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Coefficient inequality for transforms of starlike and convex functions

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ABSTRACT. The objective of this paper is to obtain an upper bound for the second Hankel functional associated with the k^{th} root transform $\left[f(z^k)\right]^{\frac{1}{k}}$ of normalized analytic function $f(z)$ belonging to starlike and convex functions, defined on the open unit disc in the complex plane, using Toeplitz determinants.

1. INTRODUCTION

Let A denote the class of all functions $f(z)$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

in the open unit disc $E = \{z : |z| < 1\}$. Let S be the subclass of A consisting of univalent functions. For a univalent function in the class A, it is well known that the n^{th} coefficient is bounded by n. The bounds for the coefficients give information about the geometric properties of these functions. For example, the bound for the second coefficient of normalized univalent function readily yields the growth and distortion properties of univalent functions. The Hankel determinant of f for $q \geq 1$ and $n \geq 1$ was defined by Pommerenke [9] as

$$H_q(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2q-2} \end{vmatrix}, (a_1 = 1).$$

This determinant has been considered by many authors in the literature. In particular cases, $q = 2, n = 1, a_1 = 1$ and $q = 2, n = 2,\ a_1 = 1$, the Hankel determinant simplifies respectively to

$$H_2(1) = \begin{vmatrix} a_1 & a_2 \\ a_2 & a_3 \end{vmatrix} = a_3 - a_2^2,$$

$$\text{and } H_2(2) = \begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix} = a_2 a_4 - a_3^2.$$

We refer to $H_2(2)$ as the second Hankel determinant. It is well known that for the univalent function of the form (1.1) the sharp inequality $H_2(1) = |a_3 - a_2^2| \leq 1$ holds true [4]. For a family $\mathcal{T}$ of functions in S, the more general problem of finding sharp estimates for the functional $|a_3 - \mu a_2^2| (\mu \in \mathbb{R}$ or $\mu \in \mathbb{C})$ is popularly known as the Fekete-Szegö

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problem for $\mathcal{T}$. Ali [3] found sharp bounds to the first four coefficients and sharp estimate for the Fekete-Szegö functional $|\gamma_3 - t\gamma_2^2|$, where t is real for the inverse function of f defined as $f^{-1}(w) = w + \sum_{n=2}^{\infty} \gamma_n w^n$ when it belongs to the class of strongly starlike functions of order α $(0 < \alpha \leq 1)$ denoted by $\widetilde{ST}(\alpha)$. Janteng, Halim and Darus [6] obtained the second Hankel determinant and sharp upper bounds for the familiar subclasses of S, namely, starlike and convex functions denoted by ST and CV and have shown that $|a_2a_4 - a_3^2| \leq 1$ and $|a_2a_4 - a_3^2| \leq \frac{1}{8}$ respectively. Recently, several authors have defined certain subclasses of univalent, multivalent analytic functions and obtained sharp upper bounds to the second Hankel determinant for the functions in these classes. R. M. Ali, S. K. Lee, V. Ravichandran and S. Supramaniam [2] obtained sharp bound for the Fekete-Szegö functional denoted by $|b_{2k+1} - \mu b_{k+1}^2|$ associated with the k^{th} root transform $\left[f(z^k)\right]^{\frac{1}{k}}$ of the function given in (1.1), belonging to certain subclasses of S. The k^{th} root transform for the function f given in (1.1) is defined as

$$F(z) := \left[f(z^k)\right]^{\frac{1}{k}} = z + \sum_{n=1}^{\infty} b_{kn+1} z^{kn+1}. \tag{1.2}$$

In this paper, we obtain an upper bound to the functional $|b_{k+1}b_{3k+1} - b_{2k+1}^2|$, called the second Hankel determinant for the k^{th} root transform for the function f when it belongs to the familiar subclasses ST and CV of S, defined as follows.

Definition 1.1. A function $f(z) \in A$ is said to be in ST, if it satisfies the condition

$$Re\left\{\frac{zf'(z)}{f(z)}\right\} > 0, \ \forall z \in E. \tag{1.3}$$

Definition 1.2. A function $f(z) \in A$ is said to be in CV, if and only if

$$Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > 0, \ \forall z \in E. \tag{1.4}$$

From the Definitions 1.1 and 1.2, we observe that, there exists an Alexander type Theorem [1], which relates the classes ST and CV, stated as follows.

$$f \in CV \Leftrightarrow zf' \in ST.$$

Some preliminary Lemmas required for proving our results are as follows:

2. Preliminary results

Let $\mathbb{P}$ denote the class of functions consisting of p, such that

$$p(z) = 1 + c_1z + c_2z^2 + c_3z^3 + ... = 1 + \sum_{n=1}^{\infty} c_n z^n, \tag{2.5}$$

which are regular in the open unit disc E and satisfy Re$p(z) > 0$, for any $z \in E$. Here $p(z)$ is called the Carathéodory function [4].

Lemma 2.1. ([8, 10]) *If $p \in \mathbb{P}$, then $|c_k| \leq 2$, for each $k \geq 1$ and the inequality is sharp for the function $\frac{1+z}{1-z}$.*

Lemma 2.2. ([5]) *The power series for $p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n$ given in (2.5) converges in the open unit disc E to a function in $\mathbb{P}$ if and only if the Toeplitz determinants*

$$D_n = \begin{vmatrix} 2 & c_1 & c_2 & \cdots & c_n \\ c_{-1} & 2 & c_1 & \cdots & c_{n-1} \\ c_{-2} & c_{-1} & 2 & \cdots & c_{n-2} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ c_{-n} & c_{-n+1} & c_{-n+2} & \cdots & 2 \end{vmatrix}, n = 1, 2, 3....$$

and $c_{-k} = \overline{c}_k$, are all non-negative. They are strictly positive except for $p(z) = \sum_{k=1}^{m} \rho_k p_0(e^{it_k} z)$, $\rho_k > 0$, t_k real and $t_k \neq t_j$, for $k \neq j$, where $p_0(z) = \frac{1+z}{1-z}$; in this case $D_n > 0$ for $n < (m-1)$ and $D_n \doteq 0$ for $n \geq m$.

This necessary and sufficient condition found in [5] is due to Carathéodory and Toeplitz. We may assume without restriction that $c_1 > 0$. On using Lemma 2.2, for $n = 2$, we have

$$D_2 = \begin{vmatrix} 2 & c_1 & c_2 \\ \overline{c}_1 & 2 & c_1 \\ \overline{c}_2 & \overline{c}_1 & 2 \end{vmatrix} = [8 + 2Re\{c_1^2 c_2\} - 2 \mid c_2 \mid^2 - 4|c_1|^2] \geq 0,$$

which is equivalent to

$$2c_2 = \{c_1^2 + x(4 - c_1^2)\}, \text{ for some } x, |x| \leq 1. \tag{2.6}$$

For $n = 3$,

$$D_3 = \begin{vmatrix} 2 & c_1 & c_2 & c_3 \\ \overline{c}_1 & 2 & c_1 & c_2 \\ \overline{c}_2 & \overline{c}_1 & 2 & c_1 \\ \overline{c}_3 & \overline{c}_2 & \overline{c}_1 & 2 \end{vmatrix} \geq 0$$

and is equivalent to

$$|(4c_3 - 4c_1c_2 + c_1^3)(4 - c_1^2) + c_1(2c_2 - c_1^2)^2| \leq 2(4 - c_1^2)^2 - 2|(2c_2 - c_1^2)|^2. \tag{2.7}$$

Simplifying the relations (2.6) and (2.7), we get

$$4c_3 = \{c_1^3 + 2c_1(4 - c_1^2)x - c_1(4 - c_1^2)x^2 + 2(4 - c_1^2)(1 - |x|^2)z\}, \text{ with } |z| \leq 1. \tag{2.8}$$

In obtaining our results, we refer to the classical method devised by Libera and Zlotkiewicz [7].

3. Main results

Theorem 3.1. *If f given by (1.1) belongs to ST and F is the k^{th} root transformation of f given by (1.2) then*

$$|b_{k+1} b_{3k+1} - b_{2k+1}^2| \leq \frac{1}{k^2}$$

and the inequality is sharp.

Proof. For $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in ST$, there exists an analytic function $p \in \mathbb{P}$ in the open unit disc E with $p(0) = 1$ and $Re[p(z)] > 0$ such that

$$\frac{zf'(z)}{f(z)} = p(z) \Leftrightarrow zf'(z) = p(z)f(z). \tag{3.9}$$

Using the series representations for $f(z), f'(z)$ and $p(z)$ in (3.9), we have

$$z\left\{1+\sum_{n=2}^{\infty} na_n z^{n-1}\right\} = \left\{1+\sum_{n=1}^{\infty} c_n z^n\right\}\left\{z+\sum_{n=2}^{\infty} a_n z^n\right\}. \tag{3.10}$$

Upon Simplification, on equating the coefficients in (3.10) yields,

$$a_2 = c_1;\ a_3 = \frac{1}{2}(c_1^2+c_2);\ a_4 = \frac{1}{3}(c_3+\frac{3}{2}c_1c_2+\frac{c_1^3}{2}). \tag{3.11}$$

For a function f given by (1.1), a computation shows that

$$\begin{aligned}\left[f(z^k)\right]^{\frac{1}{k}} &= \left[z^k+\sum_{n=2}^{\infty} a_n z^{nk}\right]^{\frac{1}{k}} \\ &= \Big[z+\frac{1}{k}a_2 z^{k+1}+\Big\{\frac{1}{k}a_3+\frac{(1-k)}{2k^2}a_2^2\Big\}z^{2k+1} \\ &+\Big\{\frac{1}{k}a_4+\frac{(1-k)}{k^2}a_2a_3+\frac{(1-k)(1-2k)}{6k^3}a_2^3\Big\}z^{3k+1}\ +\ \cdots\Big].\end{aligned} \tag{3.12}$$

Simplifying the expressions (1.2) and (3.12) along with (3.11), we get

$$\begin{aligned} b_{k+1} &= \frac{c_1}{k}\ ;\ b_{2k+1} = \frac{1}{2k^2}(kc_2+c_1^2)\ ; \\ b_{3k+1} &= \frac{1}{6k^3}(2k^2c_3+3kc_1c_2+c_1^3). \end{aligned} \tag{3.13}$$

Substituting the values of b_{k+1}, b_{2k+1} and b_{3k+1} from (3.13) in the second Hankel determinant $|\ b_{k+1}b_{3k+1}-b_{2k+1}^2\ |$ to the k^{th} transformation for the function $f\in ST$, which simplifies to

$$|b_{k+1}b_{3k+1}-b_{2k+1}^2| = \frac{1}{12k^4}|4k^2c_1c_3-3k^2c_2^2-c_1^4|. \tag{3.14}$$

From Lemma 2.2, substituting the values of c_2 and c_3 from (2.6) and (2.8) respectively, on the right-hand side of the expression (3.14), we have

$$\begin{aligned}|b_{k+1}b_{3k+1}-b_{2k+1}^2| = \Big|\ &4k^2c_1\times\frac{1}{4}\big\{c_1^3+2c_1(4-c_1^2)x-c_1(4-c_1^2)x^2 \\ &+2(4-c_1^2)(1-|x|^2)z\big\}-3k^2\times\frac{1}{2}\big\{c_1^2+x(4-c_1^2)\big\}-c_1^4\Big|.\end{aligned}$$

Using triangle inequality and the fact that $|z|<1$, we get

$$\begin{aligned}|b_{k+1}b_{3k+1}-b_{2k+1}^2| \le \big|&(k^2-4)c_1^4+8k^2c_1(4-c_1^2)+2k^2c_1^2(4-c_1^2)|x| \\ &-(c_1+2)(c_1+6)k^2(4-c_1^2)|x|^2\big|.\end{aligned} \tag{3.15}$$

Since $c_1=c\in[0,2]$, noting that $(c_1+a)(c_1+b)\ge(c_1-a)(c_1-b)$, where $a,b\ge 0$, applying triangle inequality and replacing $|x|$ by μ on the right-hand side of (3.15), we obtain

$$\begin{aligned}4|b_{k+1}b_{3k+1}-b_{2k+1}^2| &\le \big[\ \big(4-k^2\big)c^4+8k^2c(4-c^2)+2k^2c^2(4-c^2)\mu \\ &\quad +(c-2)(c-6)k^2(4-c^2)\mu^2\ \big] \\ &= F(c,\mu),\ \text{for } 0\le\mu=|x|\le 1,\end{aligned} \tag{3.16}$$

$$\begin{aligned}\text{where } F(c,\mu) = &\big(4-k^2\big)c^4+8k^2c(4-c^2)+2k^2c^2(4-c^2)\mu \\ &+(c-2)(c-6)k^2(4-c^2)\mu^2.\end{aligned} \tag{3.17}$$

We next maximize the function $F(c, \mu)$ on the closed region $[0, 2] \times [0, 1]$. Differentiating $F(c, \mu)$ given in (3.17) partially with respect to μ, we get

$$\frac{\partial F}{\partial \mu} = 2k^2\{c^2 + (c-2)(c-6)\mu\}(4-c^2). \tag{3.18}$$

For $0 < \mu < 1$, for fixed c with $0 < c < 2$, from (3.18), we observe that $\frac{\partial F}{\partial \mu} > 0$. Therefore, $F(c, \mu)$ is an increasing function of μ and hence it cannot have a maximum value at any point in the interior of the closed region $[0, 2] \times [0, 1]$. Further, for fixed $c \in [0, 2]$, we have

$$\max_{0 \le \mu \le 1} F(c, \mu) = F(c, 1) = G(c). \tag{3.19}$$

Simplifying the relations (3.17) and (3.19), we get

$$G(c) = -4(k^2 - 1)c^4 + 48k^2, \tag{3.20}$$

$$G'(c) = -16(k^2 - 1)c^3. \tag{3.21}$$

From the expression (3.21), we observe that $G'(c) \le 0$ for every $c \in [0, 2]$ and for all values of k. Therefore, $G(c)$ is a decreasing function of c in the interval $[0, 2]$, whose maximum value occurs at $c = 0$ only and from (3.20), it is given by

$$\max_{0 \le c \le 2} G(0) = 48k^2. \tag{3.22}$$

Simplifying the expressions (3.16) and (3.22) along with (3.14), we obtain

$$|b_{k+1}b_{3k+1} - b_{2k+1}^2| \le \frac{1}{k^2}. \tag{3.23}$$

Choosing $c_1 = c = 0$ and selecting $x = 1$ in (2.6) and (2.8), we find that $c_2 = 2$ and $c_3 = 0$. Using these values in (3.14), we observe that equality is attained which shows that our result is sharp and for these values, we derive the extremal function, given by

$$\frac{zf'(z)}{f(z)} = \frac{1 + z^2}{1 - z^2} = 1 + 2z^2 + 2z^4 + ... \tag{3.24}$$

This completes the proof of our Theorem 3.1. □

Remark 3.1. In particular, for $k = 1$ in (3.23), the result coincides with that of Janteng, Halim and Darus [6].

Theorem 3.2. *If f given by (1.1) belongs to CV and F is the k^{th} root transformation of f given by (1.2) then*

$$| b_{k+1}b_{3k+1} - b_{2k+1}^2 | \le \frac{(2k^2 + 1)}{3k^2(5k^2 + 3)}.$$

Proof. For $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in CV$, from the Definition 1.2, there exists an analytic function $p \in \mathbb{P}$ in the unit disc E with $p(0) = 1$ and $Rep(z) > 0$ such that

$$\left\{1 + \frac{zf''(z)}{f'(z)}\right\} = p(z) \Leftrightarrow f'(z) + zf''(z) = p(z)f'(z). \tag{3.25}$$

Replacing $f'(z)$, $f''(z)$ and $p(z)$ with their equivalent expressions in (3.25), we have

$$\left\{1 + \sum_{n=2}^{\infty} n a_n z^{n-1}\right\} + z\left\{\sum_{n=2}^{\infty} n(n-1)a_n z^{n-2}\right\}$$
$$= \left\{1 + \sum_{n=1}^{\infty} c_n z^n\right\}\left\{z + \sum_{n=2}^{\infty} n a_n z^{n-1}\right\}. \tag{3.26}$$

Simplifying and equating the coefficients in (3.26) yields,

$$a_2=\frac{c_1}{2};\ a_3=\frac{1}{6}(c_1^2+c_2);\ a_4=\frac{1}{12}(c_3+\frac{3}{2}c_1c_2+\frac{c_1^3}{2}). \tag{3.27}$$

Applying the same procedure as described in Theorem 3.1, a computation shows that

$$b_{k+1}=\frac{c_1}{2k}\ ;\ b_{2k+1}=\frac{1}{24k^2}[4kc_2+(k+3)c_1^2]\ ;$$
$$b_{3k+1}=\frac{1}{48k^3}[4k^2c_3+2k(k+2)c_1c_2+(1+k)c_1^3]. \tag{3.28}$$

Further, we have

$$|b_{k+1}b_{3k+1}-b_{2k+1}^2|=\frac{1}{576k^4}|24k^2c_1c_3+4k^2c_1^2c_2-16k^2c_2^2-(k^2+3)c_1^4|. \tag{3.29}$$

From Lemma 2.2, substituting the values of c_2 and c_3 from (2.6) and (2.8) respectively on the right-hand side of (3.29) and applying the same procedure as described in Theorem 3.1, we obtain

$$|24k^2c_1c_3+4k^2c_1^2c_2-16k^2c_2^2-(k^2+3)c_1^4|\leq\Big|3(k^2-1)c_1^4+12k^2c_1(4-c_1^2)+$$
$$6k^2c_1^2(4-c_1^2)|x|-2k^2(c_1+2)(c_1+4)(4-c_1^2)|x|^2\Big|. \tag{3.30}$$

Since $c_1=c\in[0,2]$, applying the same procedure as described in Theorem 3.1, we get

$$|24k^2c_1c_3+4k^2c_1^2c_2-16k^2c_2^2-(k^2+3)c_1^4|\leq\big[\ 3(k^2-1)c^4+12k^2c(4-c^2)+$$
$$6k^2c^2(4-c^2)\mu+2k^2(c-2)(c-4)(4-c^2)\mu^2\ \big] \tag{3.31}$$
$$=F(c,\mu),\ \text{for } 0\leq\mu=|x|\leq 1.$$

Using the same procedure as described in Theorem 3.1, we observe that $\frac{\partial F}{\partial\mu}>0$ and further we have

$$G(c)=-(5k^2+3)c^4+16k^2c^2+64k^2, \tag{3.32}$$
$$G'(c)=-4(5k^2+3)c^3+32k^2c, \tag{3.33}$$
$$G''(c)=-12(5k^2+3)c^2+32k^2. \tag{3.34}$$

For optimum value of c, consider $G'(c)=0$. From (3.33), we get

$$-4\left[(5k^2+3)c^2-8k^2\right]c=0. \tag{3.35}$$

Let us have the following cases.

Case1: For $c=0$, from (3.34), we get $G''(0)=32k^2>0$, for each value of k. Therefore, by the second derivative test $G(c)$ has minimum value at $c=0$.

Case2: For $c\neq 0$ and for every value of k, from (3.35), we obtain

$$c^2=\frac{8k^2}{5k^2+3}\in[0,2]. \tag{3.36}$$

Substituting the value of c^2 in (3.34), it simplifies to $G''(c)=-64k^2<0,$ for each k, which implies that $G(c)$ has maximum value at c, where c^2 is given by (3.36). Using the value of c^2 in (3.32), the maximum value of $G(c)$ is obtained to be

$$G_{max}=G(c^2)=\frac{192k^2(2k^2+1)}{(5k^2+3)}. \tag{3.37}$$

Simplifying the expressions (3.31) and (3.37) along with (3.29), we obtain

$$|b_{k+1}b_{3k+1}-b_{2k+1}^2|\leq\frac{(2k^2+1)}{3k^2(5k^2+3)}. \tag{3.38}$$

This completes the proof of our Theorem 3.2.

Remark 3.2. Choosing $k = 1$ in (3.38), the result coincides with that of Janteng, Halim and Darus [6] and the inequality is sharp.

□

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