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Coefficient bounds for certain subclasses of bi-univalent functions

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ABSTRACT. In this paper we discuss some newly constructed subclasses of bi-univalent functions and establish bounds for the coefficients of the functions in the subclasses $S_\Sigma(\lambda, \alpha)$ and $S_\Sigma(\lambda, \beta)$.

1. INTRODUCTION

Let A denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

which are analytic in the open unit disk $U = \{z : |z| < 1\}$, and let S be the subclass of A consisting of the functions of the form (1.1) which are also univalent in U.

The Koebe one-quarter theorem [6] states that the image of U under every function f from S contains a disk of radius $\frac{1}{4}$. Thus every such univalent function has an inverse f^{-1} which satisfies

$$f^{-1}(f(z)) = z \ \ (z \in U)$$

and

$$f\left(f^{-1}(w)\right) = w \ \left(|w| < r_0(f)\ ,\ r_0(f) \geq \frac{1}{4}\right),$$

where

$$f^{-1}(w) = w \ - a_2 w^2 + \left(2a_2^2 - a_3\right) w^3 - \left(5a_2^3 - 5a_2 a_3 + a_4\right) w^4 + \cdots .$$

A function $f \in A$ is said to be bi-univalent in U if both f and f^{-1} are univalent in U. Let Σ denote the class of bi-univalent functions defined in the unit disk U.

For a brief history and interesting examples in the class Σ, see [14]. Examples of functions in the class Σ are

$$\frac{z}{1-z}, \ \ -\log(1-z), \ \ \frac{1}{2}\log\left(\frac{1+z}{1-z}\right), \ldots$$

However, the familiar Koebe function is not a member of Σ. Other common examples of functions in S such as

$$z - \frac{z^2}{2} \text{ and } \frac{z}{1-z^2}$$

are also not members of Σ (see [14]). Lewin [10] studied the class of bi-univalent functions, obtaining the bound 1.51 for modulus of the second coefficient $|a_2|$. Subsequently, Brannan and Clunie [3] conjectured that $|a_2| \leq \sqrt{2}$ for $f \in \Sigma$. Later, Netanyahu [12] showed

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that $\max|a_2| = \frac{4}{3}$ if $f(z) \in \Sigma$. Brannan and Taha [4] introduced certain subclasses of the bi-univalent function class Σ similar to the familiar subclasses $S^{\star}(\beta)$ and $K(\beta)$ of starlike and convex functions of order β $(0 \leq \beta < 1)$, respectively (see [12]). Thus, following Brannan and Taha [4], a function $f(z) \in A$ is said to be in the class $S^{\star}_{\Sigma}(\alpha)$ of strongly bi-starlike functions of order α $(0 < \alpha \leq 1)$ if each of the following two conditions:

$$f \in \Sigma, \quad \left|\arg\left(\frac{zf^{'}(z)}{f(z)}\right)\right| < \frac{\alpha\pi}{2} \quad (0 < \alpha \leq 1,\ z \in U)$$

and

$$\left|\arg\left(\frac{wg^{'}(w)}{g(w)}\right)\right| < \frac{\alpha\pi}{2} \quad (0 < \alpha \leq 1,\ w \in U)$$

is satisfied. It is said to be in the class $K_{\Sigma}(\alpha)$ of strongly bi-convex functions of order α $(0 < \alpha < 1)$ if each of the following two conditions:

$$f \in \Sigma, \quad \left|\arg\left(1 + \frac{zf^{'}(z)}{f(z)}\right)\right| < \frac{\alpha\pi}{2} \quad (0 < \alpha \leq 1,\ z \in U)$$

and

$$\left|\arg\left(1 + \frac{wg^{'}(w)}{g(w)}\right)\right| < \frac{\alpha\pi}{2} \quad (0 < \alpha \leq 1,\ w \in U)$$

is satisfied, where g is the extension of f^{-1} to U.

The classes $S^{\star}_{\Sigma}(\alpha)$ and $K_{\Sigma}(\alpha)$ of bi-starlike functions of order α and bi-convex functions of order α, corresponding to the function classes $S^{\star}(\alpha)$ and $K(\alpha)$, were also introduced analogously. For each of the function classes $S^{\star}_{\Sigma}(\alpha)$ and $K_{\Sigma}(\alpha)$, they found non-sharp estimates on the initial coefficients. In fact, the aforecited work of Srivastava et al. [14] essentially revived the investigation of various subclasses of the bi-univalent function class Σ in recent years. Recently, many authors investigated bounds for various subclasses of bi-univalent functions ([1], [2], [7], [11], [14], [15], [16]). Not much is known about the bounds on the general coefficient $|a_n|$ for $n \geq 4$. In the literature, there exist only a few works determining the general coefficient bounds $|a_n|$ for the analytic bi-univalent functions [5], [8], [9]). The coefficient estimate problem for each of $|a_n|$ $(n \in \mathbb{N}\backslash\{1,2\};\ \mathbb{N} = \{1,2,3,...\})$ is still an open problem.

The aim of the this paper is to introduce two new subclasses of the function class Σ and find estimates on the coefficients $|a_2|$ and $|a_3|$ for functions in these new subclasses of the functions.

We remind the following lemma which will be useful to derive our basic results.

Lemma 1.1. [13] *If $p(z) = 1 + p_1 z + p_2 z^2 + p_3 z^3 + \cdots$ is an analytic function in U with positive real part, then*

$$|p_n| \leq 2 \qquad (n \in \mathbb{N} = \{1,2,\ldots\})$$

and

$$\left|p_2 - \frac{p_1^2}{2}\right| \leq 2 - \frac{|p_1|^2}{2}. \tag{1.2}$$

2. Coefficient bounds for the function class $S_\Sigma(\lambda,\alpha)$

Definition 2.1. A function $f \in \Sigma$ is said to be in the class $S_\Sigma(\lambda,\alpha)$ if the following two conditions

$$f \in \Sigma, \quad \left|\arg \tfrac{1}{2}\left(\tfrac{zf'(z)}{f(z)} + \left(\tfrac{zf'(z)}{f(z)}\right)^{\frac{1}{\lambda}}\right)\right| < \frac{\alpha\pi}{2} \quad (0<\alpha\le 1,\ 0<\lambda\le 1,\ \ z\in U) \tag{2.3}$$

and

$$\left|\arg \tfrac{1}{2}\left(\tfrac{wg'(w)}{g(w)} + \left(\tfrac{wg'(w)}{g(w)}\right)^{\frac{1}{\lambda}}\right)\right| < \frac{\alpha\pi}{2} \quad (0<\alpha\le 1,\ 0<\lambda\le 1,\ w\in U) \tag{2.4}$$

are satisfied, where $g = f^{-1}$.

Theorem 2.1. *Let f given by (1.1) be in the class $S_\Sigma(\lambda,\alpha)$, $0<\alpha\le 1$. Then*

$$|a_2| \le \frac{4\alpha\lambda}{\sqrt{\alpha(3\lambda^2+1)+(\lambda+1)^2}}$$

and

$$|a_3| \le \frac{2\alpha\lambda}{\lambda+1} + \frac{16\alpha^2\lambda^2}{(\lambda+1)^2}.$$

Proof. Let $f \in S_\Sigma(\lambda,\alpha)$. Then

$$\frac{1}{2}\left(\frac{zf'(z)}{f(z)} + \left(\frac{zf'(z)}{f(z)}\right)^{\frac{1}{\lambda}}\right) = [p(z)]^\alpha \tag{2.5}$$

$$\frac{1}{2}\left(\frac{wg'(w)}{g(w)} + \left(\frac{wg'(w)}{g(w)}\right)^{\frac{1}{\lambda}}\right) = [q(w)]^\alpha \tag{2.6}$$

where $g = f^{-1}$, $p, q \in P$ (i.e., are polynomials) and have the forms

$$p(z) = 1 + p_1 z + p_2 z^2 + \cdots$$

and

$$q(w) = 1 + q_1 w + q_2 w^2 + \cdots .$$

Now, equating the coefficients in (2.5) and (2.6), we get

$$\frac{\lambda+1}{2\lambda} a_2 = \alpha p_1, \tag{2.7}$$

$$\frac{\lambda+1}{2\lambda}\left(2a_3 - a_2^2\right) + \frac{1-\lambda}{4\lambda^2} a_2^2 = \alpha p_2 + \frac{\alpha(\alpha-1)}{2} p_1^2, \tag{2.8}$$

and

$$-\frac{\lambda+1}{2\lambda} a_2 = \alpha q_1, \tag{2.9}$$

$$\frac{\lambda+1}{2\lambda}\left(3a_2^2 - 2a_3\right) + \frac{1-\lambda}{4\lambda^2} a_2^2 = \alpha q_2 + \frac{\alpha(\alpha-1)}{2} q_1^2. \tag{2.10}$$

From (2.7) and (2.9) we obtain

$$p_1 = -q_1. \tag{2.11}$$

and

$$\frac{(\lambda+1)^2}{2\lambda^2} a_2^2 = \alpha^2 (p_1^2 + q_1^2). \tag{2.12}$$

Also by (2.8), (2.10) and (2.12) we have

$$\tfrac{2\lambda^2+\lambda+1}{2\lambda^2}a_2^2 = \alpha\left(p_2+q_2\right)+\tfrac{\alpha(\alpha-1)}{2}(p_1^2+q_1^2).$$

$$= \alpha\left(p_2+q_2\right)+\tfrac{\alpha(\alpha-1)}{2}\tfrac{(\lambda+1)^2}{2\lambda^2\alpha^2}a_2^2,$$

and therefore, we get

$$a_2^2 = \frac{4\alpha^2\lambda^2\left(p_2+q_2\right)}{2\alpha(2\lambda^2+\lambda+1)-(\alpha-1)(\lambda+1)^2}. \tag{2.13}$$

Applying Lemma 1.1 for the coefficients p_2 and q_2, we obtain

$$|a_2| \leq \frac{4\alpha\lambda}{\sqrt{\alpha(3\lambda^2+1)+(\lambda+1)^2}}.$$

Next, in order to find the bound on $|a_3|$, by subtracting (2.10) from (2.8), we obtain

$$\frac{2\left(\lambda+1\right)}{\lambda}\left(a_3-a_2^2\right) = \alpha\left(p_2-q_2\right)+\frac{\alpha(\alpha-1)}{2}(p_1^2-q_1^2).$$

Then, in view of (1.2) and (2.12) , we have

$$|a_3| \leq \frac{2\alpha\lambda}{\lambda+1}+\frac{16\alpha^2\lambda^2}{(\lambda+1)^2}.$$

□

3. Coefficient bounds for the functions in the class $S_\Sigma(\lambda,\beta)$

Definition 3.2. A function $f\in\Sigma$ is said to be in the class $S_\Sigma(\lambda,\beta)$ if the following conditions

$$f\in\Sigma,\ \ \mathrm{Re}\left(\tfrac{1}{2}\left(\tfrac{zf'(z)}{f(z)}+\left(\tfrac{zf'(z)}{f(z)}\right)^{\frac{1}{\lambda}}\right)\right)>\beta \quad (0\leq\beta<1,\ 0<\lambda\leq1,\ z\in U) \tag{3.14}$$

and

$$\mathrm{Re}\left(\tfrac{1}{2}\left(\tfrac{wg'(w)}{g(w)}+\left(\tfrac{wg'(w)}{g(w)}\right)^{\frac{1}{\lambda}}\right)\right)>\beta \quad (0\leq\beta<1,\ 0<\lambda\leq1,\ w\in U) \tag{3.15}$$

are satisfied, where we denoted $g=f^{-1}$.

Theorem 3.2. *Let f given by (1.1) be in the class $S_\Sigma(\lambda,\beta),\ 0\leq\beta<1$. Then*

$$|a_2| \leq \sqrt{\frac{8\lambda^2\left(1-\beta\right)}{2\lambda^2+\lambda+1}}$$

and

$$|a_3| \leq \frac{2\lambda(1-\beta)}{\lambda+1}+\frac{16\lambda^2}{(\lambda+1)^2}.$$

Proof. Let $f\in S_\Sigma(\lambda,\beta)$. Then

$$\frac{1}{2}\left(\frac{zf'(z)}{f(z)}+\left(\frac{zf'(z)}{f(z)}\right)^{\frac{1}{\lambda}}\right)=\beta+(1-\beta)p(z) \tag{3.16}$$

$$\frac{1}{2}\left(\frac{wg'(w)}{g(w)}+\left(\frac{wg'(w)}{g(w)}\right)^{\frac{1}{\lambda}}\right)=\beta+(1-\beta)q(w) \tag{3.17}$$

where $p,q\in P$ and $g=f^{\,-1}$.

It follows from (3.16) and (3.17) that

$$\frac{\lambda+1}{2\lambda}a_2 = (1-\beta)p_1, \tag{3.18}$$

$$\frac{\lambda+1}{2\lambda}\left(2a_3 - a_2^2\right) + \frac{1-\lambda}{4\lambda^2}a_2^2 = (1-\beta)p_2, \tag{3.19}$$

and

$$-\frac{\lambda+1}{2\lambda}a_2 = (1-\beta)q_1, \tag{3.20}$$

$$\frac{\lambda+1}{2\lambda}\left(3a_2^2 - 2a_3\right) + \frac{1-\lambda}{4\lambda^2}a_2^2 = (1-\beta)q_2. \tag{3.21}$$

From (3.19) and (3.21) we obtain

$$p_1 = -q_1. \tag{3.22}$$

and

$$\frac{(\lambda+1)^2}{2\lambda^2}a_2^2 = (1-\beta)^2(p_1^2 + q_1^2). \tag{3.23}$$

Also from (3.19), (3.21) and (3.22) we have

$$\frac{2\lambda^2+\lambda+1}{2\lambda^2}a_2^2 = (1-\beta)\left(p_2+q_2\right).$$

Therefore, we have

$$a_2^2 = \frac{2(1-\beta)\lambda^2\left(p_2+q_2\right)}{2\lambda^2+\lambda+1}. \tag{3.24}$$

Appying Lemma 1.1. for the coefficients p_2 and q_2, we obtain

$$|a_2| \le \sqrt{\frac{8\lambda^2\left(1-\beta\right)}{2\lambda^2+\lambda+1}}.$$

Next, in order to find the bound on $|a_3|$, by subtracting (3.21) from (3.19), we obtain

$$\frac{2\left(\lambda+1\right)}{\lambda}\left(a_3 - a_2^2\right) = (1-\beta)\left(p_2 - q_2\right).$$

Then, in view of (1.2) and (3.23) , we have

$$|a_3| \le \frac{2(1-\beta)\lambda}{\lambda+1} + \frac{16(1-\beta)^2\lambda^2}{(\lambda+1)^2}.$$

□

Taking $\lambda = 1$ in Theorems 2.1 and 3.1, one can get the following corollaries.

Corollary 3.1. *Let f given by (1.1) be in the class $S_\Sigma(\alpha),\ 0 < \alpha \le 1$. Then*

$$|a_2| \le \frac{2\alpha}{\sqrt{\alpha+1}}$$

and

$$|a_3| \le \alpha + 4\alpha^2.$$

Corollary 3.2. *Let f given by (1.1) be in the class $S_\Sigma(\beta),\ 0 \le \beta < 1$. Then*

$$|a_2| \le \sqrt{2\left(1-\beta\right)}$$

and

$$|a_3| \le (1-\beta) + 4(1-\beta)^2.$$

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