CREAT. MATH. INFORM. **24** (2015), No. 1, 53 - 60
Online version at http://creative-mathematics.ubm.ro/
Print Edition: ISSN 1584 - 286X Online Edition: ISSN 1843 - 441X

On some weak separation axioms

SABIR HUSSAIN

ABSTRACT. We introduce and explore weak separation axioms namely γ-semi-T_i(for $i = 0, 1, 2$) spaces. We also define and discuss γ-semi-D_i(for $i = 0, 1, 2$) spaces and develop the relations between these spaces. Moreover, we initiate the concept of γ-S-continuous function and discuss the behavior of γ-semi-D_1 space under γ-S-continuous function.

1. INTRODUCTION

One of the most important and interesting area of mathematics named as Topology. The influence of which is evident in almost every other branch of mathematics. Kasahara S. [14] introduced and discussed an operation γ of a topology τ into the power set P(X) of a space X . Ogata H. [17], initiated γ-open sets and explored the related topological properties. Many researchers worked on the investigations of H. Ogata and introduced many concepts in the literature.

Hussain S. and Ahmad B. [1-13] generalized many classical notions of topology by exploring and investigating many interesting properties of γ-operations on topological spaces. Levine N. [15] introduced semi-open sets in topological spaces. Several topologist generalized many old concepts of topology using semi open sets. In [7], Hussain S., et. al initiated and discussed γ-semi-open sets in topological spaces as a generalization of γ-open sets. In [2], [6] they discussed and investigated γ-semi-closure (interior) and γ-semi-continuous functions. They also obtained some properties of pre γ-semi-open sets using properties of minimal γ-semi-open sets [10] and maximal γ-semi-open sets [13] in topological spaces. Recently, Hussain S. [9], discussed and explored the properties of γ-semi-R_0 spaces in terms of γ-semi-open sets which contains the γ-semi-closure of each of its singletons.

In this paper, we introduce and explore weak separation axioms namely γ-semi-T_i(for $i = 0, 1, 2$) spaces. We also define and discuss γ-semi-D_i(for $i = 0, 1, 2$) spaces and develop the relations between these spaces. Moreover, we initiate the concept of γ-S-continuous function and discuss the behavior of γ-semi-D_1 space under γ-S-continuous function.

First, we recall some definitions and results used in this paper. Hereafter, we shall write a space in place of a topological space.

2. PRELIMINARIES

Throughout the present paper X denotes the topological spaces.

Definition [14]. An operation $\gamma : \tau \rightarrow$ P(X) is a function from τ to the power set of X such that V $\subseteq V^\gamma$, for each V $\in \tau$, where V^γ denotes the value of γ at V. The operations defined

Received: 03.12.2014. In revised form: 05.03.2015. Accepted: 31.03.2015
2010 *Mathematics Subject Classification.* 54A05, 54A20, 54C08, 54D10.
Key words and phrases. *γ-semi-open (closed), γ-semi-closure(interior) , γ-semi-D_i (for $i = 0, , 1, 2$), γ-semi-T_i (for $i = 0, , 1, 2$), γ-S-continuous function.*

by γ(G) = G, γ(G) = cl(G) and γ(G) = intcl(G) are examples of operation γ.
Definition [14]. Let A$\subseteq$ X. A point $x \in A$ is said to be a γ-interior point of A, if there exists an open nbd N of x such that $N^\gamma \subseteq$ A and we denote the set of all such points by int_γ(A). Thus

$$int_\gamma(A) = \{x \in A : x \in N \in \tau \text{ and } N^\gamma \subseteq A\} \subseteq A.$$

Note that A is γ-open [14] iff A = int_γ(A). A set A is called γ- closed [1] iff X-A is γ-open.
Definition [17]. A point $x \in X$ is called a γ-closure point of A$\subseteq$ X, if $U^\gamma \cap A \neq \phi$, for each open nbd U of x. The set of all γ-closure points of A is called γ-closure of A and is denoted by cl_γ(A). A subset A of X is called γ-closed, if $cl_\gamma(A) \subseteq A$. Note that $cl_\gamma(A)$ is contained in every γ-closed superset of A.
Definition [14]. An operation γ on τ is said be regular, if for any open nbds U,V of x $\in$ X, there exists an open nbd W of x such that $U^\gamma \cap V^\gamma \supseteq W^\gamma$.
Definition [7]. A subset A of a space X is said to be a γ-semi-open set, if there exists a γ-open set O such that $O \subseteq A \subseteq cl_\gamma(O)$. The set of all γ-semi-open sets is denoted by $SO_\gamma(X)$. A is γ-semi-closed iff $X - A$ is γ-semi-open in X.
Definition [2]. Let A be a subset of a space X. The intersection of all γ-semi-closed sets containing A is called γ-semi-closure of A and is denoted by $scl_\gamma(A)$.
Note that A is γ-semi-closed if and only if $scl_\gamma(A) = A$.
Definition [2]. Let A be a subset of a space X. The union of γ-semi-open subsets of A is called γ-semi-interior of A and is denoted by $sint_\gamma(A)$.
Definition [11]. A subset A of X is said to be γ-semi-nbd of a point $x \in X$, if there exists a γ-semi-open set U such that $x \in U \subseteq A$.
Lemma [11]. Let A be a subset of a space X. Then $x \in scl_\gamma(A)$ if and only if for any γ-semi-nbd N_x of x in X, $A \cap N_x \neq \phi$.

3. γ-SEMI-D-SETS

Definition 3.1. Let X be a space and $A \subseteq X$. Then A is called γ-semi-D-set, if there exists two γ-semi-open sets U_1 and V_1 such that $U_1 \neq X$ and $A = U_1 \backslash V1$.

Clearly every γ-semi-open set $U_1 \neq X$ is a γ-semi-D-set, if $A = U_1$ and $V_1 = \phi$.

Example 3.1. Let X= $\{a, b, c\}$, $\tau = \{\phi, X, \{a\}, \{a, b\}\}$ be topology on X. For $b \in X$, define an operation $\gamma : \tau \to P(X)$ by

$$\gamma(A) = A^\gamma = \begin{cases} cl(A), & \text{if } b \in A \\ A, & \text{if } b \notin A \end{cases}$$

It is easy to show that $SO_\gamma(X) = \{\phi, X, \{a\}, \{a, b\}, \{a, c\}\}$, where $SO_\gamma(X)$ represents γ-semi-open sets in X. Clearly nonempty γ-semi-D sets are $\{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, X$.

Definition 3.2. A space X is said to be γ-semi-D_0, if for any $x, y \in X$ such that $x \neq y$, there exists a γ-semi-D-set U of X such that $x \in U$ and $y \notin U$ or a γ-semi-D-set V of X such that $y \in V$ and $x \notin V$.

Definition 3.3. A space X is said to be γ-semi-D_1, if for any $x, y \in X$ such that $x \neq y$, there exists a γ-semi-D-set U of X such that $x \in U$ and $y \notin U$ and a γ-semi-D-set V of X such that $y \in V$ and $x \notin V$.

Definition 3.4. A space X is said to be γ-semi-D_2, if for any $x, y \in X$ such that $x \neq y$, there exist γ-semi-D-sets U and V of X such that $x \in U$, $y \in V$ and $U \cap V = \phi$.

Definition 3.5. A space X is said to be γ-semi-T_0, if for any $x, y \in X$ such that $x \neq y$, there exists a γ-semi-open set U of X such that $x \in U$ and $y \notin U$ or a γ-semi open set V of X such that $y \in V$ and $x \notin V$.

Definition 3.6. A space X is said to be γ-semi-T_1, if for any $x, y \in X$ such that $x \neq y$, there exists a γ-semi-open set U of X such that $x \in U$ and $y \notin U$ and a γ-semi-open set V of X such that $y \in V$ and $x \notin V$.

Definition 3.7. A space X is said to be γ-semi-T_2, if for any $x, y \in X$ such that $x \neq y$, there exist γ-semi-open sets U and V of X such that $x \in U$, $y \in V$ and $U \cap V = \phi$.

Example 3.2. In above example 3.1, clearly space X is γ-semi-$D_i (i = 0, 1, 2)$ space but the space X is not γ-semi-T_i ($i = 0, 1, 2)$ space.

Now we characterize γ-semi-T_0 in the following Theorem:

Theorem 3.1. *A space X is γ-semi-T_0 if and only if $scl_\gamma(\{x\}) \neq scl_\gamma(\{y\})$, for each $x, y \in X$ such that $x \neq y$.*

Proof. ($\Rightarrow$) Suppose that $scl_\gamma(\{x\}) \neq scl_\gamma(\{y\})$, for $x, y \in X$ such that $x \neq y$. Let $p \in X$ such that $p \in scl_\gamma(\{x\})$ and $p \notin scl_\gamma(\{y\})$. We claim that $x \notin scl_\gamma(\{y\})$. Contrarily suppose that, if $x \in scl_\gamma(\{y\})$ then $scl_\gamma(\{x\}) \subseteq scl_\gamma(\{y\})$. This is contradiction to the fact that $p \notin scl_\gamma(\{y\})$. Consequently $x \in [scl_\gamma(\{y\})]^c$, where $[scl_\gamma(\{y\})]^c$ is a γ-semi-open set and $y \notin [scl_\gamma(\{y\})]^c$.
($\Leftarrow$) Suppose that X be a γ-semi-T_0 space and $x, y \in X$ such that $x \neq y$. Then by definition, there exists a γ-semi-open set U containing x or y. Let we suppose that $x \in U$ but $y \notin U$. Then U^c is a γ-semi-closed set such that $x \notin U$ but $y \in U$. Since $scl_\gamma(\{y\})$ is the smallest γ-semi-closed set such that $y \in scl_\gamma(\{y\})$ [2], $scl_\gamma(\{y\}) \subseteq U^c$ and thus $x \notin scl_\gamma(\{y\})$. Hence $scl_\gamma(\{x\}) \neq scl_\gamma(\{y\})$. This completes the proof. □

Proposition 3.1. *Let $\gamma : \tau \to P(X)$ be a regular operation. If X is γ-semi-T_1, then for each $x \in X$, $\{x\}$ is γ-semi-closed.*

Proof. Suppose that X is γ-semi-T_1 and $x \in X$. Let $y \in \{x\}^c$. Then $x \neq y$ and by definition of γ-semi-T_1 space, there exists a γ-semi-open set G_y such that $y \in G_y$ and $x \notin G_y$. In consequence, $y \in G_y \subseteq \{c\}^c$ implies that $\{x\}^c = \bigcup\{G_y : y \in \{x\}^c\}$. Since γ is regular operation, so $\{x\}^c$ is γ-semi-open and thus $\{x\}$ is γ-semi-closed. This completes the proof. □

Proposition 3.2. *Let X be a space. If for each $x \in X$, $\{x\}$ is γ-semi-closed. Then X is γ-semi-T_1.*

Proof. Suppose that for each $x \in X$, $\{x\}$ is γ-semi-closed. Let $p, q \in X$ such that $p \neq q$ implies $q \in \{p\}^c$. Hence $\{p\}^c$ is a γ-semi-open set such that $q \in \{p\}^c$ and $p \notin \{p\}^c$. In similar way $\{q\}^c$ is a γ-semi-open set such that $p \in \{q\}^c$ and $q \notin \{q\}^c$. Hence X is a γ-semi-T_1 space. This completes the proof. □

Combining Propositions 3.1 and 3.2, we have:

Proposition 3.3. *Let $\gamma : \tau \to P(X)$ be a regular operation. A space X is γ-semi-T_1 if and only if for each $x \in X$, $\{x\}$ is γ-semi-closed.*

From Proposition 3.3 and by definitions of γ-semi-D_i (for $i = 0, 1, 2)$ sets and γ-semi-T_i(for $i = 0, 1, 2)$ spaces, we obtain the following:

(A) γ-semi-$T_2 \Rightarrow \gamma$-semi-$T_1 \Rightarrow \gamma$-semi-T_0
(B) γ-semi-$T_i \Rightarrow \gamma$-semi-D_i; for $i = 0, 1, 2$
(C) γ-semi-$D_2 \Rightarrow \gamma$-semi-$D_1 \Rightarrow \gamma$-semi-D_0

Theorem 3.2. *Let X be a space. Then*
(1) X is γ-semi-$D_0 \Leftrightarrow X$ is γ-semi-T_0.
(2) X is γ-semi-$D_1 \Leftrightarrow X$ is γ-semi-D_2.

Proof. (1) ($\Leftarrow$) This follows from above implication (B).
($\Rightarrow$) Suppose that X be a γ-semi-D_0 space. Then for each $x, y \in X$ such that $x \neq y$, there exists γ-semi-D-set U such that U contains any one of x or y. Let us assume that $x \in U$, $y \notin U$. Consider $U = V_1 \backslash V_2$, where V_1 and V_2 are γ-semi-open sets and $V_1 \neq X$. Then $x \in V_1$ and for $y \notin U$, we consider two cases.:
Case i. $y \notin V_1$. In this case, $x \in V_1$ and $y \notin V_1$.
Case ii. $y \in V_1$ and $y \in V_2$. In this case $y \in V_2$ and $x \notin V_2$.
This implies that X is a γ-semi-T_0 space.
(2)($\Leftarrow$) This follows from above implication (C).
($\Rightarrow$) Suppose that X be a γ-semi-D_1 space. Then for each $x, y \in X$ such that $x \neq y$, there exist γ-semi-D-sets U and V such that $x \in U$, $y \notin U$ and $y \in V$ and $x \notin V$. Let us assume the γ-semi-open sets G_1, G_2, G_3 and G_4 such that $U_1 = G_1 \backslash G_2$ and $U_2 = G_3 \backslash G_4$. From $x \notin U_2$, we have either $x \notin G_3$ or $x \in G_3$ and $x \in G_4$. We consider two possibilities:
(i). If $x \notin G_3$. Since $y \in U_1$ then either $y \in G_1$ and $y \in G_2$ or $y \notin G_1$. If $y \in G_1$ and $y \in G_2$. Then $x \in G_1 \backslash G_2$, $y \in G_2$ and $(G_1 \backslash G_2) \cap G_2 = \phi$.
If $y \notin G_1$. Since $x \in G_1 \backslash G_2$ implies $x \in G_1 \backslash (G_2 \cup G_3)$ and $y \in G_3 \backslash G_4$ implies $y \in G_3 \backslash (G_1 \cup G_4)$. Clearly, $(G_1 \backslash (G_2 \cup G_3)) \cap (G_3 \backslash (G_1 \cup G_4)) = \phi$.
(ii). If $x \in G_3$ and $x \in G_4$. Then $y \in G_3 \backslash G_4$, $x \in G_4$ and $(G_3 \backslash G_4) \cap G_4 = \phi$.
This implies that the space X is γ-semi-D_2 space. Hence the proof. □

The proof of the following proposition directly follows form the implication (C) and Theorem 3.2.

Proposition 3.4. *Let X be a space. The X is γ-semi-D_1 implies X is γ-semi-T_0.*

Definition 3.8. Let X be a space and $x \in X$. Then x is said to be a γ-semi-neat point of X, if x has only one γ-semi-nbd which is X itself.

Theorem 3.3. *Let X be a space. If X is a γ-semi-T_0 space, then the following are equivalent:*
(1) X is a γ-semi-D_1 space.
(2) There exists no any γ-semi-neat point of X.

Proof. (1) $\Rightarrow$ (2). Suppose that X is γ-semi-D_1, then for each $x, y \in X$ such that $x \neq y$, there exists γ-semi-D-set, say, $G = G_1 \backslash G_2$ of X and so is G_1. Which implies that $G_1 \neq X$. Hence x is not a γ-semi-neat point. Since x is arbitrary, this proves (2).
(2) $\Rightarrow$ (1). Suppose that X is γ-semi-T_0. Then by definition, for each $x, y \in X$ and $x \neq y$, there exist γ-semi-open-nbd G such that $x \in G$ and $y \notin G$. So G is a γ-semi-D-set such that $G \neq X$. If there is no γ-semi-neat point of X, then $y \in X$ is not a γ-semi-neat point. Thus there exists a γ-semi-open-nbd U of y such that $U \neq X$. Thus $(U \backslash G)$ with $x \notin (U \backslash G)$ is a γ-semi-D-set such that $y \in (U \backslash G)$ and $x \notin (U \backslash G)$. This implies that X is a γ-semi-D-set. Hence the proof. □

Definition 3.9. Let X be a space. Then X is said to be γ-semi-symmetric, if for $x, y \in X$, $x \in scl_\gamma(\{y\})$ implies $y \in scl_\gamma(\{x\})$.

Definition 3.10. Let X be a space and $A \subseteq X$. Then A is said to be a (γ, γ)-semi-generalized closed set (in short (γ, γ)-sg-closed), if $scl_\gamma(A) \subseteq G$, for γ-semi-open set G in X and $A \subseteq G$.

Proposition 3.5. *Let X be a space and $A \subseteq X$. If A is γ-semi-closed, then it is (γ, γ)-sg-closed.*

Theorem 3.4. *Let X be a space. The X is γ-semi-symmetric if and only if for each $x \in X$, $\{x\}$ is (γ, γ)-sg-closed.*

Proof. ($\Leftarrow$) Suppose that $x \in scl_\gamma(\{y\})$. Assume contrarily that $y \notin scl_\gamma(\{x\})$. This implies that $y \in (scl_\gamma(\{x\}))^c$. Therefore $\{y\} \subseteq (scl_\gamma(\{x\}))^c$. Hence $scl_\gamma(\{y\}) \subseteq (scl_\gamma(\{x\}))^c$. So $x \in (scl_\gamma(\{x\}))^c$. A contradiction. This gives as desired.
($\Rightarrow$) Contrarily suppose that for $x \in X$ and a γ-semi-open set G in X such that $\{x\} \subseteq G$ and $scl_\gamma(\{x\}) \nsubseteq G$. This implies that $scl_\gamma(\{x\}) \cap G^c \neq \phi$. Let us suppose that $y \in (scl_\gamma(\{x\}) \cap G^c)$. Here we have $x \in scl_\gamma(\{y\})$. Thus we have $scl_\gamma(\{y\}) \subseteq G^c$ and $x \notin G$. This is a contradiction to the fact. Hence the proof is completed. □

Theorem 3.5. *Let X be a space and $\gamma : \tau \to P(X)$ be a regular operation. If X is a γ-semi-T_1 then it is γ-semi-symmetric.*

Proof. By Proposition 3.3, X is γ-semi-T_1 implies that for each $x \in X$, $\{x\}$ is γ-semi-closed. So by Proposition 3.5, $\{x\}$ is (γ, γ)-sg-closed. Hence by Theorem 3.4, $\{x\}$ is γ-semi-symmetric. Hence the proof.

□

Theorem 3.6. *Let X be a space and $\gamma : \tau \to P(X)$ be a regular operation. Then the following statements are equivalent:*
(1) X is γ-semi-symmetric and γ-semi-T_0.
(2) X is γ-semi-T_1.

Proof. $(2) \Rightarrow (1)$. This follows directly from Theorem 3.5 and Implication (A).
$(1) \Rightarrow (2)$. Since X is γ-semi-T_0, so for $x, y \in X$ such that $x \neq y$, we have γ-semi-open sets U (say) such that $x \in U \subseteq (\{y\})^c$. Then $x \notin scl_\gamma(\{y\})$ and hence $y \notin scl_\gamma(\{x\})$. Therefore, there exists a γ-semi-open set V such that $y \in V \subseteq (\{x\})^c$. Hence X is a γ-semi-T_1 space. This completes the proof. □

The following theorem follows from Implication (A), Theorems 3.2, 3.6 and Proposition 3.3.

Theorem 3.7. *Let X be a space and $\gamma : \tau \to P(X)$ be a regular operation. If X is γ-semi-symmetric, then the following statements are equivalent:*
(1) X is γ-semi-T_0.
(2) X is γ-semi-D_1.
(3) X is γ-semi-T_1.

4. γ-S-Continuous functions

Definition 4.11. Let X and Y be two spaces. A function $f : X \to Y$ is γ-S-continuous, if and only if for any γ-semi-open sets U in Y, $f^{-1}(U)$ is γ-semi-open in X.

Example 4.3. Let $X = \{a, b, c\}$, $\tau = \{\phi, X, \{a\}, \{a, b\}\}$ be topology on X. For $b \in X$, define an operation $\gamma : \tau \to P(X)$ by

$$\gamma(A) = A^\gamma = \begin{cases} cl(A), & \text{if } b \in A \\ A, & \text{if } b \notin A \end{cases}$$

Clearly $SO_\gamma(X) = \{\phi, X, \{a\}, \{a,b\}, \{a,c\}\}$, where $SO_\gamma(X)$ represents γ-semi-open sets in X. The function $f : X \to X$ is defined by setting $f(a) = f(b) = a$ and $f(c) = b$. Calculations show that f is γ-S-continuous function.

Theorem 4.8. *Let X and Y be two spaces. A function $f : X \to Y$ is γ-S-continuous, if for each $x \in X$ and each γ-semi-open set B such that $f(x) \in B$, there exists a γ-semi-open set $A \in X$ such that $f(A) \subseteq B$. Where γ is a regular operation.*

Proof. Suppose that f is γ-S-continuous. Therefore for γ-semi-open set B in Y, $f^{-1}(B)$ is γ-semi-open in X. We prove that for each γ-semi-open set B containing $f(x)$, there exists γ-semi-open set A in X such that $x \in A$ and $f(A) \subseteq B$. Let $x \in f^{-1}(B) \in SO_\gamma(X)$ and $A = f^{-1}(B)$. Then $x \in A$ and $f(A) \subseteq ff^{-1}(B) \subseteq B$, where B is γ-semi-open.

Conversely, let B be γ-semi-open set in Y. We prove that inverse image of γ-semi-open set in Y is γ-semi-open set in X. Let $x \in f^{-1}(B)$. Then $f(x) \in B$. Thus there exists an $A_x \in SO_\gamma(X)$ such that $x \in A_x$ and $f(A_x) \in B$. Then $x \in A_x \subseteq f^{-1}(B)$ and $f^{-1}(B) = \bigcup_{x \in f^{-1}(B)} A_x$ implies $f^{-1}(B) \in SO_\gamma(X)$, since γ is regular operation [2]. This proves that f is γ-S-continuous. Hence the proof. □

Definition 4.12. Let X be a space. A net $(x_i)_{i \in I}$ is said to be γ-S-converges to $x \in X$, if for each γ-semi-open set G with $x \in G$, there exists $i_0 \in I$ such that $i \geq i_0$ implies $x_i \in G$.

Definition 4.13. Let X be a space. A filterbase Γ in X, γ-S-converges to $x \in X$, if for each γ-semi-open set G with $x \in G$, there exists $F \in \Gamma$ such that $F \subseteq G$.

Definition 4.14. Let X be a space. A filterbase Γ in X, γ-S-accumulates to $x \in X$, if for each γ-semi-open set G with $x \in G$ and each $F \in \Gamma$, $F \cap G \neq \phi$.

The following theorem directly follows from the above definitions:

Theorem 4.9. *If a filterbase Γ in X, γ-S-converges to $x \in X$, then Γ γ-S-accumulates to x.*

The proof of the following theorem is easy and thus omitted:

Theorem 4.10. *Let X and Y be two spaces and $f : X \to Y$ be a function. The the following statements are equivalent:*
(1) f is γ-S-continuous.
(2) $f(\Gamma)$ γ-S-converges to $f(x)$, for each $x \in X$ and each filterbase Γ which γ-S-converges to x.
(3) The net $(f(x_i))_{i \in I}$ of Y, γ-S-converges to $f(x)$ in Y, for each x in X and each net $(x_i)_{i \in I}$ in X which γ-S-converges to x.

Definition 4.15. Let X and Y be two spaces. A function $f : X \to Y$ is γ-S-D-continuous if and only if for any γ-semi-D set U in Y, $f^{-1}(U)$ is γ-semi-D in Y.

Theorem 4.11. *Let X and Y be two spaces. A function $f : X \to Y$ is γ-S-D-continuous, if for each $x \in X$ and each γ-semi-D set U such that $f(x) \in U$, there exists a γ-semi-D set $G \in X$ such that $f(G) \subseteq U$. Where γ is a regular operation.*

Definition 4.16. Let X be a space. A net $(x_i)_{i \in I}$ is said to be γ-S-D-converges to $x \in X$, if for each γ-semi-D sets G with $x \in G$, there exists $i_0 \in I$ such that $i \geq i_0$ implies $x_i \in G$.

Definition 4.17. Let X be a space. A filterbase Γ in X, γ-S-D-converges to $x \in X$, if for each γ-semi-D set G with $x \in G$, there exists $F \in \Gamma$ such that $F \subseteq G$.

Theorem 4.12. *Let X and Y be two spaces and $f : X \to Y$ be a function. The the following statements are equivalent:*

(1) f is γ-S-D-continuous.
(2) $f(\Gamma)$ γ-S-D-converges to $f(x)$, for each $x \in X$ and each filterbase Γ which γ-S-D-converges to x.
(3) The net $(f(x_i))_{i \in I}$ of Y, γ-S-D-converges to $f(x)$ in Y, for each x in X and each net $(x_i)_{i \in I}$ in X which γ-S-D-converges to x.

Theorem 4.13. *Let X and Y be two spaces and $f : X \to Y$ be a function. If f is a γ-S-continuous surjective function, then $f^{-1}(U)$ is a γ-semi-D set in X, for each γ-semi-D set U in Y.*

Proof. Let U be a γ-semi-D set in Y. Then by definition , there exist γ-semi-open sets V_1 and V_2 in Y such that $V_1 \neq Y$ and $U = V_1 \backslash V_2$. Since f is γ-S-continuous, so $f^{-1}(V_1)$ and $f^{-1}(V_2)$ are γ-semi-open in X. Also since f is surjective, $V_1 \neq Y$ implies $f^{-1}(V_1) \neq X$. Thus $f^{-1}(U) = f^{-1}(V_1) \backslash f^{-1}(V_2)$ is a γ-semi-D set. This completes the proof. □

Theorem 4.14. *Let X and Y be two spaces and $f : X \to Y$ is a γ-S-continuous bijective function. If Y is a γ-semi-D_1 space then X is a γ-semi-D_1 space.*

Proof. Let $x, y \in X$ such that $x \neq y$. Since Y is γ-semi-D_1 and f is injective, therefore there exist γ-semi-D sets U and V in Y such that $f(x) \in U$, $f(y) \in V$ and $f(y) \notin U$, $f(x) \notin V$. Thus $f^{-1}(U)$ and $f^{-1}(V)$ are γ-semi-D sets in X such that $x \in f^{-1}(U)$ and $y \in f^{-1}(V)$, by Theorem 4.13. This shows that X is a γ-semi-D_1 space. Hence the proof. □

Theorem 4.15. *Let X and Y be two spaces. Then X is γ-semi-D_1 if and only if for $x, y \in X$ with $x \neq y$, there exists a γ-S-continuous surjective function $f : X \to Y$, where Y is a γ-semi-D_1 space such that $f(x) \neq f(y)$.*

Proof. ($\Rightarrow$) The proof of necessity follows by taking the identity function on X such that $x, y \in X$ and $x \neq y$.
($\Leftarrow$) Suppose $x, y \in X$ such that $x \neq y$. By hypothesis, there exist a γ-S-continuous surjective function $f : X \to Y$, where Y is a γ-semi-D_1 space such that $f(x) \neq f(y)$. Therefore, there exist γ-semi-D sets U and V in Y such that $f(x) \in U$, $f(y) \in V$ and $U \cap V = \phi$. Since f is γ-S-continuous and surjective, by Theorem 4.14, $f^{-1}(U)$ and $f^{-1}(V)$ are γ-semi-D sets in X such that $x \in f^{-1}(U)$, $y \in f^{-1}(V)$ and $f^{-1}(U) \cap f^{-1}(V) = \phi$. Thus by Theorem 4.13, X is a γ-semi-D_1 space. This completes the proof. □

Acknowledgement. The author is thankful to the editor-in-chief, Prof. Vasile Berinde for his remarks towards the improvement of this paper.

References

[1] Ahmad, B. and Hussain, S., *Properties of γ-Operations on Topological Spaces*, Aligarh Bull. Math., **22** (2003), No. 1, 45–51
[2] Ahmad, B. and Hussain, S., *γ-Semi-Open Sets in Topological Spaces II*, Sotheast Asian Bull. Math., **34** (2010), No. 5, 997–1008
[3] Ahmad, B. and Hussain, S., *γ-Convergence in Topological Spaces*, Southeast Asian Bull. Math., **29** (2005), 832–842
[4] Ahmad, B. and Hussain, S., *γ^*-Regular and γ-Normal Space*, Math. Today., **22** (2006), No. 1, 37–44
[5] Ahmad, B. and Hussain, S., *On γ-s-Closed Subspaces*, Far East Jr. Math. Sci., **31** (2008), No. 2, 279–291
[6] Ahmad, B., Hussain, S. and Noiri, T., *On Some Mappings in Topological Space*, Eur. J. Pure Appl. Math., **1** (2008), 22–29
[7] Ahmad, B., Hussain, S. and Noiri, T., *γ-Semi-Open Sets in Topological Spaces*, Asian Eur. J. Math., **3** (2010), No. 3, 427–433 DOI: 10.1142/S1793557110000337
[8] Ahmad, B., Hussain, S. and Noiri, T., *On γ-Semi Continuous Functions*, Punjab University Jr. Math., **42** (2010), 57–65
[9] Hussain, S., *On Some Properties of γ-semi-R_0 Spaces*, , Creat. Math. Inform., **23** (2014), No. 2, 203–207

[10] Hussain, S. and Ahmad, B., *Applications of Minimal γ-Semi-Open Sets,* , Fasciculli Mathematici, **46** (2011), 65–76
[11] Hussain, S. and Ahmad, B., *On γ-s-Closed Spaces*, Sci. Magna Jr., **3** (2007), No. 4, 89–93
[12] Hussain, S. and Ahmad, B., *On γ-s-Regular Spaces and Almost γ-s-Continuous Functions*, Lobackevskii. J. Math., **30** (2009), No. 4, 263–268. DOI: 10.1134/51995080209040039
[13] Hussain, S. and Alghamdi, M. A., *Some Properties of Weak form of γ-Semi-Open Sets*, Tamkang Jr. Math., **42** (2010), 57–65 DOI:10.5556/j.tkjm.43.2012.329-338
[14] Kasahara, S., *Operation-Compact Spaces*, Math. Japon., **24** (1979), 97–105
[15] Levine, N., *Semi-open sets and semi continuity in topological spaces*, Amer. Math. Monthly, **70** (1963), 36–41
[16] Levine, N., *Generalized closed sets in topological spaces*, Rend. Ser. Math. Palerno., (2) **19** (1970), 89–96
[17] Ogata, H., *Operations on Topological Spaces and Associated Topology*, Math. Japon., **36** (1991), No. 1, 175–184
[18] Umehara, J., Maki, H. and Noir, T., *Bi operations on topological spaces and some separation axioms*, Mem Fac. Sci. Kochi University Ser. A. Math., **13** (1992), 45–59
[19] Umehara, J., Maki, H. and Noir, T., *A certain bi operations on topological spaces*, Mem Fac. Sci. Kochi University Ser. A. Math., **15** (1994), 41–49

Department of Mathematics
College of Science
Qassim University
P. O. Box 6644, Buraydah 51452, Saudi Arabia
E-mail address: sabiriub@yahoo.com; sh.hussain@qu.edu.sa