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Bienlargements on generalized topological spaces

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ABSTRACT. The aim of this paper is to introduce and study the concept of bienlargement on generalized topological spaces.

1. INTRODUCTION AND PRELIMINARIES

Let X be a nonempty set and μ be a collection of subsets of X. Then μ is called a generalized topology on X if $\emptyset \in \mu$ and $G_i \in \mu$ for $i \in I \neq \emptyset$ implies $\underset{i\in I}{\cup} G_i \in \mu$. We call the pair (X, μ) a generalized topological spaces on X. The members of μ are called μ-open sets [2] and the complement of a μ-open set is called a μ-closed set. The generalized-closure of a set A of X, denoted by $c_\mu(A)$, is the intersection of all μ-closed sets containing A and the generalized-interior of A, denoted by $i_\mu(A)$, is the union of μ-open sets included in A. A mapping $\kappa : \mu \to P(X)$ is called an enlargement [4] on X if $M \subset \kappa M$ (= $\kappa(M)$) whenever $M \in \mu$. Let μ be a generalized topology on X and $\kappa : \mu \to P(X)$ an enlargement of μ. Let us say that a subset $A \subset X$ is κ_μ-open [4] if and only if $x \in A$ implies the existence of a μ-open set M such that $x \in M$ and $\kappa M \subset A$. The collection of all κ_μ-open sets is a generalized topology on X and is denoted by κ_μ [4]. A subset $A \subset X$ is said to be κ_μ-closed if and only if $X \backslash A$ is κ_μ-open [4]. The set c_{κ_μ} (briefly $c_\kappa A$) is defined in [4] as the following:

$$c_\kappa(A) = \{x \in X : \kappa(M) \cap A \neq \emptyset \text{ for every } \mu\text{-open set M containing x}\}.$$

The aim of this paper is introduce and study the concept of bienlargement on generalized topological spaces.

2. ON $(\kappa, \kappa')_\mu$-OPEN SETS

In this section, in similar form as in [6], we introduce the concept of $(\kappa, \kappa')_\mu$-open set induced by two enlargements κ, κ' of an generalized topology μ, and study some basic properties of this sets.

Definition 2.1. [1] An enlargement κ on μ is said to be regular if for any μ-open sets U, V containing $x \in X$, there exists a μ-open set W containing x such that $\kappa(U) \cap \kappa(V) \supset \kappa(W)$.

Definition 2.2. [1] An enlargement κ on μ is said to be open if for every μ-open set U containing each $x \in X$, there exists a κ_μ-open set B such that $x \in B$ and $\kappa(U) \supset B$.

Lemma 2.1. [4] *Let A be a generalized topological space (X, μ), then we have $A \subset c_\mu(A) \subset c_\kappa(A) \subset c_{\kappa_\mu}(A)$.*

In the sequel κ, κ' are enlargements defined from μ to the power set of X and λ, λ' are enlargements defined from ν to the power set of Y.

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Definition 2.3. A subset A of a generalized topological space (X, μ) is said to be $(\kappa, \kappa')_\mu$-open if for each $x \in A$ there exist μ-open sets U and V containing x such that $\kappa(U) \cup \kappa'(V) \subset A$.

Proposition 2.1. *Let A be a subset of a generalized topological space (X, μ). Then we have the following*

(1) *A is $(\kappa, \kappa')_\mu$-open if and only if A is κ_μ-open and κ'_μ-open.*
(2) *If A is $(\kappa, \kappa')_\mu$-open , then A is μ-open.*
(3) *If A_i is $(\kappa, \kappa')_\mu$-open for every $i \in \Lambda$, then $\cup\{A_i : i \in \Lambda\}$ is $(\kappa, \kappa')_\mu$-open.*
(4) *The following statements are equivalent:*
   (a) A is $(\kappa, \kappa)_\mu$-open.
   (b) A is κ_μ-open.
   (c) A is $(\kappa, id)_\mu$-open, where $id : \mu \to P(X)$ is the identity enlargement, that is, $id(A) = A$ for every $A \in \mu$.

Proof. (1) (Necessity) Let $x \in A$. Then there exist μ-open sets U and V containing x such that $\kappa(U) \cup \kappa'(V) \subset A$, and so $\kappa(U) \subset A$ and $\kappa'(V) \subset A$. This implies that A is κ_μ-open and κ'_μ-open.
(Sufficiency) Let $x \in A$. It follows from assumptions that there exist μ-open sets U and V containing x such that $\kappa(U) \subset A$ and $\kappa'(V) \subset A$ and so $\kappa(U) \cup \kappa'(V) \subset A$. Therefore A is $(\kappa, \kappa')_\mu$-open.
(2) Since $\kappa_\mu \subset \mu$ [[4], Proposition 1.2] and A is κ_μ-open by (1), A is μ-open.
(3) Let $x \in \cup\{A_i : i \in \Lambda\}$. Then there exist μ-open sets W and S containing x such that $\kappa(W) \cup \kappa'(S) \subset A_i$ for some $i \in \Lambda$. Therefore $\kappa(W) \cup \kappa'(S) \subset \cup\{A_i : i \in \Lambda\}$ and so $\cup\{A_i : i \in \Lambda\}$ is $(\kappa, \kappa')_\mu$-open.
(4) (a)$\Leftrightarrow$(b) is shown by setting $\kappa' = \kappa$ in (1).
(b)$\Leftrightarrow$(c) is shown by definitions. □

Definition 2.4. $(\kappa, \kappa')_\mu$ denotes the set of all $(\kappa, \kappa')_\mu$-open sets of (X, μ).

According the Definition 2.4, the following relation is shown by Proposition 2.1,

$$(\kappa, \kappa')_\mu = \kappa_\mu \cap \kappa'_\mu \subset \mu \quad (\star)$$

Definition 2.5. A generalized topological space (X, μ) is called (κ, κ')-regular if for each point x of X and every μ-open set U containing x there exist μ-open sets W and S containing x such that $\kappa(W) \cup \kappa'(S) \subset U$.

Lemma 2.2. *Let $\kappa : \mu \to P(X)$ be an enlargement on a generalized topological space (X, μ). Then, (X, μ) is a κ-regular space if and only if $\mu = \kappa_\mu$ holds.*

Proof. Necessity: It is sufficient to prove that $\kappa_\mu \supset \mu$. Let A be a μ-open set. For any $x \in A$, there exists a μ-open set U containing x such that $U \subset A$. By the κ-regularity, there exists a μ-open set W containing x such that $\kappa(W) \subset U$. Thus, for each $x \in A$, we have $x \in W$ and $\kappa(W) \subset A$. Then A is κ_μ-open. Therefore, we have $\mu \subset \kappa_\mu$.
Sufficiency: For each $x \in X$ and for each μ-open set V containing x since $V \in \mu = \kappa_\mu$ there exists a μ-open set W containing x such that $\kappa(W) \subset V$. This implies that (X, μ) is κ-regular. □

Proposition 2.2. *Let $\kappa : \mu \to P(X)$ and $\kappa' : \mu \to P(X)$ be two enlargements on a generalized topological space (X, μ). Then,*

(1) *(X, μ) is (κ, κ')-regular if and only if $(\kappa, \kappa')_\mu = \mu$ holds.*
(2) *(X, μ) is (κ, κ')-regular if and only if it is κ-regular and κ'-regular.*

(3) *The following statements are equivalent:*
   (a) (X, μ) *is* (κ, κ)*-regular.*
   (b) (X, μ) *is* κ*-regular.*
   (c) (X, μ) *is* (κ, id)*-regular.*

Proof. (1)(Necessity) By using $(\star)$ see Definition 2.4, it is sufficient to prove $\mu \subset (\kappa, \kappa')_\mu$. Let $A \in \mu$ and $x \in A$. There exist μ-open sets W and S containing x such that $\kappa(W) \cup \kappa'(S) \subset A$. This implies that A is $(\kappa, \kappa')_\mu$-open.
(Sufficiency) Let $x \in X$ and let U be a μ-open set containing x. Since U is $(\kappa, \kappa')_\mu$-open, it is shown that (X, μ) is (κ, κ')-regular.
(2) By using (1) and $(\star)$, (X, μ) is (κ, κ')-regular if and only if $\mu = \kappa_\mu = \kappa'_\mu = (\kappa, \kappa')_\mu$ holds. By using Lemma 2.2 the proof is complete.
(3) Since $(\kappa, \kappa')_\mu = \kappa_\mu = (\kappa, id)_\mu \subset \mu$ holds in general, the equivalences are proved by using (1). □

Definition 2.6. A subset F of a generalized topological space (X, μ) is said to be $(\kappa, \kappa')_\mu$-closed if its complement $X \backslash F$ is $(\kappa, \kappa')_\mu$-open. Let $\mathcal{F}_{(\kappa,\kappa')}$ be the set of all $(\kappa, \kappa')_\mu$-closed sets of (X, μ), that is, $\mathcal{F}_{(\kappa,\kappa')} = \{F : X \backslash F \in (\kappa, \kappa')_\mu\}$.

Definition 2.7. For a subset A of (X, μ) and $(\kappa, \kappa')_\mu$, $c_{(\kappa,\kappa')_\mu}(A)$ denotes the intersection of all $(\kappa, \kappa')_\mu$-closed sets containing A, that is $c_{(\kappa,\kappa')_\mu}(A) = \cap\{F : A \subset F \text{ for all } F \in \mathcal{F}_{(\kappa,\kappa')}\}$.

Proposition 2.3. *Let* $\kappa : \mu \to P(X)$ *and* $\kappa' : \mu \to P(X)$ *be two enlargements and* A *a subset of* X.
   (1) *For a point* x *of* X, $x \in c_{(\kappa,\kappa')_\mu}(A)$ *if and only if* $V \cap A \neq \emptyset$ *for every* $(\kappa, \kappa')_\mu$*-open set* V *containing* x.
   (2) $A \subset c_{(\kappa,\kappa')_\mu}(A)$.
   (3) $A \in \mathcal{F}_{(\kappa,\kappa')}$ *if and only if* $c_{(\kappa,\kappa')_\mu}(A) = A$.
   (4) *The set* $c_{(\kappa,\kappa')_\mu}(A)$ *is a* $(\kappa, \kappa')_\mu$*-closed set of* (X, μ)
   (5) *if* $A \subset B$, *then* $c_{(\kappa,\kappa')_\mu}(A) \subset c_{(\kappa,\kappa')_\mu}(B)$.

Proof. (1). Denote $E = \{y \in X : V \cap A \neq \emptyset \text{ for every } V \in (\kappa, \kappa')_\mu \text{ such that } y \in V\}$. We shall prove that $c_{(\kappa,\kappa')_\mu}(A) = E$. Let $x \notin E$. Then there exists a $(\kappa, \kappa')_\mu$-open set V containing x such that $V \cap A = \emptyset$. This implies that $X \backslash V$ is $(\kappa, \kappa')_\mu$-closed and $A \subset X \backslash V$. Hence $c_{(\kappa,\kappa')_\mu}(A) \subset X \backslash V$. It follows that $x \notin c_{(\kappa,\kappa')_\mu}(A)$. Thus we have that $c_{(\kappa,\kappa')_\mu}(A) \subset E$. Conversely, let $x \notin c_{(\kappa,\kappa')_\mu}(A)$. Then there exists a $(\kappa, \kappa')_\mu$-closed set F such that $A \subset F$ and $x \notin F$. Then we have that $x \in X \backslash F$, $X \backslash F \in (\kappa, \kappa')_\mu$ and $(X \backslash F) \cap A = \emptyset$. This implies that $x \notin E$. Hence $E \subset c_{(\kappa,\kappa')_\mu}(A)$. Therefore $c_{(\kappa,\kappa')_\mu}(A) = E$.
(2). Follows from Definition 2.7.
(3). Necessity: It is straightforward by definitions. Sufficiency: It suffices to prove that $X \backslash A$ is $(\kappa, \kappa')_\mu$-open. Let $x \in X \backslash A$. It follows from the assumption that $x \notin c_{(\kappa,\kappa')_\mu}(A)$ and so there exists a $(\kappa, \kappa')_\mu$-closed set F containing A such that $x \notin F$. This implies that there exists a $(\kappa, \kappa')_\mu$-open set $X \backslash F$ containing x such that $X \backslash A \supset X \backslash F$. Hence $X \backslash A$ is $(\kappa, \kappa')_\mu$-open by Proposition 2.1(3).
(4). Let $x \in c_{(\kappa,\kappa')_\mu}(c_{(\kappa,\kappa')_\mu}(A))$ and V be any $(\kappa, \kappa')_\mu$-open set containing x. By (1), $V \cap c_{(\kappa,\kappa')_\mu}(A) \neq \emptyset$ and hence $V \cap A \neq \emptyset$. This shows that $x \in c_{(\kappa,\kappa')_\mu}(A)$. Therefore, by using (1) and (2) we obtain $c_{(\kappa,\kappa')_\mu}(c_{(\kappa,\kappa')_\mu}(A)) = c_{(\kappa,\kappa')_\mu}(A)$ and hence by (3) the proof completes. (5). Follows from Definition 2.7. □

Definition 2.8. For a subset A of (X,μ),

$c_{(\kappa,\kappa')}(A) = \{x \in X : (\kappa(U) \cup W^{\kappa'}) \cap A \neq \emptyset,$ for every μ -open sets U, Wsuch that $x \in U \cap W\}$.

Theorem 2.1. *Let A be a subset of (X,μ). Then $c_{(\kappa,\kappa')_\mu}(A) = c_\kappa(A) \cup c_{\kappa'}(A)$ holds, where $c_\kappa(A)$ and $c_{\kappa'}(A)$ are κ-closure and κ'-closure of A, respectively.*

Proof. By Definition 2.8, it is shown that the following statements (1)-(5) are equivalent:

(1) $x \notin c_{(\kappa,\kappa')}(A)$.
(2) There exist μ-open sets U and W containing x such that $(\kappa(U) \cup \kappa'(W)) \cap A = \emptyset$.
(3) There exist μ-open sets U and W containing x such that $\kappa(U) \cap A = \emptyset$ and $\kappa'(W) \cap A = \emptyset$.
(4) $x \notin c_\kappa(A)$ and $x \notin c_{\kappa'}(A)$.
(5) $x \notin c_\kappa(A) \cup c_{\kappa'}(A)$.

□

Theorem 2.2. *Let $\kappa : \mu \to P(X)$ and $\kappa' : \mu \to P(X)$ be two enlargements and A a subset of X. Then we have the following:*

(1) A is $(\kappa,\kappa')_\mu$-closed if and only if $c_{(\kappa,\kappa')}(A) = A$.
(2) $c_{(\kappa,\kappa')_\mu}(A) = A$ if and only if $c_{(\kappa,\kappa')}(A) = A$.
(3) A is $(\kappa,\kappa')_\mu$-open if and only if $c_{(\kappa,\kappa')}(X \backslash A) = X \backslash A$.

Proof. (1) Necessity: It suffices to prove that $c_{(\kappa,\kappa')}(A) \subset A$. Let $x \notin A$. Then, its complement $X \backslash A$ is a $(\kappa,\kappa')_\mu$-open set containing x. There exist μ-open sets U and W containing x such that $\kappa(U) \cup \kappa'(W) \subset X \backslash A$ and so $(\kappa(U) \cup \kappa'(W)) \cap A = \emptyset$. This shows that $x \notin \mathrm{Cl}_{(\kappa,\kappa')}(A)$.
Sufficiency: Let $x \in X \backslash A$. Since $x \notin c_{(\kappa,\kappa')}(A)$, there exist μ-open sets U and W containing x such that $(\kappa(U) \cup \kappa'(W)) \cap A = \emptyset$ and so $\kappa(U) \cup \kappa'(W) \subset X \backslash A$. This shows that $X \backslash A$ is $(\kappa,\kappa')_\mu$-open; hence A is (κ,κ')-μ-closed.
(2) It is proved by (1) and Proposition 2.3 (3).
(3) The proof is obvious from (1) and Definition 2.6. □

Proposition 2.4. *For a subset A of a generalized topological space (X,μ), the following properties hold:*

(1) $A \subset c_\mu(A) \subset c_{(\kappa,\kappa')}(A) \subset c_{(\kappa,\kappa')_\mu}(A)$.
(2) If (X,μ) is (κ,κ')-regular, then $c_\mu(A) = c_{(\kappa,\kappa')}(A) = c_{(\kappa,\kappa')_\mu}(A)$.
(3) $c_{(\kappa,\kappa')}(A)$ is a μ-closed subset of (X,μ).
(4) $c_{(\kappa,\kappa')_\mu}(c_{(\kappa,\kappa')}(A)) = c_{(\kappa,\kappa')_\mu}(A) = c_{(\kappa,\kappa')}(c_{(\kappa,\kappa')_\mu}(A))$.

Proof. (1) By Theorem 2.1 and Lemma 2.1, it is shown that $c_{(\kappa,\kappa')}(A) = c_\kappa(A) \cup c_{\kappa'}(A) \supset c_\mu(A)$. It follows from Definition 2.7, that $c_{(\kappa,\kappa')_\mu}(A) \supset c_{\kappa_\mu}(A) \supset c_\kappa(A)$ and $c_{(\kappa,\kappa')_\mu}(A) \supset c_{\kappa'}(A)$ similarly. This shows that $c_{(\kappa,\kappa')}(A) \subset c_{(\kappa,\kappa')_\mu}(A)$ by Theorem 2.1.
(2) By Proposition 2.2 $\mu = (\kappa,\kappa')_\mu$ and hence $c_{(\kappa,\kappa')_\mu}(A) = c_\mu(A)$. By using (1) it is shown that $c_\mu(A) = c_{(\kappa,\kappa')}(A) = c_{(\kappa,\kappa')_\mu}(A)$.
(3) It follows from Theorem 2.1 and Proposition 1.3 of [4] that $c_\mu(c_{(\kappa,\kappa')}(A)) = c_\mu(c_\kappa(A)) \cup c_\mu(c_{\kappa'}(A)) = c_{(\kappa,\kappa')_\mu}(A)$.
(4) By Proposition 2.3 (4) and Theorem 2.2 (2), we have that $c_{(\kappa,\kappa')_\mu}(A) = c_{(\kappa,\kappa')}(c_{(\kappa,\kappa')_\mu}(A))$. It follows from (1) and Proposition 2.3 (5) that $c_{(\kappa,\kappa')}(A) \subset c_{(\kappa,\kappa')_\mu}(A) \subset c_{(\kappa,\kappa')_\mu}(c_{(\kappa,\kappa')}(A))$. By using these inclusions and Proposition 2.3 (5), (3) we obtain $c_{(\kappa,\kappa')_\mu}(c_{(\kappa,\kappa')}(A)) \subset c_{(\kappa,\kappa')_\mu}(A) \subset c_{(\kappa,\kappa')_\mu}(c_{(\kappa,\kappa')}(A))$ and hence $c_{(\kappa,\kappa')_\mu}(A) = c_{(\kappa,\kappa')_\mu}(c_{(\kappa,\kappa')}(A))$. □

3. On $((\kappa,\kappa'),(\lambda,\lambda'))$-continuous functions

In this section, we introduce a new type of continuity considering two enlargements of a generalized topology in both the domain and range of a function. As in [5] and [7], we give several characterizations for this new type of continuity.

Definition 3.9. A function $f:(X,\mu)\to(Y,\nu)$ is said to be $((\kappa,\kappa'),(\lambda,\lambda'))$-continuous if for each point x of X and each pair of μ-open sets U and V containing x, there exist ν-open sets W and S containing $f(x)$ such that $f(\kappa(U)\cup\kappa'(V))\subset\lambda(W)\cup\lambda'(S)$.

Theorem 3.3. *Let $f:(X,\mu)\to(Y,\nu)$ be a $((\kappa,\kappa'),(\lambda,\lambda'))$-continuous function. Then*

(1) $f(c_{(\kappa,\kappa')}(A))\subset c_{(\lambda,\lambda')}(f(A))$ holds for every subset A of X.

(2) For any (λ,λ')-closed set B of (Y,ν), $f^{-1}(B)$ is (κ,κ')-closed in (X,τ).

Proof. (1) Let $y\in f(c_{(\kappa,\kappa')}(A))$ and let W and S be any ν-open sets containing y. Then there exist μ-open sets U and V containing x such that $f(x)=y$ and $f(\kappa(U)\cup\kappa'(V))\subset\lambda(W)\cup\lambda'(S)$. Since $x\in c_{(\kappa,\kappa')}(A)$, we have $(\kappa(U)\cup\kappa'(V))\cap A\neq\emptyset$. Hence $\emptyset\neq f((\kappa(U)\cup\kappa'(V))\cap A)\subset f(\kappa(U)\cup\kappa'(V))\cap f(A)\subset(\lambda(W)\cup\lambda'(S))\cap f(A)$. This implies that $y\in c_{(\lambda,\lambda')}(f(A))$. Therefore $f(c_{(\kappa,\kappa')}(A))\subset c_{(\lambda,\lambda')}f((A))$.
(2) Let B be a (λ,λ')-closed subset of (Y,ν). Then $c_{(\lambda,\lambda')}(B)=B$. By using (1) we have $f(c_{(\kappa,\kappa')}(f^{-1}(B)))\subset c_{(\lambda,\lambda')}(f(f^{-1}(B)))\subset c_{(\lambda,\lambda')}(B)=B$. Therefore, we have $c_{(\kappa,\kappa')}(f^{-1}(B))\subset f^{-1}(B)$. Hence $f^{-1}(B)=c_{(\kappa,\kappa')}(f^{-1}(B))$ implies that $f^{-1}(B)$ is (κ,κ')-closed in (X,τ). □

Theorem 3.4. *If a function $f:(X,\mu)\to(Y,\nu)$ is $((\kappa,\kappa'),(\lambda,\lambda'))$-continuous on X, then for every (λ,λ')-open subset V in Y, $f^{-1}(V)$ is (κ,κ')-open in X.*

Proof. Suppose that f is $((\kappa,\kappa'),(\lambda,\lambda'))$-continuous on X. Let V be any (λ,λ')-open subset of Y. We show that $f^{-1}(V)$ is (κ,κ')-open in X. Let $x\in f^{-1}(V)$. Then $f(x)\in V$, which is (λ,λ')-open in Y. Since V is (λ,λ')-open, there exist ν-open sets W and S containing $f(x)$ such that $\lambda(W)\cup\lambda'(S)\subset V$. Now by the $((\kappa,\kappa'),(\lambda,\lambda'))$-continuity of f, there exist μ-open sets U and T containing x such that $f(\kappa(U)\cup\kappa'(T))\subset\lambda(W)\cup\lambda'(S)$. Thus $f(\kappa(U)\cup\kappa'(T))\subset V$ implies $\kappa(U)\cup\kappa'(T)\subset f^{-1}(V)$. This implies that $f^{-1}(V)$ is (κ,κ')-open in X. □

Theorem 3.5. *A function $f:(X,\mu)\to(Y,\nu)$ is $((\kappa,\kappa'),(\lambda,\lambda'))$-continuous on X if for every (λ,λ')-open subset V in Y, $f^{-1}(V)$ is (κ,κ')-open in X, where λ and λ' are open enlargements.*

Proof. For any $x\in X$, let V and W be ν-open sets containing $f(x)$. Since λ and λ' are open, there exists a λ-open set B_1 and a λ'-open set B_2 such that

(1) $f(x)\in B_1\subset\lambda(V)$ and
(2) $f(x)\in B_2\subset\lambda'(W)$.

Since B_1 and B_2 are λ-open and λ'-open respectively, $f^{-1}(B_1)$ and $f^{-1}(B_2)$ are κ-open and κ'-open in X and $x\in f^{-1}(B_1)$ and $x\in f^{-1}(B_2)$. This gives that there exist μ-open sets U and S containing x such that $\kappa(U)\subset f^{-1}(B_1)$ and $\kappa'(S)\subset f^{-1}(B_2)$. By (1) and (2) we have $\kappa(U)\subset f^{-1}(\lambda(V))$ and $\kappa'(S)\subset f^{-1}(\kappa'(W))$. This implies that $\kappa(U)\cup\kappa'(S)\subset f^{-1}(\lambda(V))\cup f^{-1}(\lambda'(W))\subset f^{-1}(\lambda(V)\cup\lambda'(W))$ gives $f(\kappa(U)\cup\kappa'(S))\subset f(f^{-1}(\lambda(V)\cup\lambda'(W)))\subset(\lambda(V)\cup\lambda'(W))$. This shows that f is $((\kappa,\kappa'),(\lambda,\lambda'))$-continuous. □

Theorem 3.6. *Let $f:(X,\mu)\to(Y,\nu)$ be a function and λ, λ' be open enlargements. Then f is $((\kappa,\kappa'),(\lambda,\lambda'))$-continuous if and only if for every (λ,λ')-open subset V in Y, $f^{-1}(V)$ is (κ,κ')-open in X.*

Proof. The proof follows from Theorems 3.4 and 3.5. □

Theorem 3.7. *If for every (λ, λ')-closed subset C in Y, $f^{-1}(C)$ is (κ, κ')-closed in X, then the function $f : (X, \mu) \to (Y, \nu)$ is $((\kappa, \kappa'), (\lambda, \lambda'))$-continuous on X, where λ and λ' are open enlargements.*

Proof. Suppose that for any (λ, λ')-closed set C in Y, $f^{-1}(C)$ is (κ, κ')-closed in X. Let V be any (λ,λ')-open subset of Y. Then $Y\backslash V$ is (λ, λ')-closed and $f^{-1}(Y\backslash V) = f^{-1}(Y)\backslash f^{-1}(V) = X\backslash f^{-1}(V)$ gives $X\backslash f^{-1}(V)$ is (κ, κ')-closed in X. That is, $f^{-1}(V)$ is (κ, κ')-open in X. Thus by Theorem 3.6, f is $((\kappa, \kappa'), (\lambda, \lambda'))$-continuous. □

Theorem 3.8. *Let $f : (X, \mu) \to (Y, \nu)$ be a function and λ, λ' be open enlargements. Then f is $((\kappa, \kappa'), (\lambda, \lambda'))$-continuous if and only if for every (λ, λ')-closed subset C in Y, $f^{-1}(C)$ is (κ, κ')-closed in X.*

Proof. The proof follows from Theorem 3.7 and Theorem 1.2 of [3]. □

Theorem 3.9. *If $f(c_{(\kappa,\kappa')}(A)) \subset c_{(\lambda,\lambda')}(f(A))$ holds for every subset A of X. Then $f : (X, \mu) \to (Y, \nu)$ is $((\kappa, \kappa'), (\lambda, \lambda'))$-continuous, where λ, λ' be open enlargements.*

Proof. Suppose that for any subset A of X, $f(c_{(\kappa,\kappa')}(A)) \subset c_{(\lambda,\lambda')}(f(A))$. We apply Theorem 3.7 to prove that f is $((\kappa, \kappa'), (\lambda, \lambda'))$-continuous. Let C be (λ, λ')-closed subset of Y and $A = f^{-1}(C)$. We show that A is (κ, κ')-closed in X by proving that $c_{(\kappa,\kappa')}(A) = A$. By assumption, $f(c_{(\kappa,\kappa')}(A)) = f(c_{(\kappa,\kappa')}(f^{-1}(C))) \subset c_{(\lambda,\lambda')}(f(f^{-1}(C))) \subset c_{(\lambda,\lambda')}(C) = C$, because C is (λ, λ')-closed. So that $c_{(\kappa,\kappa')}(A) \subset f^{-1}(f(c_{(\kappa,\kappa')}(A))) \subset f^{-1}(C) = A \subset c_{(\kappa,\kappa')}(A)$. Hence $f^{-1}(C) = A = c_{(\kappa,\kappa')}(A)$ and hence A is (κ, κ')-closed in X. Thus f is $((\kappa, \kappa'), (\lambda, \lambda'))$-continuous. □

Theorem 3.10. *Let $f : (X, \mu) \to (Y, \nu)$ be a function and λ, λ' be μ-open enlargements. Then f is $((\kappa, \kappa'), (\lambda, \lambda'))$-continuous if and only if $f(c_{(\kappa,\kappa')}(A)) \subset c_{(\lambda,\lambda')}(f(A))$ holds for every subset A of X.*

Proof. The proof follows from Theorem 3.9 and Theorem 1.2 of [2]. □

For a subset A of a generalized topological space (X, μ), we denote:

(1) $i_{(\kappa,\kappa')}(A) = \{x \in A : \text{there exist } U, V \in \mu \text{ containing } x \text{ such that } \kappa(U) \cup \kappa'(V) \subset A\}$.
(2) $b_{(\kappa,\kappa')}(A) = \{x \in X : x \notin i_{(\kappa,\kappa')}(A) \text{ or } x \notin i_{(\kappa,\kappa')}(X\backslash A)\}$.
(3) $e_{(\kappa,\kappa')}(A) = i_{(\kappa,\kappa')}(X\backslash A)$.

Theorem 3.11. *For a function $f : (X, \mu) \to (Y, \nu)$, the following properties are equivalent:*

(1) f is $((\kappa, \kappa'), (\lambda, \lambda'))$-continuous.
(2) $f^{-1}(i_{(\lambda,\lambda')}(B)) \subset i_{(\kappa,\kappa')}(f^{-1}(B))$ for any subset B of Y.
(3) $c_{(\kappa,\kappa')}(f^{-1}(B)) \subset f^{-1}(c_{(\lambda,\lambda')}(B))$ for any subset B of Y.
(4) $b_{(\kappa,\kappa')}f^{-1}(B) \subset f^{-1}(b_{(\lambda,\lambda')}(B))$ for any subset B of Y.

Proof. (1)⇒ (2): Let B be any subset of Y. Suppose that f is $((\kappa, \kappa'), (\lambda, \lambda'))$-continuous and $x \in f^{-1}(i_{(\lambda,\lambda')}(B))$. Then $f(x) \in i_{(\lambda,\lambda')}(B)$. Then $f(x) \in i_{(\lambda,\lambda')}(B)$. Therefore, there exist ν-open sets U and V containing $f(x)$ such that $\lambda(U) \cup \lambda'(V) \subset B$. Since f is $((\kappa, \kappa'), (\lambda, \lambda'))$-continuous, there exist μ-open sets W and T containing x such that $f(\kappa(W) \cup \kappa'(T)) \subset \lambda(U) \cup \lambda'(V)$. Then $f(\kappa(W) \cup \kappa'(T)) \subset B$. This gives $\kappa(W) \cup \kappa'(T) \subset f^{-1}(B)$ which implies $x \in i_{(\kappa,\kappa')}(f^{-1}(B))$.
(2)⇒(1): For each $x \in X$, let V and W be ν-open sets containing $f(x)$. Since λ and λ' are open enlargements, there exists a λ-open set S_1 and a λ'-open set S_2 such that

$f(x) \in S_1 \subset \lambda(V)$ and $f(x) \in S_2 \subset \lambda'(V)$. This implies $f(x) \in S_1 \cup S_2 \subset \lambda(V) \cup \lambda'(W)$. Since S_1 is λ-open and S_2 is λ'-open, $S_1 \cup S_2$ is (λ, λ')-open. Therefore $i_{(\lambda,\lambda')}(S_1 \cup S_2)$. Put $S_3 = S_1 \cup S_2$. Then $i_{(\lambda,\lambda')}(S_3) = S_3$. Now $f^{-1}(S_3) = f^{-1}(i_{(\lambda,\lambda')}(S_3)) \subset i_{(\kappa,\kappa')}(f^{-1}(S_3))$ or $f^{-1}(S_3) \subset i_{(\kappa,\kappa')}(f^{-1}(S_3))$. This proves that $f^{-1}(S_3)$ is (κ, κ')-open. In consequence, there exist μ-open sets U and T containing x such that $\kappa(U) \cup T^{\kappa'} \subset f^{-1}(S_3)$. This gives $f(\kappa(U) \cup \kappa'(T)) \subset f(f^{-1}(S_3)) \subset S_3 = S_1 \cup S_2 \subset \lambda(V) \cup \lambda'(W)$. Consequently, we have $f(\kappa(U) \cup \kappa'(T)) \subset \lambda(V) \cup \lambda'(W)$.
(2)⇒(3). Let B be any subset of Y. Then we have $c_{(\kappa,\kappa')}(f^{-1}(B)) = c_{(\kappa,\kappa')}(X \backslash (X \backslash f^{-1}(B)))$ $= X \backslash i_{(\kappa,\kappa')}(X \backslash f^{-1}(B)) = X \backslash i_{(\kappa,\kappa')}(f^{-1}(Y) \backslash f^{-1}(B)) = X \backslash i_{(\kappa,\kappa')}(f^{-1}(Y \backslash B)) \subset X \backslash f^{-1}$ $(i_{(\lambda,\lambda')}(Y \backslash B)) = f^{-1}(Y \backslash i_{(\lambda,\lambda')}(Y \backslash B)) = f^{-1}(c_{(\lambda,\lambda')}(B))$. Consequently, we have $c_{(\kappa,\kappa')}(f^{-1}(B)) = f^{-1}(c_{(\lambda,\lambda')}(B))$.
(3)⇒(4). Let B be any subset of Y. Then $b_{(\kappa,\kappa')}(f^{-1}(B)) = c_{(\kappa,\kappa')}(f^{-1}(B)) \cap c_{(\kappa,\kappa')}(X \backslash f^{-1}(B))$ $= c_{(\kappa,\kappa')}(f^{-1}(B)) \cap c_{(\kappa,\kappa')} f^{-1}(Y \backslash B) \subset f^{-1}(c_{(\lambda,\lambda')}(B)) \cap f^{-1}(c_{(\lambda,\lambda')}(Y \backslash B)) = f^{-1}((c_{(\lambda,\lambda')}(B))$ $\cap c_{(\lambda,\lambda')}(Y \backslash B)) = f^{-1}(b_{\lambda,\lambda')}(B))$. Consequently, we have $bd_{(\kappa,\kappa')}(f^{-1}(B)) \subset f^{-1}(b_{(\kappa,\kappa')}(B)$.
(4)⇒(2). Let B be any subset of Y. Then $f^{-1}(i_{(\lambda,\lambda')}(B)) = f^{-1}(B \backslash b_{(\lambda,\lambda')}(B)) = f^{-1}(B) \backslash f^{-1}$ $(b_{(\lambda,\lambda')}(B)) \subset f^{-1}(B) \backslash b_{(\kappa,\kappa')}(f^{-1}(B)) = i_{(\kappa,\kappa')}(f^{-1}(B))$. Thus $f^{-1}(i_{(\lambda,\lambda')}(B)) \subset i_{(\kappa,\kappa')}(f^{-1}(B))$. □

Theorem 3.12. *Let $f : (X, \mu) \to (Y, \nu)$ be a $((\kappa, \kappa'), (\lambda, \lambda'))$-continuous function and A be any subset of X. Then*

(1) $i_{(\lambda,\lambda')}(f(A)) \subset f(i_{(\kappa,\kappa')}(A))$ if f is injective.
(2) $f(b_{(\kappa,\kappa')}(A)) \subset b_{(\lambda,\lambda')}(f(A))$.
(3) $e_{(\lambda,\lambda')}(f(A)) \subset f(e_{(\kappa,\kappa')}(A))$ if f is bijective.

Proof. (1). Let $y \in i_{(\lambda,\lambda')}(f(A))$. Then there exist ν-open sets U and V containing $y \in Y$ such that $\lambda(U) \cup \lambda'(V) \subset f(A)$ and $y = f(x)$. By $((\kappa, \kappa'), (\lambda, \lambda'))$-continuity of f, there exist μ-open sets W and T containing x such that $f(\kappa(W) \cup \kappa'(T)) \subset \lambda(U) \cup \lambda'(V)$. Thus $f(\kappa(W) \cup \kappa'(T)) \subset f(A)$. Since f is injective, $\kappa(W) \cup \kappa'(T) \subset A$. This gives that $x \in i_{(\kappa,\kappa')}(A)$. So $f(x) \in f(i_{(\kappa,\kappa')}(A))$. Thus $i_{(\lambda,\lambda')}(f(A)) \subset f(i_{(\kappa,\kappa')}(B))$.
(2). By Theorem 3.3, we have $f(b_{(\kappa,\kappa')}(A)) = f(c_{(\kappa,\kappa')}(A) \cap c_{(\kappa,\kappa')}(X \backslash A)) \subset f(c_{(\kappa,\kappa')}(A)) \cap f(c_{(\kappa,\kappa')}(X \backslash A)) \subset c_{(\lambda,\lambda')}(f(A)) \cap c_{(\lambda,\lambda')}(f(X \backslash A)) = b_{(\lambda,\lambda')}(f(A))$. Thus $f(b_{(\kappa,\kappa')}(B)) \subset b_{(\lambda,\lambda')}(f(A))$.
(3) By (1), we have $e_{(\lambda,\lambda')}(f(A)) = i_{(\lambda,\lambda')}(Y \backslash f(A)) = i_{(\lambda,\lambda')}(f(X) \backslash f(A)) \subset i_{(\lambda,\lambda')}(f(X \backslash A)) \subset f(i_{(\kappa,\kappa')}(X \backslash A)) = f(e_{(\kappa,\kappa')}(A))$. □

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