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A Banach algebra which is generated by idempotents

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ABSTRACT. In this paper we show that the Banach algebra $C_0(X)$, where X is a locally compact Hausdorff space, is generated by idempotents if and only if X is totally disconnected.

1. MAIN RESULTS

Let X be topological space and $C(X)$ denote the space of all continuous complex-valued functions on X. Then it is well-known that if X is compact, the space $C(X)$ is a Banach algebra with pointwise multiplication and the sup-norm. We say that the continuous function f on locally compact Hausdorff space X vanishes at infinity if for every $\varepsilon > 0$ the set $\{x : \ |f(x)| \geq \varepsilon\}$ is compact, and define

$$C_0(X) = \{f \in C(X) : \ f \ \ vanishes \ \ at \ \ infinity\}.$$

Then $C_0(X)$ is a Banach algebra with pointwise multiplication and the sup-norm and $C_0(X) = C(X)$, if X is compact. The Banach algebra $C_0(X)$ is one of the example in Harmonic analysis which has important properties. We say that a subset $\mathcal{A}$ of $C(X)$ vanishes at $x \in X$, if $f(x) = 0$ for all $f \in \mathcal{A}$ and that it separates points, i.e, if for all distinct $x, y \in X$ there exists $f \in \mathcal{A}$ such that $f(x) \neq f(y)$.

The support of $f \in C(X)$ is the smallest closed set outside of which f vanishes, that is, the closer of $\{x : \ f(x) \neq 0\}$.

The collection of all continuous complex-valued functions on X, whose support is compact is denoted by $C_c(X)$.

It is well-known that for locally compact Hausdorff space X, $C_0(X)$ is the closure of $C_c(X)$, in the uniform norm [6].

A topological space X is called totally disconnected if for every distinct $x, y \in X$, there exist disjoint open sets G_1 and G_2 such that $x \in G_1$, $y \in G_2$ and $X = G_1 \cup G_2$, [11].

Let G be a locally compact group and $B(G)$ denote the Fourier-Stieltjes algebra of G. It is shown [7] that the closed algebra $B_I(G)$, where $B_I(G)$ denote the closure of the span of all idempotents in G, generated by the idempotents in $B(G)$.

Bart et al. in [1] studied the relation between logarithmic residues and sums of idempotents in the Banach algebra generated by the compact operators and the identity in the case when the underlying Banach space is infinite dimensional. Krupnik et al. in [9] discussed on certain Banach algebra generated by two idempotents which are C^*-algebras.

The aim of this paper is to show that the Banach algebra $C_0(X)$, where X is a locally compact Hausdorff space, generated by idempotents if and only if X is totally disconnected.

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The reader is referred to [2], [3], [4], [5], [8] and [10] for more information about the Banach algebras generated by two idempotents.

For the proof of the main Theorem we need the Urysohn's Lemma, which it's proof contained in [6].

Urysohn's Lemma: Let X be an locally compact Hausdorff space and $K \subset U \subset X$ where K is compact and U is open. Then there exists $f \in C(X, [0,1])$ such that $f = 1$ on K and $f = 0$ outside of a compact subset of U.

Theorem 1.1. *For a locally compact Hausdorff space X, the Banach algebra $C_0(X)$ is generated by idempotents if and only if X is totally disconnected.*

Proof. Suppose $C_0(X)$ is generated by idempotents and let $x, y \in X$. Then by Urysohn's Lemma there exists $f \in C_0(X)$ such that $f(x) = 1$ and $f(y) = 0$. Since every element of the self-adjoint subalgebra generated by idempotents is of the form

$$F = \sum_{i=1}^{k} \alpha_i f_i, \qquad (\dagger)$$

for some $f_i \in C_0(X)$ and $\alpha_i \in \mathbb{C}$, thus there is a sequence F_n of elements of the form ($\dagger$), such that $F_n \longrightarrow f$ uniformly on X. Hence,

$$\lim_n F_n(x) = 1, \quad and \qquad \lim_n F_n(y) = 0.$$

So there exists a number N such that $|F_N(x)| > 1/2$ and $|F_N(y)| < 1/2$. Take

$$G_1 = F_N^{-1}(\{z \in \mathbb{C} : \ |z| > 1/2\}) \ \ and \ \ G_2 = F_N^{-1}(\{z \in \mathbb{C} : \ |z| < 1/2\}).$$

Then $x \in G_1$, $y \in G_2$, $X = G_1 \cup G_2$ and $G_1 \cap G_2 = \varnothing$. Thus, X is totally disconnected.

Conversely, suppose that X is totally disconnected. Let $x \neq y$, $x \in G_1$, $y \in G_2$, where G_1 and G_2 are open sets and $X = G_1 \cup G_2$. Then the continuous function $f(x) = 1$ for $x \in G_1$ and $f(x) = 0$ for $x \in G_2$, separates x and y. So the closed self-adjoint subalgebra generated by idempotent, is $C_0(X)$, by the Stone-Weierstrass Theorem [6]. ☐

Theorem 1.2. *If $C_c(X)$, for locally compact Hausdorff space X, is generated by idempotents, then X is totally disconnected.*

Proof. Suppose $C_c(X)$ is generated by idempotents and let $x, y \in X$. Then by Theorem 2.12 of [12] there exists $f \in C_c(X)$ such that $f(x) = 1$ and $f(y) = 0$. The rest of proof is similar to the proof of Theorem 1.1. ☐

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REFERENCES

[1] Bart , H., Ehrhardt, T. and Silbermann, B., *Logarithmic residues in the Banach algebra generated by the compact operators and the identity*, Math. Nachr., **268** (2004), 3–30

[2] Bottcher, A. and Spitkovsky, I. M., *Classification of the finite-dimensional algebras generated by two tightly coupled idempotents*, Linear Algebra Appl., **439** (2013), 538–551

[3] Bottcher, A. and Spitkovsky, I. M., *Group inversion in certain finite-dimensional algebras generated by two idempotents* , Indag. Math. (N.S), **23** (2012), 715–732

[4] Bottcher, A. and Spitkovsky, I. M., *On certain finite-dimensional algebras generated by two idempotents*, Linear Algebra Appl., **435** (2011), 1823–1836

[5] Finck, T., Roch, S. and Silbermann, B., *Banach algebras generated by two idempotents and one flip*, Math. Nachr., **216** (2000), 73–94

[6] Folland, G. B., *Real Analysis: modern techniques and their application*, Second edition, New York, 1999

[7] Ilie, M. and Spronk, N., *The algebra generated by idempotents in a Fourier-Stieltjes algebra*, Houston J. Math., **33** (2007), No. 4, 1131–1145

[8] Krupnik, N., *Symmetrization of the symbol in Banach algebras generated by idempotents*, Integral Eq. Operator Theory., **31** (1998), 470–481

[9] Krupnik, N., Roch, S. and Silbermann, B., *On C*-algebras generated by idempotents*, J. Func. Anal., **137** (1996), 303–319

[10] Krupnik, N. and Spigel, E., *Invertibility symbol for a Banach algebra generated by two idempotents and a shift*, Integral Eq. Operator Theory., **17** (1993), 567–578

[11] Munkers, J. R., *Topology*, Second edition, Upper Saddle River: Prentice Hall, 2000

[12] Rudin, W., *Real and Complex Analysis*, New York, 1970

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