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Note on a Schurer-Stancu-type operator

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ABSTRACT. The aim of this paper is to introduce a class of operators of Schurer-Stancu-type with the property that the test functions e_0 and e_1 are reproduced. Also, in our approach, a theorem of error approximation and a Voronovskaja-type theorem for this operators are obtained. Finally, we study the convergence of the iterates for our new class of operators.

1. INTRODUCTION

Let $\mathbb{N}$ be a set of positive integers, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and α, β real numbers with $0 \leq \alpha \leq \beta$. We denote by e_j the monomial of j degree, $j \in \mathbb{N}_0$. For any $m \in \mathbb{N}$ and $p \in \mathbb{N}$ fixed, F. Schurer [11] has introduced the linear positive operator

$$(\tilde{B}_{m,p}f)(x) = \sum_{k=0}^{m+p} \binom{m+p}{k} x^k (1-x)^{m+p-k} f\left(\frac{k}{m}\right), \tag{1.1}$$

defined for any $f \in C([0, 1+p])$ and $x \in [0,1]$. The author proved that if $f \in C([0,1+p])$, then $\tilde{B}_{m,p}(f) \longrightarrow f$ uniform on $[0,1]$.

For any $m \in \mathbb{N}$ and $0 \leq \alpha \leq \beta$, Stancu [12] has introduced the linear positive operator

$$\left(P_m^{(\alpha,\beta)} f\right)(x) = \sum_{k=0}^{m} \binom{m}{k} x^k (1-x)^{m-k} f\left(\frac{k+\alpha}{m+\beta}\right), \tag{1.2}$$

defined for any $f \in C([0,1])$ and $x \in [0,1]$. The author proved that if $f \in C([0,1])$ then $P_m^{(\alpha,\beta)}(f) \longrightarrow f$ uniform on $[0,1]$.

In 2006, Dan Bărbosu [3] introduced for any $m \in \mathbb{N}, p \in \mathbb{N}$ fixed, $\alpha, \beta \in \mathbb{R}$, $0 \leq \alpha \leq \beta$ an operator of Schurer-Stancu type by the form

$$(F_{m,p}^{(\alpha,\beta)} f)(x) = \sum_{k=0}^{m+p} \binom{m+p}{k} x^k (1-x)^{m+p-k} f\left(\frac{k+\alpha}{m+\beta}\right), \tag{1.3}$$

defined for any $f \in C([0,1+p])$ and $x \in [0,1]$. The author proved that if $f \in C([0,1+p])$, then $F_{m,p}^{(\alpha,\beta)}(f) \longrightarrow f$ uniform on $[0,1]$. Note that the above operators from (1.1)-(1.3) preserve the test function e_0.

In the study of above classes a Bohman-Korovkin theorem was used. Following the ideas from [1], [4]-[8] and [10], in this paper we introduced a general class which preserves the test function e_0 and e_1. For our operators a convergence theorem and Voronovskaja-type theorem are obtained. The paper is organized as follows. In Section 2 we have recall some results obtained in [9] which are essentially for obtaining the main results of this paper. Section 3 is devoted to the construction of a class of linear and positive operators

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that preserves the test functions e_0 and e_1. In concordance with [2], in Section 4 we study the convergence of the iterates of the new class of operators.

2. Preliminaries

In this section, we recall some notions and results which we will use in what follows. We consider I, J real intervals with the property $I \cap J \neq \emptyset$, let $E(I), F(J)$ be certain subsets of a space of all real functions defined on I, respectively J. Set

$$B(I) = \{f | f : I \to \mathbb{R}, f \text{ bounded on } I\},$$

$$C(I) = \{f | f : I \to \mathbb{R}, f \text{ continuous on } I\}$$

and

$$C_B(I) = B(I) \cap C(I).$$

For $x \in I$, we consider the function $\psi_x : I \to \mathbb{R}, \psi_x(t) = t - x, t \in I$. For any $m \in \mathbb{N}_0$, we consider the functions $\varphi_{m,k} : J \to \mathbb{R}$, with the property that $\varphi_{m,k}(x) \geq 0$ for any $x \in J$, $k \in \{0, 1, 2, ..., m + p\}$ and the linear positive functionals $A_{m,k} : E(I) \to \mathbb{R}$, $k \in \{0, 1, 2, ..., m + p\}$. For $m \in \mathbb{N}$ we define the operator $L_m : E(I) \to F(J)$ by

$$(L_m f)(x) = \sum_{k=0}^{m+p} \varphi_{m,k}(x) A_{m,k}(f). \tag{2.4}$$

Remark 2.1. The operators $(L_m)_{m \in \mathbb{N}}$ are linear and positive on $E(I \cap J)$.

For any $f \in E(I), x \in I \cap J$, $m \in \mathbb{N}$ and $i \in \mathbb{N}_0$, we define $T_{m,i}$ by

$$(T_{m,i} L_m)(x) = m^i (L_m \psi_x^i)(x) = m^i \sum_{k=0}^{m+p} \varphi_{m,k}(x) A_{m,k}(\psi_x^i) \tag{2.5}$$

In the following, let s be a fixed even natural number and we suppose that the operators $L_m, m \in \mathbb{N}$ verifies the following conditions:
there exists the smallest $\alpha_s, \alpha_{s+2} \in [0, \infty)$ such that

$$\lim_{m \longrightarrow \infty} \frac{(T_{m,j} L_m)(x)}{m^{\alpha_j}} = B_j(x) \in \mathbb{R}, \tag{2.6}$$

for any $x \in I \cap J, j \in \{s, s + 2\}$ and

$$\alpha_{s+2} < \alpha_s + 2. \tag{2.7}$$

If $I \subset \mathbb{R}$ is an given interval and $f \in C_B(I)$, then, the first order modulus of smoothness of f is the function $\omega(f; \cdot) : [0, +\infty) \to \mathbb{R}$ defined for any $\delta \geq 0$ by $\omega(f, \delta) = sup\{|f(x') - f(x'')| : x', x'' \in I, |x' - x''| \leq \delta\}$.

Theorem 2.1. [9] *Let $f : I \longrightarrow \mathbb{R}$ be a function. If $x \in I \cap J$ and f is s times derivable function on I, the function $f^{(s)}$ is continuous in x, then*

$$\lim_{m \longrightarrow \infty} m^{s-\alpha_s} \Big((L_m f)(x) - \sum_{i=0}^{s} \frac{f^{(i)}(x)}{m^i i!} (T_{m,i} L_m)(x) \Big) = 0. \tag{2.8}$$

If f is a s times differentiable function on I, the function $f^{(s)}$ is continuous on I and there exists $m(s) \in \mathbb{N}$ and $k_j \in \mathbb{R}$ such that for any natural number $m \geq m(s)$ and for any $x \in I \cap J$ we have

$$\frac{(T_{m,j} L_m)(x)}{m^{\alpha_j}} \leq k_j, \tag{2.9}$$

where $j \in \{s, s+2\}$, then the convergence given in (2.8) is uniformly on $I \cap J$ and

$$m^{s-\alpha_s}\left|(L_m f)(x) - \sum_{i=0}^{s}\frac{f^{(i)}(x)}{m^i i!}(T_{m,i}L_m)(x)\right| \le \tag{2.10}$$

$$\le \frac{1}{s!}(k_s + k_{s+2})\omega\left(f^{(s)}; \frac{1}{\sqrt{m^{2+\alpha_s-\alpha_{s+2}}}}\right),$$

for any $x \in I \cap J$ and $m \ge m(s)$.

Now, let α, β be fixed real numbers and $p \in \mathbb{N}$ fixed with the property that $0 < \alpha+p < \beta$. We observe that if $m, m_0 \in \mathbb{N}, m \ge m_0$, then

$$\left[\frac{\alpha}{m_0+\beta}, \frac{m_0+p+\alpha}{m_0+\beta}\right] \subset \left[\frac{\alpha}{m+\beta}, \frac{m+p+\alpha}{m+\beta}\right]. \tag{2.11}$$

Moreover, for $\alpha = \beta = p = 0$ the interval $\left[\frac{\alpha}{m+\beta}; \frac{m+p+\alpha}{m+\beta}\right]$ becomes [0,1].

3. A NEW CLASS OF OPERATORS

Let $p \in \mathbb{N}$ fixed, $0 < \alpha + p < \beta$ and the functions $a_m, b_m : J \longrightarrow \mathbb{R}$ such that $a_m(x) \ge 0, b_m(x) \ge 0$ for any $x \in J, m \in \mathbb{N}$ and $I = [0, 1+p]$. We define the operator of the following form

$$\left(S_{m,p}^{(\alpha,\beta)} f\right)(x) = \sum_{k=0}^{m+p}\binom{m+p}{k}a_m^k(x)b_m^{m+p-k}(x)f\left(\frac{k+\alpha}{m+\beta}\right), \tag{3.12}$$

for any $m \in \mathbb{N}, x \in J$ and $f \in E([0, 1+p])$, here $E([0, 1+p])$ is the linear space of all real valued functions defined on $[0, 1+p]$. In what follows, we impose the additional condition to be fulfilled by our operators

$$\left(S_{m,p}^{(\alpha,\beta)} e_0\right)(x) = 1, m \in \mathbb{N}, x \in J. \tag{3.13}$$

We get

$$(a_m(x) + b_m(x))^{m+p} = 1, m \in \mathbb{N}, x \in J, \tag{3.14}$$

from where

$$a_m(x) + b_m(x) = 1, m \in \mathbb{N}, x \in J. \tag{3.15}$$

The second condition will be read as follows

$$\left(S_{m,p}^{(\alpha,\beta)} e_1\right)(x) = x, m \in \mathbb{N}, x \in J. \tag{3.16}$$

We get

$$\frac{a_m(x)(m+p)}{m+\beta}(a_m(x) + b_m(x))^{m+p-1} + \frac{\alpha}{m+\beta}(a_m(x) + b_m(x))^{m+p} = x, \tag{3.17}$$

$m \in \mathbb{N}, x \in J$. From (3.14), (3.15) and (3.17) it follows

$$a_m(x) = \frac{(m+\beta)x - \alpha}{m+p} \tag{3.18}$$

and

$$b_m(x) = \frac{m+p-(m+\beta)x+\alpha}{m+p} \tag{3.19}$$

for any $m \in \mathbb{N}$ and $x \in J$. We fix $m_0 \in \mathbb{N}$ and let $\mathbb{N}_1 = \{m \in \mathbb{N}_0 | m \ge m_0\}$.

Lemma 3.1. *Let $m \in \mathbb{N}_1$, then the following relations are equivalent*
(i) $a_m(x) \geq 0$ and $b_m(x) \geq 0$;
(ii) $x \in J$ and $J = \left[\frac{\alpha}{m_0+\beta}, \frac{m_0+p+\alpha}{m_0+\beta}\right]$.

Proof. Taking into account (2.11), after some calculus, it follows immediately. □

Now, considering the relations (3.18) and (3.19), the operator (3.12) becomes

$$\left(S_{m,p}^{(\alpha,\beta)} f\right)(x) = \frac{1}{(m+p)^{m+p}} \sum_{k=0}^{m+p} \binom{m+p}{k} \left((m+\beta)x - \alpha\right)^k \cdot \tag{3.20}$$

$$\cdot (m+p-(m+\beta)x+\alpha)^{m+p-k} \cdot f\left(\frac{k+\alpha}{m+\beta}\right)$$

for any $m \in \mathbb{N}_1, x \in J$ and $f \in E([0, 1+p])$.
From (3.20) we have

$$\left(S_{m,p}^{(\alpha,\beta)} e_2\right)(x) = \frac{m+p-1}{m+p} x^2 + \left(\frac{2(m+p-1)\alpha}{(m+\beta)(m+p)} + \frac{1}{m+\beta} + \frac{2\alpha}{m+\beta}\right) x + \tag{3.21}$$

$$+ \left(\frac{m+p-1}{(m+\beta)^2(m+p)} - \frac{1}{(m+\beta)^2}\right)\alpha^2 - \frac{1}{(m+\beta)^2}\alpha,$$

for any $m \in \mathbb{N}_1, x \in J, f \in E([0, 1+p])$. Coming back to Theorem 2.1 and Remark 2.1 for our operator (3.12) we have $I = [0, 1+p]$, $E([0, 1+p]) = C([0, 1+p])$,

$$\varphi_{m,k} = \frac{1}{(m+p)^{m+p}} \binom{m+p}{k} \left((m+\beta)x - \alpha\right)^k \cdot \tag{3.22}$$

$$\cdot \left(m+p-(m+\beta)x+\alpha\right)^{m+p-k}$$

and

$$A_{m,k}(f) = f\left(\frac{k+\alpha}{m+\beta}\right), \tag{3.23}$$

for any $m \in \mathbb{N}_1, x \in \left[\frac{\alpha}{m_0+\beta}, \frac{m_0+p+\alpha}{m_0+\beta}\right]$ and $f \in C([0, 1+p])$.

Remark 3.2. If $x \in \left[\frac{\alpha}{m_0+\beta}, \frac{m_0+p+\alpha}{m_0+\beta}\right]$ and $m \in \mathbb{N}_1$, then the operators $S_{m,p}^{(\alpha,\beta)}$ are linear and positive for $f \in C([0, 1+p])$.

Lemma 3.2. *For $m \in \mathbb{N}_1$ and $x \in \left[\frac{\alpha}{m_0+\beta}, \frac{m_0+p+\alpha}{m_0+\beta}\right]$, the following identities*

$$\left(T_{m,0} S_{m,p}^{(\alpha,\beta)}\right)(x) = 1, \tag{3.24}$$

$$\left(T_{m,1} S_{m,p}^{(\alpha,\beta)}\right)(x) = 0 \tag{3.25}$$

and

$$\left(T_{m,2} S_{m,p}^{(\alpha,\beta)}\right)(x) = m^2 \left(\left(S_{m,p}^{(\alpha,\beta)} e_2\right)(x) - x^2\right) \tag{3.26}$$

hold.

Proof. We take (2.5), (3.13) and (3.16) into account. □

In concordance with Theorem 2.1, from (3.24)-(3.26) we obtain $k_0 = 1$, $m(0) = 1$, $k_2 = \frac{16\alpha^2+8\alpha+5}{4}$, $\alpha_0 = 0$ and $\alpha_2 = 1$.
The main results are obtained from the next theorems

Theorem 3.2. *If $f : [0, 1 + p] \longrightarrow \mathbb{R}$ is a continuous function on $[0, 1]$, then we have*

$$\lim_{m \longrightarrow \infty} S_{m,p}^{(\alpha,\beta)} = f \tag{3.27}$$

uniformly on $J = \left[\frac{\alpha}{m_0+\beta}, \frac{m_0+p+\alpha}{m_0+\beta}\right]$.

There exists $m^* = max(m_0, m(0))$ *such that*

$$\left|\left(S_{m,p}^{(\alpha,\beta)} f\right)(x) - f(x)\right| \leq \frac{9 + 16\alpha^2 + 8\alpha}{4} \cdot \omega\left(f, \frac{1}{\sqrt{m}}\right), \tag{3.28}$$

for any $x \in \left[\frac{\alpha}{m_0+\beta}, \frac{m_0+p+\alpha}{m_0+\beta}\right], m \in \mathbb{N}, m \geq m^*$.

Proof. We used (3.24) and Theorem 2.1 to obtain (3.27). From the relation (3.26) we obtain that

$$\frac{(T_{m,2} S_{m,p}^{\alpha,\beta})(x)}{m} \leq x(4\alpha + 1 - x) + 1 \leq \frac{16\alpha^2 + 8\alpha + 5}{4}. \tag{3.29}$$

Taking into account (3.24) and (3.29) we obtain (3.28). □

Theorem 3.3. *If $f : [0, 1 + p] \longrightarrow \mathbb{R}$ is a continuous function on $[0, 1]$ and is two times differentiable on $[0, 1]$ having the second-order derivative continuous on $[0, 1]$, then we have*

$$\lim_{m \longrightarrow \infty} m\left(\left(S_{m,p}^{(\alpha,\beta)} f\right)(x) - f(x)\right) = \frac{f^{(2)}(x)}{2} x(4\alpha + 1 - x). \tag{3.30}$$

Proof. Using that

$$\lim_{m \longrightarrow \infty} \frac{T_{m,2} S_{m,p}^{\alpha,\beta}}{m} = x(4\alpha + 1 - x)$$

and taking into account Theorem 2.1 and Lemma 3.2, we obtain the relation (3.30). □

Theorem 3.3 is a Voronovskaja-type theorem.

4. Application

Following [2], for our new operator, we obtain

Theorem 4.4. *Let $I = [c, d]$, $f : I \to \mathbb{R}$, $S_{m,p}^{(\alpha,\beta)}, m \in \mathbb{N}_1$ be defined by (3.8) such that $\varphi_{m,0}(c) = \varphi_{m,m+p}(d) = 1$. If $f \in C([c, d])$, then the iterates sequence $\left(\left(S_{m,p}^{(\alpha,\beta)}\right)^n\right)_{n\geq 1}$ verifies*

$$\lim_{n \longrightarrow \infty} \left(\left(S_{m,p}^{(\alpha,\beta)}\right)^n f\right)(x) = f(c) + \frac{f(d) - f(c)}{d - c}(x - c), \tag{4.31}$$

uniformly on $[c, d]$, where $c = \frac{\alpha}{m+\beta}$, $d = \frac{m+p+\alpha}{m+\beta}$.

Proof. At first we define

$$X_{\gamma,\delta} := \{f \in C([c, d]) | f(c) = \gamma, f(d) = \delta\}, (\gamma, \delta) \in \mathbb{R} \times \mathbb{R}.$$

Clearly, every $X_{\gamma,\delta}$ is a closed subset of $C([c, d])$ and the system $X_{\gamma,\delta}, (\gamma, \delta) \in \mathbb{R}\times\mathbb{R}$, makes up a partition of this space. Since $\varphi_{m,0}(c) = \varphi_{m,m+p}(d) = 1$, the relation (3.13) implies $\left(S_{m,p}^{(\alpha,\beta)} f\right)(c) = f(c)$ and $\left(S_{m,p}^{(\alpha,\beta)} f\right)(d) = f(d)$, in other words for all $(\gamma, \delta) \in \mathbb{R} \times \mathbb{R}$ and $m \in \mathbb{N}_1$, $X_{\gamma,\delta}$ is an invariant subset of $S_{m,p}^{(\alpha,\beta)}$.

Further on, we prove that $S_{m,p}^{(\alpha,\beta)}|_{X_{\gamma,\delta}} : X_{\gamma,\delta} \longrightarrow X_{\gamma,\delta}$ is a contraction for every $(\gamma,\delta) \in \mathbb{R}\times\mathbb{R}$ and $m \in \mathbb{N}_1$. Let us denote $u_m := \min_{x\in[c,d]}(\varphi_{m,0}(x)+\varphi_{m,m+p}(x))$. If f and g belong to $X_{\gamma,\delta}$ then, for every $x \in [c,d]$, we can write

$$\begin{aligned}\left|\left(S_{m,p}^{(\alpha,\beta)}f\right)(x)-\left(S_{m,p}^{(\alpha,\beta)}g\right)(x)\right| &= \left|\sum_{k=1}^{m+p-1}\varphi_{m,k}(x)(f-g)(x_{m,k})\right| \\ &\leq \sum_{k=1}^{m+p-1}\varphi_{m,k}(x)\|f-g\|_\infty \\ &= (1-\varphi_{m,0}(x)-\varphi_{m,m+p}(x))\|f-g\|_\infty \\ &\leq (1-u_m)\|f-g\|_\infty,\end{aligned}$$

where $x_{m,k}=\frac{k+\alpha}{m+\beta}$, $k \in \{0,...,m+p\}$.

Consequently we obtain $\left\|\left(S_{m,p}^{(\alpha,\beta)}f\right)(x)-\left(S_{m,p}^{(\alpha,\beta)}g\right)(x)\right\|_\infty \leq (1-u_m)\|f-g\|_\infty$. In our case, from Lemma 3.1 we have $u_m > 0$, which guarantees our statement. On the other hand, the function $p^*_{\gamma,\delta} := \gamma + ((\delta-\gamma)/(d-c))(e_1-c)$ belongs to $X_{\gamma,\delta}$ and since $S_{m,p}^{(\alpha,\beta)}$ reproduces the affine functions $p^*_{\gamma,\delta}$ is a fixed point of $S_{m,p}^{(\alpha,\beta)}$.

For any $f \in C([c,d])$ one has $f \in X_{f(c),f(d)}$ and, by using the contraction principle, we get $\lim_{n\longrightarrow\infty}\left(S_{m,p}^{(\alpha,\beta)}\right)^n f = p^*_{f(c),f(d)}$. We obtained the desired result in (4.31). □

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