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Common fixed point and best approximation results for subcompatible mappings in hyperbolic ordered metric spaces

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ABSTRACT. In the present paper we establish a common fixed point theorem and apply it to find new best approximation results for ordered subcompatible mappings in the hyperbolic ordered metric space. Our results unify, generalize and complement various known results.

1. INTRODUCTION

The Banach fixed point theorem for contraction mappings has been generalized and extended in many directions. Existence of fixed points in ordered metric spaces was first investigated in 2004 by Ran and Reurings [15] and then by Nieto and Lopez [12]. In 2009, Doric [5] proved some fixed point theorems for generalized (ψ, φ)-weakly contractive mappings in ordered metric spaces. Recently, Radenović and Kadelburg [14] presented a result for generalized weak contractive mappings in ordered metric spaces (see also [17, 18] and references mentioned therein). Recently, Khamsi and Khan [9] studied some inequalities in hyperbolic metric spaces which lay foundation for a new mathematical field: the application of geometric theory of Banach spaces to fixed point theory.

The aim of this article is to study common fixed points of subcompatible mappings in the frame work of hyperbolic ordered metric spaces. Some results on best approximation for these mappings are also established which in turn extend and strengthen various known results.

2. PRELIMINARIES

Let (X, d) be a metric space. A path joining x $\in$ X to y $\in$ X is a map c from a closed interval $[0, l] \subset R$ to X such that $c(0) = x, c(l) = y$ and $d(c(s), c(t)) = |s-t|$ for all $s, t \in [0, l]$. In particular, c is an isometry and $d(x, y) = l$. The image of c is called a metric segment joining x and y. When it is unique, the metric segment is denoted by $[x, y]$. We shall denote by $(1-\lambda)x \oplus \lambda y$, the unique point z of $[x, y]$ which satisfies $d(x, z) = \lambda d(x, y)$ and $d(z, y) = (1-\lambda)d(x, y)$.

Such metric spaces are usually called convex metric spaces (see Takahashi [19] and Khan at el. [10]). Moreover, if we have for all $p, x, y \in X$

$$d\left(\frac{1}{2}p \oplus \frac{1}{2}x, \frac{1}{2}p \oplus \frac{1}{2}y\right) \leq \frac{1}{2}d(x, y),$$

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then X is called a hyperbolic metric space. It is easy to check that in this case for all $x, y, z, w \in X$ and $\lambda \in [0, 1]$

$$d((1-\lambda)x \oplus \lambda y, (1-\lambda)z \oplus \lambda w) \leq (1-\lambda)d(x, z) + \lambda d(y, w).$$

Obviously, normed linear spaces are hyperbolic spaces [9].

Definition 2.1. A subset Y of a hyperbolic ordered metric space X is said to be an ordered convex if Y includes every metric segment joining any two of its comparable points.

Definition 2.2. A subset Y of a hyperbolic ordered metric space X is said to be an ordered q-starshaped if there exists $q \in Y$ such that Y includes every metric segment joining any of its point comparable with q.

Definition 2.3. Let X be a hyperbolic ordered metric space. Then X is said to satisfy property (I) if $(1-\lambda)x \oplus \lambda y \leq (1-\lambda)z \oplus \lambda w$ for all $x, y, z, w \in X$ with $x \leq z$ and $y \leq w$.

Definition 2.4. A self mapping f on an ordered convex subset Y of a hyperbolic ordered metric space X is said to be affine if $f((1-\lambda)x \oplus \lambda y) = (1-\lambda)fx \oplus \lambda fy$ for all comparable elements $x, y \in Y$ and $\lambda \in [0, 1]$.

Definition 2.5. Let f be a self mapping on X. A point $x \in X$ is called a fixed point of f if $f(x) = x$. We denote the set of fixed points of f by $Fix(f)$.

Definition 2.6. Let f and g be two self mappings on X. A point $x \in X$ is called a common fixed point of pair (f, g) if $x = fx = gx$. If $w = fx = gx$ for some $x \in X$, then w is called a point of coincidence of f and g.

Definition 2.7. [3] Let $(X, \leq)$ be an ordered set. A pair (f, g) on X is said to be:
(i) weakly compatible if f and g commute at their coincidence points.
(i) weakly increasing if for all $x \in X$, we have $fx \leq gfx$ and $gx \leq fgx$,
(ii) partially weakly increasing if $fx \leq gfx$, for all $x \in X$.

Remark 2.1. A pair (f, g) is weakly increasing if and only if the ordered pair (f, g) and (g, f) are partially weakly increasing.

Example 2.1. Let $X = [0, 1]$ be endowed with usual ordering. Let $f, g : X \to X$ be defined by $fx = x^2$ and $gx = x^{\frac{1}{2}}$. Then $fx = x^2 \leq x = gfx$ for all $x \in X$. Thus (f, g) is partially weakly increasing. But $gx = x^{\frac{1}{2}} \nleq x = fgx$ for $x \in (0, 1)$. So (g, f) is not partially weakly increasing.

Definition 2.8. Let $(X, \leq)$ be an ordered set. A mapping f is called weak annihilator of g if $fgx \leq x$ for all $x \in X$.

Example 2.2. Let $X = [0, 1]$ be endowed with usual ordering. Define $f, g : X \to X$ by $fx = x^2$ and $gx = x^3$. Then $fgx = x^6 \leq x$ for all $x \in X$. Thus f is a weak annihilator of g.

Definition 2.9. Let $(X, \leq)$ be an ordered set. A self mapping f on X is called dominating mapping if $x \leq fx$ for each $x \in X$.

Example 2.3. Let $X = [0, 1]$ be endowed with usual ordering. Let $f : X \to X$ be defined by $fx = x^{\frac{1}{3}}$. Then $x \leq x^{\frac{1}{3}} = fx$ for all $x \in X$. Thus f is a dominating mapping.

Example 2.4. Let $X = [0, \infty)$ be endowed with usual ordering. Define $f : X \to X$ by

$$fx = \begin{cases} x^{\frac{1}{n}} & \text{for } x \in [0, 1), \\ x^n & \text{for } x \in [1, \infty), \end{cases}$$

$n \in \mathbb{N}$. Then $x \leq fx$ for all $x \in X$. Hence f is a dominating mapping.

Definition 2.10. Let X be an ordered set and f, g be self mappings on X. Then the pair (f, g) is said to be order limit preserving if $gx_0 \leq fx_0$, for all sequences $\{x_n\}$ in X with $gx_n \leq fx_n$ and $x_n \to x_0$.

Definition 2.11. Let X be a hyperbolic ordered metric space, Y be an ordered q-starshaped subset of X, S and T be two self mappings on X and $q \in Fix(S)$. Then T is said to be:
(i) ordered S-contraction if there exists $k \in (0, 1)$ such that
$d(Tx, Ty) \leq kd(Sx, Sy)$; for $x, y \in Y$ with $x \leq y$.
(ii) ordered asymptotically S-nonexpansive if there exists a sequence $\{k_n\} \subset [1, \infty)$ with $\lim_{n\to\infty} k_n = 1$ such that $d(T^n x, T^n y) \leq k_n d(Sx, Sy)$ for each $x, y \in Y$ with $x \leq y$ and each $n \in \mathbb{N}$. If $S = I$ (identity map) then T is ordered asymptotically nonexpansive mapping. If $k_n = 1$, for all $n \in \mathbb{N}$, then T is known as ordered S-nonexpansive mapping.
(iii) ordered R-weakly commuting if there exists a real number $R > 0$ such that
$d(TSx, STx) \leq Rd(Tx, Sx)$ for all $x \in Y$.
(iv) ordered R-subweakly commuting [4] if there exists a real number $R > 0$ such that
$d(TSx, STx) \leq Rd(Sx, Y_q^{T(x)})$ where
$Y_q^{T(x)} = \{y_\lambda\colon\ y_\lambda = (1-\lambda)q \oplus \lambda Tx$ and $\lambda \in [0, 1], q \leq x$ or $x \leq q$ for all $x \in Y\}$.
(v) ordered uniformly R-subweakly commuting [4] if there exists a real number $R > 0$ such that $d(T^n Sx, T^n Sy) \leq Rd(Sx,\ Y_q^{T^n(x)})$ for all $x \in Y$.
(vi) ordered C_q- commuting [5] if $STx = TSx$ for all $x \in C_q(S, T)$, where
$C_q(S, T) = U\{C(S, T_k) : 0 \leq k \leq 1\}$ and $T_k(x) = (1-k)q \oplus kTx$.
(vii) ordered uniformly C_q-commuting, if $S\,T^n x = T^n\, Sx$ for all $x \in C_q(S, T)$ and $n \in \mathbb{N}$.
(viii) uniformly asymptotically regular on Y if, for each $\eta > 0$, there exists $\mathbb{N}(\eta) = \mathbb{N}$ such that $d(T^n x, T^{n+1} x) < \eta$ for all $\eta \geq \mathbb{N}$ and all $x \in Y$.

Definition 2.12. [2, 16] Suppose that Y is an ordered q- starshaped subset of a hyperbolic ordered metric space X. For the self mappings S and T of Y with $q \in Fix(S)$, define $\wedge_q(S, T) = \cup\{\wedge(S,\ T_k) : 0 < k < 1\}$, where $T_k x = (1-k)q \oplus kTx$ and $\wedge(S, T_k) = \{\{x_n\} \subset Y : \lim_n Sx_n = \lim_n T_k x_n = t \in Y\}$. Then S and T are called ordered subcompatible mappings, if $\lim_n d(STx_n, TSx_n) = 0$ for all sequences $x_n \in \wedge_q(S, T)$.

Remark 2.2. (i) Obviously, R-subweakly commuting mappings are subcompatible but converse is not true. For example, let $X = R$ be endowed with usual ordering and $Y = [1, \infty)$. Let $T, S : Y \to Y$ be defined by

$$Sx = \begin{cases} \frac{1}{2} & 0 \leq x < 1, \\ x^2 & x \geq 1 \end{cases}$$

$$Tx = \begin{cases} \frac{3}{2} & 0 \leq x < 1, \\ x & x \geq 1 \end{cases}.$$

Then Y is $\frac{1}{2}$ - starshaped with $\frac{1}{2} \in Fix(S)$ and $\wedge_q(S, T) = \emptyset$. Note that S and T are ordered subcompatible but not R- subweakly commuting for all R>0.

Definition 2.13. Let Y be a closed subset of an ordered metric space X. Let $x \in X$. Define $d(x, Y) = inf\{d(x, y) : y \in Y, y \leq x$ or $x \leq y\}$. If there exists an element $y_0 \in Y$ comparable with x such that $d(x, y_0) = d(x, Y)$, then y_0 is called an ordered best approximation to x out of Y. We denote by $P_Y(x)$, the set of all ordered best approximation to x out of Y.

In 2011, Abbas et. al. [1] proved the following common fixed point theorem:

Theorem A. Let $(X, \leq, d)$ be an ordered metric space. Let f, g, S and T be self mappings on $X, (T, f)$ and (S, g) be partially weakly increasing with $f(X) \subseteq T(X), g(X) \subseteq S(X)$ and dominating maps f and g be weak annihilator of T and S respectively. Also, for every two comparable elements $x, y \in X, d(fx, gy) \leq hM(x, y)$, where

$$M(x, y) = max\{d(Sx, Ty), d(fx, Sx), d(gy, Ty), d(Sx, gy) + d(fx, Ty)\}$$

for $h \in [0, 1)$ is satisfied. If one of $f(X), g(X), S(X)$ or $T(X)$ is complete subspace of X, then $\{f, S\}$ and $\{g, T\}$ have unique point of coincidence in X provided that for a nondecreasing sequence $\{x_n\}$ with $x_n \leq y_n$ for all n and $y_n \to u$ implies $x_n \leq u$. Moreover, if $\{f, S\}$ and $\{g, T\}$ are weakly compatible, then f, g, S and T have a common fixed point.

3. Main results

Now, we give our main result in which the existence of common fixed point of ordered subcomaptible mapping is established in hyperbolic ordered metric spaces by utilizing the notions of ordered S-contractions and ordered asymptotically S-nonexpansive mappings.

Theorem 3.1. *Let Y be a nonempty closed q- starshaped subset of a hyperbolic ordered metric space X satisfying property (I) and T and S be self mappings on Y such that $S(Y) = Y, q \in Fix(S)$ and $T(Y - \{q\}) \subset S(Y - \{q\})$. Let (T, S) be partially weakly increasing, order limit preserving, T is continuous, uniformly asymptotically regular, asymptotically S-nonexpansive and S is an affine mapping. If $cl(Y - \{q\})$ is compact and S and T are subcompatible mappings on $Y - \{q\}$, then $Fix(T) \cap Fix(S)$ is nonempty provided that for a nondecreasing sequence $\{x_n\}$ with $x_n \leq y_n$ for all n and $y_n \to u$ implies $x_n \leq u$.*

Proof. Define $T_n : Y \to Y$ by $T_n(x) = (1 - \lambda_n)q \bigoplus \lambda_n Tx$ for each $n \geq 1$, where $\lambda_n \in (0, 1)$ with $\lim_{n\to\infty} \lambda_n = 1$. Then T_n is a selfmap on Y for each $n \geq 1$. Since S is ordered affine and $T(Y) \subset S(Y)$, therefore we obtain $T_n(Y) \subset S(Y)$.
Now, subcompatibility of the pair (S, T), affinity of $S, q = Sq$ and the property (I) (in respect of any $\{x_m\} \subset K$ with $\lim_m T_n x_m = \lim_m Sx_m = t \in Y$) together imply
$0 \leq \lim_m d(T_n Sx_m, ST_n x_m)$
$= \lim_m d((1 - \lambda_n)q \bigoplus \lambda_n TSx_m$, S(d((1- λ_n)q $\bigoplus \lambda_n$, Tx_m))
$= \lim_m d((1 - \lambda_n)q \bigoplus \lambda_n TSx_m, d(1 - \lambda_n)q \bigoplus \lambda_n STx_m)$
$\leq \lim_m (1 - \lambda_n)d(q, q) + \lambda_n d(TSx_m, STx_m)$
$= \lambda_n \lim_m d(TSx_m, STx_m)$.
Hence $\{T_n\}$ and S are compatible for each n whereas $T_n(Y) \subset Y = S(Y)$, S is affine and $q \in Fix(S)$. Also for any two comparable elements x and $y \in X$, we get
$d(T_n x, T_n y) = d((1 - \lambda_n)q \bigoplus \lambda_n Tx, d((1 - \lambda_n)q \bigoplus \lambda_n Ty)$
$\leq \lambda_n d(Tx, Ty)$
$\leq \lambda_n d(Sx, Sy)$.
Now by Theorem A, there exists $x_n \in Y$ such that x_n is a common fixed point of S and T_n for each $n \geq 1$ i.e.,
$x_n = T_n x_n = Sx_n = (1 - \lambda_n)q \bigoplus \lambda_n Tx_n$,
Note that

$$d(x_n, Tx_n) = d(T_n x_n, Tx_n)$$
$$= d((1 - \lambda_n)q \bigoplus \lambda_n Tx_n, Tx_n)$$
$$= (1 - \lambda_n)d(q, Tx_n).$$

This implies that $d(x_n, Tx_n) \to 0$ as n $\to \infty$. As $cl(Y - \{q\})$ is compact and Y is closed, therefore there exists a subsequence $\{x_{ni}\}$ of $\{x_n\}$ such that $x_{ni} \to x_0 \in Y$ as $i \to \infty$. By the continuity of T, we have $T(x_0) = x_0$. Since T is dominating map, therefore $Sx_k \leq TSx_k$. As T is weak annihilator of S, T is dominating, so $TS\,x_k \leq x_k \leq T\,x_k$. Thus, $Sx_k \leq T\,x_k$ and order limit preserving property of (T, S) imply that $S\,x_0 \leq T\,x_0 = x_0$. Also, $x_0 \leq Sx_0$. Consequently, $Sx_0 = T\,x_0 = x_0$. Hence, the result follows. □

Example 3.5. Let $X = R$ be endowed with usual ordering and $Y = [0, 1]$ be a subset of R. Define $Tx = 0$ and $Sx = x$ for all $x \in Y$. Then Y satisfies the property (I), Y is convex and $1 \in Fix(S)$. Also S is continuous, affine and $S(Y) = Y$. Further, the pair (T, S) is ordered subcompatible, partially weakly increasing, order limit preserving and T is continuous. Thus all the conditions of Theorem 3.1 are satisfied. Note that $Fix(T) \cap Fix(S) = \{0\}$ which substantiates Theorem 3.1.

Now, we obtain result on best approximation as a fixed point of subcompatible mappings in the setting of hyperbolic ordered metric spaces. In particular, as an application of Theorem 3.1, we demonstrate the existence of common fixed point for one pair of maps from the set of best approximation.

Theorem 3.2. *Let M be a nonempty subset of a hyperbolic ordered metric space X and let T and S be continuous selfmaps on X such that $T(\partial M \cap M) \subset M$, ∂M stands for boundary of M and $u \in Fix(S) \cap Fix(T)$ for some $u \in X$, where u is comparable with all $x \in X$. Let (T, S) be partially weakly increasing, order limit preserving, T is uniformly asymptotically regular, asymptotically S-nonexpansive and S is affine on $P_M(u)$ with $S(P_M(u)) = P_M(u), q \in Fix(S)$ and $P_M(u)$ is q-starshaped. If $cl(P_M(u))$ is compact, $P_M(u)$ is complete and S and T are subcompatible mappings on $P_M(u) \cup \{u\}$ satisfying $d(Tx, Tu) \leq d(Sx, Su)$, then $P_M(u) \cap Fix(T) \cap Fix(S)$ is nonempty provided that for a nondecreasing sequence $\{x_n\}$ with $x_n \leq y_n$ for all n and $y_n \to u$ implies $x_n \leq u$.*

Proof. Firstly, we show that T is a self mapping on $P_M(u)$, i.e., $T : P_M(u) \to P_M(u)$. To do this, let $x \in P_M(u)$. Then $d(x, u) = d(u, M)$. Note that for any $\lambda\ in(0, 1)$

$d(y_\lambda, u) = d((1 - \lambda)u \bigoplus \lambda x, u)$

= $\lambda d(x, u) < d(x, u) = d(u, M)$.

This shows that M and $y_x^u = \{y_\lambda : y_\lambda = (1 - \lambda)u \bigoplus \lambda x\}$ are disjoint. So, $x \in \partial M \cap M$ which further implies that $Tx \in M$. Since $Sx \in P_M(u)$, u is a common fixed point of S and T, therefore by the given contractive condition, we obtain

$d(Tx, u) = d(Tx, Tu) \leq d(Sx, Su) = d(Sx, u) = d(u, M)$,

which shows that $Tx \in P_M(u)$. Thus, $P_M(u)$ is T -invariant. Hence, $T(P_M(u)) \subset P_M(u) = S(P_M(u))$.

Now the result follows from Theorem 3.1. □

Remark 3.3. **(i)** Theorems 3.1 and 3.2 extend the results in [1] to more general classes of mappings defined on a hyperbolic ordered metric space.

(ii) Theorems 3.1 and 3.2 also generalise the results in [11] to more generalized spaces, i.e., hyperbolic ordered metric spaces.

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