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The locus of generalized Toricelli-Fermat points

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ABSTRACT. In this paper, we obtain the locus of generalized Torricelli-Fermat points of a triangle.

1. INTRODUCTION

In the articles [3] and [4], being given the triangle ABC, there were considered the points B' and C', the rotated of points B, and respectively C around the point A, with the same angle α towards the outside of triangle. The lines CB' and BC' intersect each other at point P_A. Similarly, we obtain the points P_B and P_C. It has been shown that the lines AP_A, BP_B and CP_C intersect each other at a point T_α, which generalizes the Torricelli-Fermat point, obtained for $\alpha = 60°$ (Figure 1).

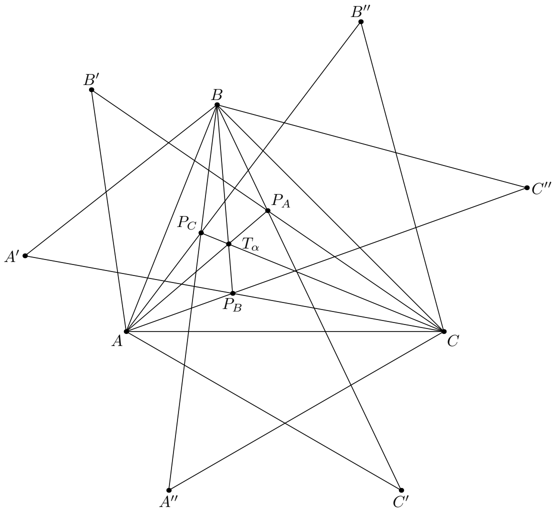

FIGURE 1

Also, considering $Q_A = BA'' \cap CA'$, $Q_B = AB'' \cap CB'$ and $Q_C = BC' \cap AC''$, the lines AQ_A, BQ_B and CQ_C intersect at one point, denoted by S_α (Figure 2).

If a, b and c are real numbers, with $a < b$ and $c > 0$, we consider the points $A(a, 0)$, $B(b, 0)$ and $C(0, c)$, so that the triangle ABC should be a scalen triangle. We denote by G the centroid of triangle ABC and by H its orthocenter.

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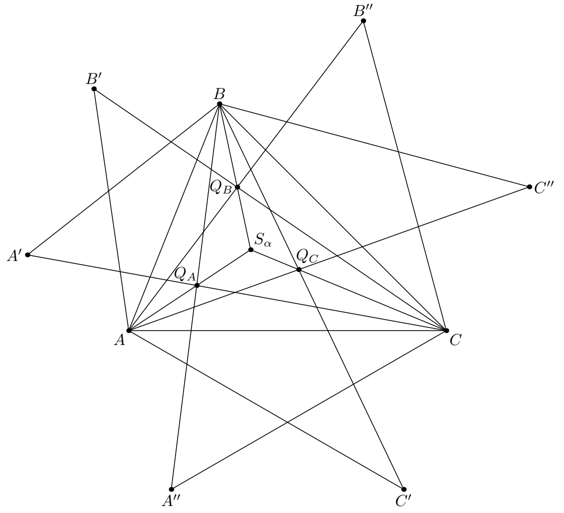


FIGURE 2

Lemma 1.1. *The equation of the rectangular hyperbola determined by the points A, B, C, G and H is given by*

$$c(a+b)x^2+2(a^2+b^2-c^2-ab)xy-c(a+b)y^2-c(a+b)^2x+ \\ (a+b)(c^2-ab)y+abc(a+b)=0. \tag{1.1}$$

Proof. We have $G\left(\frac{a+b}{3},\frac{c}{3}\right)$ and $H\left(0,-\frac{ab}{c}\right)$. The equation of hyperbola is $\alpha x^2+\beta xy-\alpha y^2+\gamma x+\delta y+\epsilon=0$. Substituting the coordinates of the five points, we find $\alpha=c(a+b)$, $\beta=2(a^2+b^2-c^2-ab)$, $\gamma=-c(a+b)^2$, $\delta=(a+b)(c^2-ab)$ and $\epsilon=abc(a+b)$.

The invariants of the conic are: $\Delta=\frac{1}{4}c(a-b)^2(a+b)(2ab+c^2-b^2)(2ab+c^2-a^2)$, $\delta=-c^2(a+b)^2-(a^2+b^2-c^2-ab)^2$ and $I=c(a+b)-c(a+b)=0$ ([2], pp. 441-447). For a scalen triangle, we find that $\Delta\neq 0$ and $\delta<0$, and the lemma is proved. □

Remark 1.1. The hyperbola in Lemma 1.1 is known in the literature as the Kiepert hyperbola ([6], [7]).

Remark 1.2. We can obtain the equations of the asimptotes of the hyperbola, $(t_1)\colon mx+ny+p=0$, where $m=\sqrt{-\delta}$, $n=\sqrt{-\delta}\cdot\frac{a^2+b^2-c^2-ab-\sqrt{-\delta}}{c(a+b)}$ and $p=\frac{a+b}{2}\left((a-b)^2-\sqrt{-\delta}\right)$, respectively $(t_2)\colon m'x+n'y+p'=0$, where $m'=-\sqrt{-\delta}$, $n'=-\sqrt{-\delta}\cdot\frac{a^2+b^2-c^2-ab+\sqrt{-\delta}}{c(a+b)}$ and $p'=\frac{a+b}{2}\left((a-b)^2+\sqrt{-\delta}\right)$.

2. MAIN RESULTS

Theorem 2.1. *The points T_α and S_α are on the hyperbola (1.1).*

Proof. The affix of intersection point of the lines AA', BB' and CC' is given by $t=\frac{mz_A+nz_B+pz_C}{m+n+p}$, where z_A, z_B and z_C are the affixes of points A, B, respectively C, and $m=\frac{A'B}{A'C}$, $n=\frac{B'C}{B'A}$ and $p=\frac{C'A}{C'B}$ ([1]). But $m=\frac{\sin C\cos\left(B-\frac{\alpha}{2}\right)}{\sin B\cos\left(C-\frac{\alpha}{2}\right)}$, $n=\frac{\sin A\cos\left(C-\frac{\alpha}{2}\right)}{\sin C\cos\left(A-\frac{\alpha}{2}\right)}$, $p=\frac{\sin B\cos\left(A-\frac{\alpha}{2}\right)}{\sin A\cos\left(B-\frac{\alpha}{2}\right)}$ ([3]), and therefore the coordinates of point T_α are:

$$\begin{cases} x(\alpha) = \frac{(a+b)(ab+c^2)\sin\alpha + c(a^2-b^2)\cos\alpha - c(a^2-b^2)}{2(a^2+b^2+c^2-ab)\sin\alpha + 2c(a-b)\cos\alpha - 4c(a-b)} \\ y(\alpha) = \frac{c(a-b)^2\sin\alpha + (a-b)(ab+c^2)\cos\alpha - (a-b)(c^2-ab)}{2(a^2+b^2+c^2-ab)\sin\alpha + 2c(a-b)\cos\alpha - 4c(a-b)}. \end{cases} \tag{2.2}$$

In case of the point S_α, we have $m = \frac{\sin C \sin(B+\alpha)}{\sin B \sin(C+\alpha)}$, $n = \frac{\sin A \sin(C+\alpha)}{\sin C \sin(A+\alpha)}$, $p = \frac{\sin B \sin(A+\alpha)}{\sin A \sin(B+\alpha)}$ ([3]). Therefore, the coordinates of point S_α are:

$$\begin{cases} x(\alpha) = \frac{(a+b)(ab+c^2)\sin 2\alpha - c(a^2-b^2)\cos 2\alpha - c(a^2-b^2)}{2(a^2+b^2+c^2-ab)\sin 2\alpha - 2c(a-b)\cos 2\alpha - 4c(a-b)} \\ y(\alpha) = \frac{c(a-b)^2\sin 2\alpha - (a-b)(ab+c^2)\cos 2\alpha - (a-b)(c^2-ab)}{2(a^2+b^2+c^2-ab)\sin 2\alpha - 2c(a-b)\cos 2\alpha - 4c(a-b)}. \end{cases} \tag{2.3}$$

By substitution, we show that these coordinates verifies equation (1.1). □

Remark 2.3. For an isosceles triangle, we have $b = -a$, $c \neq -a\sqrt{3}$, and we obtain $x(\alpha) = 0$, thus the locus is the y axis. If the triangle is equilateral, we have $b = -a$, $c = -a\sqrt{3}$, and therefore $x(\alpha) = 0$ and $y(\alpha) = -\frac{a\sqrt{3}}{3}$, thus the locus is the center of the triangle.

Lemma 2.2. *The perpendiculars from the point A at line $B'C'$, from point B at line $A''C''$ and from point C at line $A'B''$ intersect at a unique point U_α.*

Proof. We consider that the vertices of the triangle are $A(a,0)$, $B(b,0)$ şi $C(0,c)$ (Figure 3).

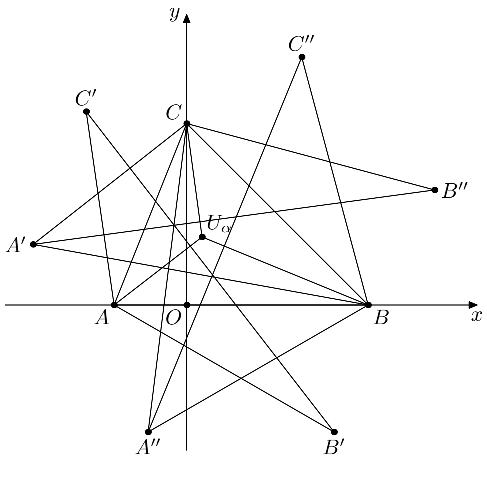


FIGURE 3

We have $A'(a\cos\alpha - c\sin\alpha, c(1-\cos\alpha) - a\sin\alpha)$ and $B''(b\cos\alpha + c\sin\alpha, c(1-\cos\alpha) + b\sin\alpha)$. The slope of $A'B''$ line is $m_{A'B''} = \frac{(a+b)\sin\alpha}{2c\sin\alpha + (b-a)\cos\alpha}$. The slope of the perpendicular from C at $A'B''$ is $-\frac{1}{m_{A'B''}} = \frac{(a-b)\cos\alpha - 2c\sin\alpha}{(a+b)\cos\alpha}$, and abscissa of intersection point of this perpendicular and the line AB is $\frac{c(a+b)\sin\alpha}{2c\sin\alpha - (a-b)\cos\alpha}$. We obtain that this perpendicular divides the side $[AB]$ in the ratio $\frac{\sin B \cos(A-\alpha)}{\sin A \cos(B-\alpha)}$. In the same way, we show that the perpendicular from A at $B'C'$ divides the side $[BC]$ in the ratio $\frac{\sin C \cos(B-\alpha)}{\sin B \cos(C-\alpha)}$, and the perpendicular from B at $A''C''$ divides the side $[AC]$ in the ratio $\frac{\sin A \cos(C-\alpha)}{\sin C \cos(A-\alpha)}$.

Since the product of three ratios is equal to 1, we deduce, based on Ceva theorem [5], that the three perpendiculars intersect at a single point.

□

Lemma 2.3. *The perpendiculars from A at line $A'A''$, from B at line $B'B''$ and from C at line $C'C''$ intersect at a single point V_α.*

Proof. We have $C'(a(1-\cos\alpha)-c\sin\alpha, c\cos\alpha-a\sin\alpha)$ and $C''(b(1-\cos\alpha)+c\sin\alpha, c\cos\alpha+b\sin\alpha)$ (Figure 4). The slope of line $C'C''$ is $m_{C'C''} = \frac{(a+b)\sin\alpha}{2c\sin\alpha+(b-a)(1-\cos\alpha)}$. The slope of perpendicular from C at $C'C''$ is $-\frac{1}{m_{C'C''}} = \frac{(a-b)(1-\cos\alpha)-2c\sin\alpha}{(a+b)\sin\alpha}$, and abscissa of point of intersection of this perpendicular and line AB is $\frac{c(a+b)\sin\alpha}{2c\sin\alpha-(a-b)(1-\cos\alpha)}$. We obtain that this perpendicular divides the side $[AB]$ in the ratio $\frac{\sin B\sin\left(A+\frac{\alpha}{2}\right)}{\sin A\sin\left(B+\frac{\alpha}{2}\right)}$.

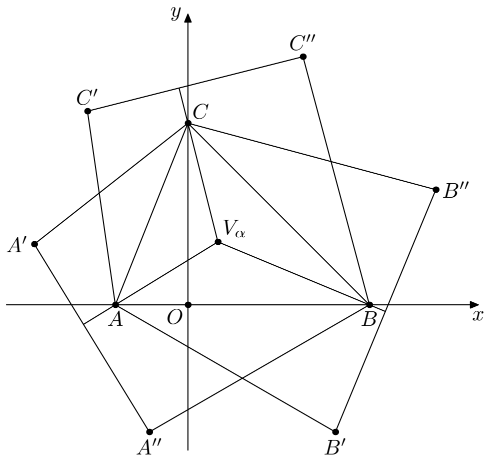


FIGURE 4

In the same way, we show that the perpendicular from A at $A'A''$ divides the side $[BC]$ in the ratio $\frac{\sin C\sin\left(B+\frac{\alpha}{2}\right)}{\sin B\sin\left(C+\frac{\alpha}{2}\right)}$, and the perpendicular from B at $B'B''$ divides the side $[AC]$ in the ratio $\frac{\sin A\sin\left(C+\frac{\alpha}{2}\right)}{\sin C\sin\left(A+\frac{\alpha}{2}\right)}$.

Since the product of three ratios is equal with 1, we deduce, base on Ceva theorem, that the three perpendiculars intersect at a single point. □

Theorem 2.2. *The points of intersection U_α and V_α are on the hyperbola (1.1), when α change from $0°$ to $360°$.*

Proof. For U_α, we have $m = \frac{\sin C\cos(B-\alpha)}{\sin B\cos(C-\alpha)}$, $n = \frac{\sin A\cos(C-\alpha)}{\sin C\cos(A-\alpha)}$, $p = \frac{\sin B\cos(A-\alpha)}{\sin A\cos(B-\alpha)}$ and its coordinates are

$$\begin{cases} x(\alpha) = \frac{(a+b)(ab+c^2)\sin 2\alpha-c(a^2-b^2)\cos 2\alpha+c(a^2-b^2)}{2(a^2+b^2+c^2-ab)\sin 2\alpha-2c(a-b)\cos 2\alpha+4c(a-b)} \\ y(\alpha) = \frac{c(a-b)^2\sin 2\alpha-(a-b)(ab+c^2)\cos 2\alpha+(a-b)(c^2-ab)}{2(a^2+b^2+c^2-ab)\sin 2\alpha-2c(a-b)\cos 2\alpha+4c(a-b)}. \end{cases} \tag{2.4}$$

For V_α, we have $m = \frac{\sin C\sin\left(B+\frac{\alpha}{2}\right)}{\sin B\sin\left(C+\frac{\alpha}{2}\right)}$, $n = \frac{\sin A\sin\left(C+\frac{\alpha}{2}\right)}{\sin C\sin\left(A+\frac{\alpha}{2}\right)}$, $p = \frac{\sin B\sin\left(A+\frac{\alpha}{2}\right)}{\sin A\sin\left(B+\frac{\alpha}{2}\right)}$ and its coordinates are

$$\begin{cases} x(\alpha) = \frac{(a+b)(ab+c^2)\sin\alpha-c(a^2-b^2)\cos\alpha-c(a^2-b^2)}{2(a^2+b^2+c^2-ab)\sin\alpha-2c(a-b)\cos\alpha-4c(a-b)} \\ y(\alpha) = \frac{c(a-b)^2\sin\alpha-(a-b)(ab+c^2)\cos\alpha-(a-b)(c^2-ab)}{2(a^2+b^2+c^2-ab)\sin\alpha-2c(a-b)\cos\alpha-4c(a-b)}. \end{cases} \tag{2.5}$$

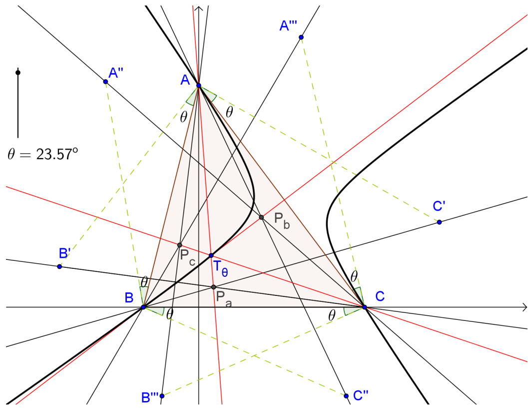
By substitution, we show that these coordinates verifies equation (1.1). □

Remark 2.4. We have $T_\alpha = S_{180^\circ - 2\alpha}$, $U_\alpha = S_{2\alpha}$ and $V_\alpha = S_{180^\circ - \alpha}$.

Remark 2.5. We have $G = S_{90^\circ}$ and $H = T_{0^\circ}$.

Remark 2.6. Torricelli point is S_{60°, and Vecten point ([5], p. 47) is S_{45°.

An image of this locus is (https://www.geogebratube.org/student/mJ4KHBYV0):



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