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Computing the Wiener index of graphs on triples

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ABSTRACT. Let S be a set of size $n \geq 8$ and V be the set of all subsets of S of size 3. Three types of intersection graphs $G_i(n), i = 0, 1, 2$, can be defined with the vertex set V whose Wiener indices will be calculated in this paper.

1. INTRODUCTION

Let $G = (V, E)$ be a simple connected graph with vertex set V and edge set E, where both are finite sets. The distance between the vertices u and v of G is denoted by $d(u, v)$ and it is the length of the shortest path joining u to v. The distance of a vertex v, denoted by $d(v)$, is the sum of all the distances between v and all vertices u of G, i. e. $d(v) = \sum_{u \in v} d(u, v)$. We always denote an edge joining the vertices u and v by $\{u, v\}$.

The Wiener index of the graph G is denoted by $W(G)$ and is defined by $W(G) = \sum_{u \in v \subseteq V} d(u, v)$, which can be written as $W(G) = \frac{1}{2} \sum_{v \in v} d(v)$.

The Wiener index is one of the oldest descriptors associated to a graph, originally a molecular graph. It is an invariant of a graph in the sense that it is unchanged under applying any automorphism of the graph, and it is called a topological index because it deals with distances between vertices. The Wiener index was first proposed in [11] and was concerned with the determination of the boiling point of paraffin. The Wiener's original definition of his index was different, but in terms of the distances between vertices is due to Hosoya [8]. Computing the Wiener index of a graph is of great interest among mathematicians and to see a few works one is referred to [1], [3], [6]. Also finding other topological indices of graphs, such as hyper-Wiener index and Szeged index, are of interest and the reader may refer to [4], [5], [10], and [9].

In this paper we will consider the so called intersection graphs defined in [7]. Let S be a set of size n, where we assume $n \geq 8$ to avoid triviality, and let V be the set of subsets of S of size 3. We define three graphs $G_i(n), i = 0, 1, 2$, as follows: the vertex set of $G_i(n)$ is S with adjacency defined by two vertices being adjacent if the sets of size 3 meet in zero, one, or two elements, respectively. We call $G_i(n), i = 0, 1, 2$, graphs on triples and our aim in this paper is to calculate the Wiener indices of these graphs.

2. PRELIMINARIES

Let $G = (V, E)$ be a simple connected graph with vertex set V and edge set E. An automorphism of G is a permutation on V which preserves adjacency. The set of all the automorphisms of G under the composition of mappings forms a group that is called the automorphism group of G and is denoted by $Aut(G)$. In [2] it is shown that if $Aut(G)$ acts

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transitively on V, then there is a simple way of calculating the Wiener index $W(G)$ of G. By transitivity of $Aut(G)$ on V we mean that if u and v are two arbitrary vertices of V, then there is $\sigma \in Aut(G)$ such that $u^\sigma = v$. In this case the graph G is called a vertex-transitive graph. We will use the following result which is proved in [2].

Result 1. Let $G = (V, E)$ be a simple connected graph. If $Aut(G)$ acts transitively on V, then $W(G) = \frac{1}{2}|V|\, d(v)$, for any $v \in V$.

Next we define the graphs whose Wiener indices are going to be calculated. Let S be a set of $n \geq 8$ elements and V be the set of all the 3-subsets of S. The graph $G_i(n)$, $i = 0, 1, 2$, has its vertex set V and two vertices (as 3-subsets) are joined if and only if they intersects in a set of cardinality i. Therefore, the number of vertices of $G_i(n)$, $i = 0, 1, 2$, is $\binom{n}{3}$ and they are regular graphs of degree $\binom{n-3}{3}$, $3\binom{n-3}{2}$ and $3(n-3)$ for $i = 0, 1, 2$, respectively. Therefore, the number of edges of these graphs are:

$$|E(G_0(n))| = \frac{1}{2}\binom{n-3}{3}\binom{n}{3},$$

$$|E(G_i(n))| = \frac{3}{2}\binom{n-3}{2}\binom{n}{3},$$

and

$$|E(G_2(n))| = \frac{3}{2}(n-3)\binom{n}{3}.$$

3. Computing the Wiener indices of $G_i(n)$, $i = 0, 1, 2$

In order to be able to use Result 1, first we prove that each of the graphs $G_i(n)$, $i = 0, 1, 2$, is a vertex-transitive graph.

Lemma 3.1. *The automorphism group of $G_i(n)$, $i = 0, 1, 2$, has a subgroup isomorphic to the symmetric group on n letters $\mathbb{S}_n$. Moreover, each of the graphs $G_i(n)$, $i = 0, 1, 2$, is a vertex-transitive graph.*

Proof. Let $\sigma : S \to S$ be a one-to-one mapping. Then σ induces a permutation on $V = S^{\{3\}}$, the set of all the 3-subsets of S, by letting: $\{a, b, c\}^\sigma = \{a^\sigma, b^\sigma, c^\sigma\}$. Now if $u = \{a, b, c\}$ and $v = \{d, e, f\}$ are two vertices in V, then $u^\sigma \cap v^\sigma = (u \cap v)^\sigma$, hence from $|u \cap v| = i$ we deduce $|u^\sigma \cap v^\sigma| = i$, proving that $\{u, v\}$ is an edge in $G_i(n)$, $i = 0, 1, 2$. This proves that $\sigma \in Aut(G_i(n))$, hence the automorphism group of $G_i(n)$ has a subgroup isomorphic to $\mathbb{S}_n$.

In order to prove the vertex transitivity of the graph, let $u = \{a, b, c\}$ and $v = \{d, e, f\}$ be two vertices of the graph $G_i(n)$, $i = 0, 1, 2$. By previous part there is a permutation σ of S_n taking u to v. But σ induces an automorphism of $G_i(n)$ such that $u^\sigma = v$, proving the vertex transitivity of each of the graphs $G_i(n)$, $i = 0, 1, 2$. □

Lemma 3.2. *Let u and v be two vertices of $G_i(n)$, $i = 0, 1, 2$. Then $d(u, v)$ is at most 2, unless $i = 2$, where $d(v, u) = 3$ also occurs.*

Proof. Without loss of generality we set $S = \{1, 2, ..., n\}$ and $v = \{1, 2, 3\}$ a vertex of $G_i(n)$, $i = 0, 1, 2$. Let u be another vertex of $G_i(n)$. If $u = v$, then $d(u, v) = 0$, hence we assume $u \neq v$. We consider three graphs $G_i(n)$, $i = 0, 1, 2$, one by one.

Case (a) $i = 0$. If $u \cap v = \varnothing$, then $d(u, v) = 1$, hence we assume $u \cap v \neq \varnothing$.

If $|u \cap v| = 1$, then we may assume $u = \{1,4,5\}$. In this case if we set $w = \{6,7,8\}$ then $\{v,w\}$ and $\{u,w\}$ are edges of $G_0(n)$, hence $d(u,v) = 2$.

If $|u \cap v| = 2$, then we may assume $u = \{1,2,4\}$. In this case if we set $w = \{5,6,7\}$, then $\{v,w\}$ and $\{u,w\}$ are edges of $G_0(n)$, hence $d(u,v) = 2$.

Case (b) $i = 1$. If $|u \cap v| = 1$, then $d(u,v) = 1$, otherwise $u \cap v \neq \varnothing$ or $|u \cap v| = 2$.

If $|u \cap v| = \varnothing$, then we may assume $u = \{4,5,6\}$, and in this case if we set $w = \{1,4,7\}$, then $\{v,w\}$ and $\{u,w\}$ are edges of $G_1(n)$, hence $d(u,v) = 2$.

If $|u \cap v| = 2$, then we may assume $u = \{1,2,4\}$, hence if $w = \{1,5,6\}$, then $\{v,w\}$ and $\{u,w\}$ are edges of $G_1(n)$, hence $d(u,v) = 2$.

Case (c) $i = 2$. If $|u \cap v| = 2$, then $d(u,v) = 1$, otherwise $u \cap v \neq \varnothing$ or $|u \cap v| = 1$.

If $|u \cap v| = \varnothing$, then we may set $u = \{4,5,6\}$, and in this case if $w = \{1,2,4\}$, $x = \{1,4,5\}$ then $\{v,w\}$, $\{w,x\}$ and $\{x,u\}$ are edges of $G_2(n)$, hence $d(u,v) = 3$.

If $|u \cap v| = 1$, then we may assume $u = \{1,4,5\}$, hence with $w = \{1,2,4\}$ we have $d(u,v) = 2$.

The proof is completed. $\square$

Theorem 3.1. *The Wiener indices of the graph $G_i(n)$, $i = 0,1,2$, are as follows:*

$$W(G_0(n)) = \frac{1}{12}\binom{n}{3}(n-3)(n^2+9n-16),$$

$$W(G_1(n)) = \frac{1}{12}\binom{n}{3}(n-3)(2n^2-9n+40),$$

$$W(G_2(n)) = \frac{1}{4}\binom{n}{3}(n-3)(n^2-3n-2).$$

Proof. By Lemma 3.1, $Aut(G_i(n))$ acts transitively on the vertices of the graph $G_i(n)$, $i = 0,1,2$. Hence by Result 1 we can write $W(G_i(n)) = \frac{1}{2}|V|\,d(v)$, where v is any vertex of $G_i(n)$, where $d(v) = \sum_{u \in v} d(u,v)$. In the following we calculate $d(v)$ for each $G_i(n)$, from which the formulae in Theorem 3.1 follows.

Case (a) $i = 0$. By Lemma 3.2 Case (a), we have $d(u,v) = 0$, 1, or 2. The number of vertices at distance 1 from v is equal to the number of 3-subsets of S which intersect v in the empty set, which is equal to $\binom{n-3}{3}$. The number of vertices at distance 2 from v are those 3-subsets S which intersect v in 1 or 2 elements, which is equal to $3\binom{n-3}{2} + 3\binom{n-3}{1}$. Therefore, $d(v) = \binom{n-3}{3} + 6\binom{n-3}{2} + 6\binom{n-3}{1} = \frac{1}{6}(n-3)(n^2+9n-16)$ and the formula follows.

Case (b) $i = 1$. By Lemma 3.2 Case (b), the vertices u at distance 1 from v are those with $|u \cap v| = 1$, and those with distance 2 from v have the property $|u \cap v| = 0$ or 2.

For a 3-subset v of S, the number of 3-subsets of S that intersect v in 1 point is $3\binom{n-3}{2}$, and the number of 3-subsets of S that intersect v in 0 or 2 points is $\binom{n-3}{3} + 3\binom{n-3}{1}$. Therefore, $d(v) = 3\binom{n-3}{2} + 2\binom{n-3}{3} + 6\binom{n-3}{1} = \frac{1}{6}(n-3)(2n^2-9n+40)$ and the formula follows.

Case (c) $i = 2$. Again by Lemma 3.2 Case (c), if $|u \cap v| = 2$, then $d(u, v) = 1$, hence the number of these u is equal to $3\binom{n-3}{1}$. If $|u \cap v| = \varnothing$, then $d(u, v) = 3$ and the number of such u is equal to $\binom{n-3}{3}$, and if $|u \cap v| = 1$, then $d(u, v) = 2$, and the number of such u is $3\binom{n-3}{2}$. Therefore, $d(v) = 3\binom{n-3}{1} + 6\binom{n-3}{2} + 3\binom{n-3}{3}$=$\frac{1}{2}(n-3)(n^2-3n-2)$ and the formula follows. The proof is completed. $\square$

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