+4}{(6n+2)^2}=\frac{1}{3}.$$

So, $\frac{1}{3}$ is a lower bound for $dcr(S_2)$. Obviously, the number $\frac{17}{49}$ as the upper bound will be achieved for $n=2$. Hence, $\frac{1}{3}\leq dcr(S_2)\leq\frac{17}{49}$. This yields that $\frac{1}{2}$ is an upper bound for both of the semigroups $dcr(S_1)$ and $dcr(S_2)$. Considering the results of the tables of Propositions A and B gives us some evidences to provide the following conjecture:

Conjecture. For every non-commutative non-commuting regular semigroup S, if $dcr(S)=\frac{1}{2}$ then the commuting regularity of the pair (x,y) results the commuting regularity of the pair (y,x). But not vice versa.

We examined the conjecture for S_1 where, $dcr(S_1)=\frac{1}{2}$. Also, our experimental results on S_2 proves the converse of this conjecture where, commuting regularity of every pair (x,y) gives the commuting regularity of (y,x), however, $dcr(S_2)<\frac{1}{2}$.

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