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Some identities involving divided differences

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ABSTRACT. To study approximation properties of linear positive operators various identities involving divided differences are used. The aim of this note is to present two types of such kind of identities. The first one was used by Abel and Ivan [Abel, U. and Ivan, M., *Some identities for the operator of Bleimamm, Butzer and Hahn involving divided differences*, Calcolo, **36** (1999), 143–160; Abel, U. and Ivan, M., *New representation of the remainder in the Bernstein approximation*, J. Math. Anal. Appl., **381** (2011), No. 2, 952–956] to derive approximation properties of Bleimann, Butzer and Hahn (BBH) operators from the corresponding properties of the classical Bernstein operators. The second type of identifies can be used to derive some approximation properties of the BBH operators from the properties of some Stancu type operators.

1. INTRODUCTION

Suppose that $I \subseteq \mathbb{R}$ is a non-empty interval, m is a positive integer and $x_1, \ldots, x_m \in I$ are distinct points. If $f \in \mathbb{R}^I$, the divided differences of f are defined recursively by

$$[x_1; f] = f(x_1),\ [x_1, x_2, \ldots, x_m; f] = \frac{[x_2, \ldots, x_m; f] - [x_1, \ldots, x_{m-1}; f]}{x_m - x_1}. \tag{1.1}$$

It is well known that the divided difference $[x_1, \ldots, x_m; f]$ is the coefficient of x^{m-1} in the $(m-1)$-th degree Lagrange polynomial interpolating the function f in the distinct points $x_1, x_2, \ldots, x_m$.
If $V(x_1, x_2, \ldots, x_m)$ denotes the Vandermonde determinant, i.e.,

$$V(x_1, x_2, \ldots, x_m) = \begin{vmatrix} 1\ x_1\ x_1^2\ \ldots\ x_1^{m-1} \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ 1\ x_m\ x_m^2\ \ldots\ x_m^{m-1} \end{vmatrix} \tag{1.2}$$

and $Wf(x_1, x_2, \ldots, x_m)$ is the determinant

$$Wf(x_1, x_2, \ldots, x_m) = \begin{vmatrix} 1\ x_1\ \ldots\ x_1^{m-2}\ f(x_1) \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ 1\ x_m\ \ldots\ x_m^{m-2}\ f(x_m) \end{vmatrix}, \tag{1.3}$$

then the divided difference $[x_1, x_2, \ldots, x_m; f]$ can be expressed under the form

$$[x_1, x_2, \ldots, x_m; f] = \frac{Wf(x_1, x_2, \ldots, x_m)}{V(x_1, x_2, \ldots, x_m)}. \tag{1.4}$$

In [12], to any $f \in \mathbb{R}^{[0,+\infty)}$ was associated the function $\widetilde{f} \in \mathbb{R}^{[0,1]}$, defined by

$$\widetilde{f}(x) = \begin{cases} (1-x) f\left(\frac{x}{1-x}\right), & x \in [0,1); \\ 0 & , x = 1, \end{cases} \tag{1.5}$$

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and were presented some relations between the divided differences of f and respectively $\widetilde{f}$. These relations were used in [1] and [12] to obtain approximation properties of Bleimann, Butzer and Hahn operators directly from the properties of Bernstein operators. They will be mentioned in Section 2, after the proof of Proposition 2.1.
In [7], to any $f \in \mathbb{R}^{[0,+\infty)}$ was associated the function $F \in \mathbb{R}^{[0,1]}$, defined by

$$F(y) = \begin{cases} f\left(\frac{y}{1-y}\right), & y \in [0,1); \\ 0 & , y = 1, \end{cases} \tag{1.6}$$

and was obtained a Voronovskaja type theorem for the operators of Bleimann, Butzer and Hahn, derived from the corresponding result for the Stancu operators $P_m^{(0,1)}$.
The purposes of this paper are the following two: first, to prove the identities presented in [1], [12] and second to establish some relationships between the divided differences of the functions f and F.

2. Main results

We start by proving the following

Proposition 2.1. ([1], [12]). *Let $f \in \mathbb{R}^{[0,+\infty)}$ be given and let $\widetilde{f} \in \mathbb{R}^{[0,1]}$ be defined by* (1.5). *The following identities*

$$(i) \left[\frac{x_1}{1-x_1}, \frac{x_2}{1-x_2}, \frac{x_3}{1-x_3}; f\right] = (1-x_1)(1-x_2)(1-x_3)[x_1, x_2, x_3; \widetilde{f}]; \tag{2.7}$$

$$(ii) \left[\frac{x_1}{1-x_1}, \frac{x_2}{1-x_2}; f\right] = -(1-x_1)(1-x_2)[x_1, x_2, 1; \widetilde{f}]; \tag{2.8}$$

$$(iii) \left[\frac{y_1}{1+y_1}, \frac{y_2}{1+y_2}, \frac{y_3}{1+y_3}; \widetilde{f}\right] = (1+y_1)(1+y_2)(1+y_3)[y_1, y_2, y_3; f\,] \tag{2.9}$$

hold, for any $x_1, x_2, x_3 \in [0,1)$ $(0 \le x_1 < x_2 < x_3 < 1)$ and any $y_1, y_2, y_3 \in [0,+\infty)$ $(0 \le y_1 < y_2 < y_3)$.

Proof. In all cases formula (1.4) will be applied.
(i) Some elementary calculations lead to

$$Wf\left(\frac{x_1}{1-x_1}, \frac{x_2}{1-x_2}, \frac{x_3}{1-x_3}\right) = \frac{1}{(1-x_1)(1-x_2)(1-x_3)} W\widetilde{f}(x_1, x_2, x_3)$$

and

$$V\left(\frac{x_1}{1-x_1}, \frac{x_2}{1-x_2}, \frac{x_3}{1-x_3}\right) = \frac{1}{(1-x_1)^2(1-x_2)^2(1-x_3)^2} V(x_1, x_2, x_3).$$

Taking the above equations into account and applying (1.4) we obtain (2.7).
(ii) By direct computation, it follows

$$W\widetilde{f}(x_1, x_2, 1) = -(1-x_1)(1-x_2) Wf\left(\frac{x_1}{1-x_1}, \frac{x_2}{1-x_2}\right)$$

and

$$V(x_1, x_2, 1) = (1-x_1)^2(1-x_2)^2 V\left(\frac{x_1}{1-x_1}, \frac{x_2}{1-x_2}\right).$$

Consequently, (1.4) leads to (2.8).

(iii) In a similar way, one obtains

$$W\widetilde{f}\left(\frac{y_1}{1+y_1},\frac{y_2}{1+y_2},\frac{y_3}{1+y_3}\right)=\frac{1}{(1+y_1)(1+y_2)(1+y_3)}Wf(y_1,y_2,y_3)$$

and

$$V\left(\frac{y_1}{1+y_1},\frac{y_2}{1+y_2},\frac{y_3}{1+y_3}\right)=\frac{1}{(1+y_1)^2(1+y_2)^2(1+y_3)^2}V(y_1,y_2,y_3).$$

Next, applying (1.4) one obtains (2.9). □

Remark 2.1. The proof of Proposition 2.1 is elementary, but it has a capital importance in order to obtain identities involving divided differences for the Bleimann, Butzer and Hahn operators from the corresponding identities in case of the Bernstein operators [1], [12]. In what follows, we recall some of the above mentioned identities.

Let $f:[0,+\infty)\to\mathbb{R}$ be a continuous function. The operator of Bleimann, Butzer and Hahn is defined in [11] by the following formula.

$$Lmf(x)=\frac{1}{(1+x)^m}\sum_{k=0}^{m}\binom{m}{k}x^kf\left(\frac{k}{n+1-k}\right),\ x\in[0,+\infty). \qquad (2.10)$$

In [12], Ivan established the following formula connecting the Bleimann, Butzer and Hahn operator and the Bernstein operator by means of a transformation independent of m.

Theorem 2.1. *For $f:[0,+\infty)\to\mathbb{R}$, the following formula is satisfied*

$$Lmf(x)=(1+x)B_{m+1}\widetilde{f}\left(\frac{x}{1+x}\right),\ x\in[0,+\infty). \qquad (2.11)$$

By virtue of (2.11) some of the best known properties of the BBH operator can easily be obtained from the properties of Bernstein operator, taking into account Proposition 2.1. The following theorems present some of them, that can be directly transferred to L_m by means of formula (2.11).

Theorem 2.2. (Averbach, [1]) *For $g\in\mathbb{R}^{[0,1]}, m\geq 1$, the following formula holds true*

$$B_mg(t)-B_{m+1}g(t)=\frac{t(1-t)}{m(m+1)}\sum_{k=0}^{m-1}p_{m-1,k}(t)\left[\frac{k}{m},\frac{k+1}{m+1},\frac{k+1}{m};g\right].$$

Theorem 2.3. (Stancu, [1]) *For $g\in\mathbb{R}^{[0,1]}, n\geq 1\, t\in[0,1]\setminus\left\{\frac{k}{m}|k=0,\dots,m\right\}$ the following formula holds true*

$$B_mg(t)-g(t)=\frac{t(1-t)}{m}\sum_{k=0}^{m-1}p_{m-1,k}(t)\left[t,\frac{k}{m},\frac{k+1}{m};g\right].$$

Theorem 2.4. (Aramă, [5]) *If $g\in C[0,1]$, then, for any $t\in[0,1]$ there exist the distinct points $t_1,t_2,t_3\in[0,1]$ such that*

$$B_mg(t)-g(t)=\frac{t(1-t)}{m}[t_1,t_2,t_3;g].$$

In [1], Abel and Ivan presented, among others, the corresponding results for the Bleimann, Butzer and Hahn operator.

Proposition 2.2. *If* $f \in \mathbb{R}^{[0,+\infty)}$ *and* $F \in \mathbb{R}^{[0,1]}$ *is defined by* (1.6), *the following identities hold true*

$$(i)\ \left[\frac{x_1}{1-x_1}, \frac{x_2}{1-x_2}, \frac{x_3}{1-x_3}; f\right] = (1-x_1)(1-x_2)(1-x_3)[x_1, x_2, x_3; (1-x)F] \tag{2.12}$$

$$(ii)\ \left[\frac{y_1}{1+y_1}, \frac{y_2}{1+y_2}, \frac{y_3}{1+y_3}; F\right] = (1+y_1)(1+y_2)(1+y_3)[y_1, y_2, y_3; (1+y)f] \tag{2.13}$$

for any $x_1, x_2, x_3 \in [0,1)$ $(0 \le x_1 < x_2 < 1)$ *and any* $y_1, y_2, y_3 \in [0,+\infty)$ $(0 \le y_1 < y_2 < y_3)$.

Proof. We shall proceed similarly to the proof of Proposition 2.1.

(i) After some elementary calculations, one gets

$$Wf\left(\frac{x_1}{1-x_1}, \frac{x_2}{1-x_2}, \frac{x_3}{1-x_3}\right) = \frac{1}{(1-x_1)(1-x_2)(1-x_3)} W((1-x)F)(x_1, x_2, x_3).$$

Taking into account that

$$V\left(\frac{x_1}{1-x_1}, \frac{x_2}{1-x_2}, \frac{x_3}{1-x_3}\right) = \frac{1}{(1-x_1)^2(1-x_2)^2(1-x_3)^2} V(x_1, x_2, x_3),$$

and applying (1.4) one obtains the desired identity (2.12).

(ii) As at (i), is follows that

$$WF\left(\frac{y_1}{1+y_1}, \frac{y_2}{1+y_2}, \frac{y_3}{1+y_3}\right) = \frac{1}{(1+y_1)(1+y_2)(1+y_3)} W((1+y)f)(y_1, y_2, y_3).$$

Now, the identity

$$V\left(\frac{y_1}{1+y_1}, \frac{y_2}{1+y_2}, \frac{y_3}{1+y_3}\right) = \frac{1}{(1+y_1)^2(1+y_2)^2(1+y_3)^2} V(y_1, y_2, y_3)$$

and (1.4) lead to (2.13). □

Let $P_m^{(0,1)} : C[0,1] \to C[0,1]$ be the Stancu operator defined by

$$P_m^{(0,1)} f(x) = \sum_{k=0}^{m} p_{m,k}(x) f\left(\frac{k}{m+1}\right). \tag{2.14}$$

In [7] the following relationship between the Bleimann, Butzer and Hahn operator and the Stancu type operator $P_m^{(0,1)}$ has been proved.

Theorem 2.5. *For any* $f : [0,+\infty) \to \mathbb{R}$, *the following formula*

$$L_m f(x) = P_m^{(0,1)} F\left(\frac{x}{1-x}\right) \tag{2.15}$$

holds true for any $x \in [0,+\infty)$, *where* F *is defined by (1.6).*

Using the equality (2.13) and the Voronovskaja theorem for the Stancu operators, in [7] the following Voronovskaja-type theorem for the Bleimann, Butzer and Hahn operators has been proved.

Theorem 2.6. *If the bounded function* $f \in B_*[0,+\infty)$ *is differentiable in some neighbourhood of a point* $x \in [0,+\infty)$ *and has the second order derivative* $f''(t)$ *and* F *is bounded, the following Voronovskaja-type formula holds*

$$\lim_{m\to\infty} m\big(L_m f(x) - f(x)\big) = \frac{x(1+x)^2}{2} f''(x), \tag{2.16}$$

Using Proposition 2.2 and Theorem 2.5 it is possible to recover identities involving divided differences for the Bleimann, Butzer and Hahn operators, corresponding to those verified by the Stancu operator.

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REFERENCES

[1] Abel, U. and Ivan, M., *Some identities for the operator of Bleimamm, Butzer and Hahn involving divided differences*, Calcolo, **36** (1999), 143–160

[2] Abel, U. and Ivan, M., *New representation of the remainder in the Bernstein approximation*, J. Math. Anal. Appl., **381** (2011), No. 2, 952–956

[3] Agratini, O., *Aproximare prin operatori liniari*, Presa Univ. Clujeană, Cluj-Napoca, 2001 (Romanian)

[4] Agratini, O., *An application of divided differences*, Aut. Comp, Appl. Math., **4** (1995), No. 2, 95–99

[5] Aramă, O., *Some properties concerning the sequence of polynomials of S. N. Bernstein (Romanian)*, Studii şi Cerc. (Cluj), **8** (1957), No. 3-4, 195–210

[6] Bărbosu, D., *Introducere în analiza numerică şi teoria aproximării*, Ed. Universităţii de Nord Baia Mare, 2009 (Romanian)

[7] Bărbosu, D., Acu, A. M. and Sofonea, F. D., *The Voronovskaja-type formula for the Bleimann, Butzer and Hahn operators*, Creative Math and Inf, **23** (2014), No. 2, 137–140

[8] Bărbosu, D., *Two dimensional divided differences revisited*, Creative Math.& Inf., **17** (2008), No. 2, 1–7

[9] Bărbosu, D. and Pop, O. T., *On the Bernstein bivariate approximation formula*, Carpathian J. Math., **24** (2008), No. 3, 293–298

[10] Bărbosu, D. and Pop, O. T., *Bivariate Schurer-Stancu operators revisited*, Carpathian J. Math., **26** (2010), No. 1, 24–35

[11] Bleimann, G., Butzer, P. L. and Hahn, L., *Bernstein-type operator approximating continuous functions on the semi-axis*, Nederl. Akad. Wetensch. Indag. Math., **42** (1980), 255–262

[12] Ivan, M., *A note on the Bleimamm, Butzer and Hahn operator*, Automat. Comput. Appl. Math., **6** (1997), No. 1, 11–15

[13] Ivan, M., *Elements of Interpolation Theory*, Mediamira Science Publisher, Cluj-Napoca (2004)

[14] Khan, R. A., *A note on a Bernstein-type operator of Bleimamm, Butzer and Hahn*, J. Approx. Theory, **53** (1988), 295–230

[15] Mercer, A. McD., *A Bernstein-type operator approximating continuous functions on the half-line*, Bull. Calcutta Math. Soc., **81** (1989), No. 2, 133–137

[16] Pop, O. T. and Bărbosu, D., *Bivariate divided differences with multiple knots*, An. Şt. Univ. Ovidius Constanţa, **17** (2009), No. 2, 181–190

[17] Stancu, D. D., *The remainder of certain linear approximation formulas in two variables*, J. SIAM Numer. Anal. Ser. B., **1** (1964), 137–163

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