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On the equal variables method applied to real variables

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ABSTRACT. As it is known, the equal variables method can be used to create and solve difficult symmetric inequalities in nonnegative variables involving the expressions $x_1 + x_2 + \cdots + x_n$, $x_1^k + x_2^k + \cdots + x_n^k$ and $f(x_1) + f(x_2) + \cdots + f(x_n)$, where k is a real constant, and f is a differentiable function on $(0, \infty)$ such that $g(x) = f'(x^{\frac{1}{k-1}})$ is strictly convex. In this paper, we extend the equal variables method to real variables.

1. INTRODUCTION

The Equal Variables Theorem (EV-Theorem) for nonnegative real variables has the following statement (see [2], [3]).

Theorem 1.1. *Let $a_1, a_2, \dots, a_n$ $(n \geq 3)$ be fixed nonnegative real numbers, and let*

$$0 \leq x_1 \leq x_2 \leq \cdots \leq x_n$$

such that

$$x_1 + x_2 + \cdots + x_n = a_1 + a_2 + \cdots + a_n$$

and

$$x_1^k + x_2^k + \cdots + x_n^k = a_1^k + a_2^k + \cdots + a_n^k,$$

where k is a real number; for $k = 0$, assume that

$$x_1 x_2 \cdots x_n = a_1 a_2 \cdots a_n > 0.$$

Let $f : \mathbb{I} \to \mathbb{R}$, where $\mathbb{I} = [0, \infty)$ when f is continuous at $x = 0$, and $\mathbb{I} = (0, \infty)$ when $f(0_+) = \pm\infty$. In addition, f is differentiable on $(0, \infty)$ and the associated function $g : (0, \infty) \to \mathbb{R}$ defined by

$$g(x) = f'\left(x^{\frac{1}{k-1}}\right)$$

is strictly convex on $(0, \infty)$. Let

$$S_n = f(x_1) + f(x_2) + \cdots + f(x_n).$$

(1) *If $k \leq 0$, then S_n is maximum for*

$$x_1 = x_2 = \cdots = x_{n-1} \leq x_n,$$

and is minimum for

$$x_1 \leq x_2 = x_3 = \cdots = x_n;$$

(2) *If $k > 0$ and either f is continuous at $x = 0$ or $f(0_+) = -\infty$, then S_n is maximum for*

$$x_1 = x_2 = \cdots = x_{n-1} \leq x_n;$$

(3) *If $k > 0$ and either f is continuous at $x = 0$ or $f(0_+) = \infty$, then S_n is minimum for*

$$x_1 = \cdots = x_{j-1} = 0, \quad x_{j+1} = \cdots = x_n, \quad j \in \{1, 2, \dots, n\}.$$

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From EV-Theorem, we can obtain some interesting particular results, which are useful in many applications.

Corollary 1.1. *Let* $a_1, a_2, \dots, a_n$ $(n \ge 3)$ *be fixed nonnegative real numbers, and let*

$$0 \le x_1 \le x_2 \le \cdots \le x_n$$

such that

$$x_1 + x_2 + \cdots + x_n = a_1 + a_2 + \cdots + a_n,$$
$$x_1^2 + x_2^2 + \cdots + x_n^2 = a_1^2 + a_2^2 + \cdots + a_n^2.$$

Let $f : \mathbb{I} \to \mathbb{R}$, *where* $\mathbb{I} = [0, \infty)$ *when* f *is continuous at* $x = 0$, *and* $\mathbb{I} = (0, \infty)$ *when* $f(0_+) = \pm\infty$. *In addition,* f *is differentiable on* $(0, \infty)$ *and the derivative* f' *is strictly convex on* $(0, \infty)$. *Let*

$$S_n = f(x_1) + f(x_2) + \cdots + f(x_n).$$

If either f *is continuous at* $x = 0$ *or* $f(0_+) = -\infty$, *then* S_n *is maximum for*

$$x_1 = x_2 = \cdots = x_{n-1} \le x_n.$$

If either f *is continuous at* $x = 0$ *or* $f(0_+) = \infty$, *then* S_n *is minimum for*

$$x_1 = \cdots = x_{j-1} = 0, \;\; x_{j+1} = \cdots = x_n, \quad j \in \{1, 2, \dots, n\}.$$

Corollary 1.2. *Let* $a_1, a_2, \dots, a_n$ $(n \ge 3)$ *be fixed positive real numbers, and let*

$$0 < x_1 \le x_2 \le \cdots \le x_n$$

such that

$$x_1 + x_2 + \cdots + x_n = a_1 + a_2 + \cdots + a_n,$$
$$x_1 x_2 \cdots x_n = a_1 a_2 \cdots a_n.$$

Let $f : (0, \infty) \to \mathbb{R}$ *be a differentiable function such that* $g : (0, \infty) \to \mathbb{R}$ *defined by*

$$g(x) = f'(1/x)$$

is strictly convex on $(0, \infty)$. *Then, the sum* $S_n = f(x_1) + f(x_2) + \cdots + f(x_n)$ *is maximum for*

$$x_1 = x_2 = \cdots = x_{n-1} \le x_n,$$

and is minimum for

$$x_1 \le x_2 = x_3 = \cdots = x_n.$$

Corollary 1.3. *Let* $a_1, a_2, \dots, a_n$ $(n \ge 3)$ *be fixed positive real numbers, and let*

$$0 < x_1 \le x_2 \le \cdots \le x_n$$

such that

$$x_1 + x_2 + \cdots + x_n = a_1 + a_2 + \cdots + a_n,$$
$$\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n} = \frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}.$$

Let $f : (0, \infty) \to \mathbb{R}$ *be a differentiable function such that* $g : (0, \infty) \to \mathbb{R}$ *defined by*

$$g(x) = f'(1/\sqrt{x})$$

is strictly convex on $(0, \infty)$; *in addition, assume that either* f *is continuous at* $x = 0$ *or* $f(0_+) = \pm\infty$. *Then, the sum* $S_n = f(x_1) + f(x_2) + \cdots + f(x_n)$ *is maximum for*

$$x_1 = x_2 = \cdots = x_{n-1} \le x_n,$$

and is minimum for

$$x_1 \le x_2 = x_3 = \cdots = x_n.$$

Notice that this paper deals with constrained optimization for real variables in a framework initiated by author in [2] for positive variables. A closely related framework of constrained optimization for positive variables can be found in [1], where Grahame Bennett gave the following result.

Theorem 1.2. *Suppose that a, b, c, d and w, x, y, z are positive numbers. Then the inequality*

$$a^p + b^p + c^p + d^p \le w^p + x^p + y^p + z^p$$

is valid whenever $|p| \ge 1$, and it reverses direction whenever $|p| \le 1$, if and only if the following five conditions are satisfied:

$$a + b + c + d = w + x + y + z,$$
$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} = \frac{1}{w} + \frac{1}{x} + \frac{1}{y} + \frac{1}{z},$$
$$abcd = wxyz,$$
$$\max\{a, b, c, d\} \le \max\{w, x, y, z\}, \qquad \min\{a, b, c, d\} \ge \min\{w, x, y, z\}.$$

In the following section, we will extend the EV-Theorem (Theorem 1.1) to the case where $x_1, x_2, \ldots, x_n$ are real numbers.

2. Main results

The main results of this paper are given by Theorem 2.3, Proposition 2.1 and Proposition 2.2.

Theorem 2.3. *Let $a_1, a_2, \ldots, a_n$ $(n \ge 3)$ be fixed real numbers, let $x_1, x_2, \ldots, x_n$ be real variables such that*

$$x_1 \le x_2 \le \cdots \le x_n,$$
$$x_1 + x_2 + \cdots + x_n = a_1 + a_2 + \cdots + a_n,$$
$$x_1^k + x_2^k + \cdots + x_n^k = a_1^k + a_2^k + \cdots + a_n^k,$$

where k is an even positive integer, and let f be a differentiable function on $\mathbb{R}$ such that the associated function $g : \mathbb{R} \to \mathbb{R}$ defined by

$$g(x) = f'\left(\sqrt[k-1]{x}\right)$$

is strictly convex on $\mathbb{R}$. Then, the sum

$$S_n = f(x_1) + f(x_2) + \cdots + f(x_n)$$

is minimum for $x_2 = x_3 = \cdots = x_n$, and is maximum for $x_1 = x_2 = \cdots = x_{n-1}$.

Taking $k = 2$ in Theorem 2.3, we obtain the following corollary.

Corollary 2.4. *Let $a_1, a_2, \ldots, a_n$ $(n \ge 3)$ be fixed real numbers, and let $x_1, x_2, \ldots, x_n$ be real variables such that*

$$x_1 \le x_2 \le \cdots \le x_n,$$
$$x_1 + x_2 + \cdots + x_n = a_1 + a_2 + \cdots + a_n,$$
$$x_1^2 + x_2^2 + \cdots + x_n^2 = a_1^2 + a_2^2 + \cdots + a_n^2.$$

If f is a differentiable function on $\mathbb{R}$ such that the derivative f' is strictly convex on $\mathbb{R}$, then the sum

$$S_n = f(x_1) + f(x_2) + \cdots + f(x_n)$$

is minimum for $x_2 = x_3 = \cdots = x_n$, and is maximum for $x_1 = x_2 = \cdots = x_{n-1}$.

For $f:\mathbb{R}\to\mathbb{R}$ defined by

$$f(t)=t^m,$$

where m is a positive odd number such that $m>k$, the associated function

$$g(x)=m\sqrt[k-1]{x^{m-1}}$$

is strictly convex on $\mathbb{R}$ because its derivative

$$g'(x)=\frac{m(m-1)}{k-1}\sqrt[k-1]{x^{m-k}}$$

is strictly increasing on $\mathbb{R}$. Thus, the following corollary holds.

Corollary 2.5. *Let $a_1,a_2,\dots,a_n$ $(n\ge 3)$ be fixed real numbers, and let $x_1,x_2,\dots,x_n$ be real variables such that*

$$x_1\le x_2\le\cdots\le x_n,$$
$$x_1+x_2+\cdots+x_n=a_1+a_2+\cdots+a_n,$$
$$x_1^k+x_2^k+\cdots+x_n^k=a_1^k+a_2^k+\cdots+a_n^k,$$

where k is an even positive integer. For any positive odd number m, $m>k$, the power sum

$$S_n=x_1^m+x_2^m+\cdots+x_n^m$$

is minimum for $x_2=x_3=\cdots=x_n$, and is maximum for $x_1=x_2=\cdots=x_{n-1}$.

In order to show the effectiveness of Theorem 2.3 and its corollaries, we will prove the inequality

$$\frac{(x_1^m+x_2^m+\cdots+x_n^m)^2}{(x_1^2+x_2^2+\cdots+x_n^2)^m}\le\frac{[(n-1)^{m-1}-1]^2}{n^m(n-1)^{m-2}},\tag{2.1}$$

where $x_1,x_2,\dots,x_n$ $(n\ge 3)$ are real numbers such that

$$x_1+x_2+\cdots+x_n=0$$

and $m\ge 3$ is an odd number. Due to homogeneity, we may set

$$x_1^2+x_2^2+\cdots+x_n^2=n(n-1),$$

when the inequality becomes

$$|x_1^m+x_2^m+\cdots+x_n^m|\le(n-1)^m-n+1.$$

Assume that $x_1\le x_2\le\cdots\le x_n$. According to Corollary 2.5, it suffices to consider the case when $n-1$ of $x_1,x_2,\dots,x_n$ are equal; that is, either $x_1=-n+1$ and $x_2=\cdots=x_n=1$, or $x_1=\cdots=x_{n-1}=-1$ and $x_n=n-1$. For each of these two cases, the desired inequality becomes an equality. Thus, the proof is completed. The equality holds for

$$\frac{-x_1}{n-1}=x_2=\cdots=x_n$$

(or any cyclic permutation).

Writing the inequality (2.1) for $n+1$ real numbers $x_1,x_2,\dots,x_{n+1}$ and setting then $x_{n+1}=-1$, we get the following result.

Let $x_1,x_2,\dots,x_n$ $(n\ge 2)$ be real numbers such that $x_1+x_2+\cdots+x_n=1$. If $m\ge 3$ is an odd number, then

$$\frac{(x_1^m+x_2^m+\cdots+x_n^m-1)^2}{(x_1^2+x_2^2+\cdots+x_n^2+1)^m}\le\frac{(n^{m-1}-1)^2}{n^{m-2}(n+1)^m},\tag{2.2}$$

with equality for $x_1=x_2=\cdots=x_n=1/n$, and also for $x_1=n$ and $x_2=\cdots=x_n=-1$ (or any cyclic permutation).

In our opinion, an extension of the EV-Theorem for real variables to other functions f than those in Theorem 2.3 is an interesting open problem. For instance, the function

$f(t) = t^4$ does not satisfy the condition in Theorem 2.3 because the associated function $g(x) = 4\sqrt[k-1]{x^3}$ is not convex on $\mathbb{R}$ when k is an even positive number. However, the following proposition holds.

Proposition 2.1. *Let $a_1, a_2, \ldots, a_n$ $(n \geq 3)$ be fixed real numbers, and let $x_1, x_2, \ldots, x_n$ be real variables such that*

$$x_1 + x_2 + \cdots + x_n = a_1 + a_2 + \cdots + a_n,$$
$$x_1^2 + x_2^2 + \cdots + x_n^2 = a_1^2 + a_2^2 + \cdots + a_n^2.$$

The power sum

$$S_n = x_1^4 + x_2^4 + \cdots + x_n^4$$

is minimum and maximum when at least $n-1$ of $x_1, x_2, \ldots, x_n$ are equal.

To give an application of Proposition 2.1, we will prove the inequality

$$\frac{x_1^4 + x_2^4 + \cdots + x_n^4}{(x_1^2 + x_2^2 + \cdots + x_n^2)^2} \geq \frac{n^2+3}{n(n^2-1)}, \tag{2.3}$$

where $n \geq 3$ is an odd number, and $x_1, x_2, \ldots, x_n$ are real numbers such that

$$x_1 + x_2 + \cdots + x_n = 0.$$

According to Proposition 2.1, it suffices to consider that

$$x_1 = \cdots = x_j := x, \quad x_{j+1} = \cdots = x_n := y, \qquad j \in \{1, 2, \ldots, n\}.$$

Therefore, we need to show that $jx + (n-j)y = 0$ implies

$$jx^4 + (n-j)y^4 \geq \frac{n^2+3}{n(n^2-1)}[jx^2 + (n-j)y^2]^2.$$

This inequality is equivalent to

$$(n-j)[(n-2j)^2 - 1]y^4 \geq 0,$$

which is true for any odd n, $n \geq 3$. Thus, the proof is completed. The equality holds for $j = (n-1)/2$ and

$$\frac{x_1}{n+1} = \cdots = \frac{x_j}{n+1} = \frac{-x_{j+1}}{n-1} = \cdots = \frac{-x_n}{n-1}$$

(or any permutation).

Writing the inequality in (2.3) for $n = 2k+1$ real numbers $x_1, x_2, \ldots, x_{2k+1}$ and setting then $x_{2k+1} = -k$, we get the following statement.

If $x_1, x_2, \ldots, x_{2k}$ $(k \geq 1)$ are real numbers such that $x_1 + x_2 + \cdots + x_{2k} = k$, then

$$\frac{x_1^4 + x_2^4 + \cdots + x_{2k}^4 + k^4}{(x_1^2 + x_2^2 + \cdots + x_{2k}^2 + k^2)^2} \geq \frac{k^2+k+1}{k(k+1)(2k+1)}, \tag{2.4}$$

with equality when k of $x_1, x_2, \ldots, x_{2k}$ are equal to $k+1$, and the other k are equal to $-k$.

Notice that for $f(t) = t^6$ and $f(t) = t^8$ (which also do not satisfy the condition in Theorem 2.3), the following proposition holds.

Proposition 2.2. *Let $a_1, a_2, \ldots, a_n$ $(n \geq 3)$ be fixed real numbers, and let $x_1, x_2, \ldots, x_n$ be real variables such that*

$$x_1 + x_2 + \cdots + x_n = a_1 + a_2 + \cdots + a_n,$$
$$x_1^2 + x_2^2 + \cdots + x_n^2 = a_1^2 + a_2^2 + \cdots + a_n^2.$$

For $m \in \{6, 8\}$, the power sum

$$S_n = x_1^m + x_2^m + \cdots + x_n^m$$

is maximum when at least $n-1$ of $x_1, x_2, \dots, x_n$ are equal.

Conjecture 2.1. *Proposition 2.2 is valid for any integer number $m \ge 3$.*

3. Proof of theorem 2.3

The proof of Theorem 2.3 is based on the following lemma.

Lemma 3.1. *Let a, b, c be fixed real numbers, not all equal, and let x, y, z be real numbers satisfying*

$$x \le y \le z, \quad x+y+z = a+b+c, \quad x^k + y^k + z^k = a^k + b^k + c^k,$$

where k is an even positive integer. Then, there exist two real numbers y_1 and y_2 such that $y_1 < y_2$ and

(1) $y \in [y_1, y_2]$;
(2) $y = y_1$ *if and only if* $x = y$;
(3) $y = y_2$ *if and only if* $y = z$.

Proof. We show first, by contradiction method, that $x < z$. Indeed, if $x = z$, then

$$x = z \;\Rightarrow\; x = y = z \;\Rightarrow\; x^k + y^k + z^k = 3\left(\frac{x+y+z}{3}\right)^k$$

$$\Rightarrow\; a^k + b^k + c^k = 3\left(\frac{a+b+c}{3}\right)^k \;\Rightarrow\; a = b = c,$$

which is false. Notice that the last implication follows from Jensen's inequality

$$a^k + b^k + c^k \ge 3\left(\frac{a+b+c}{3}\right)^k,$$

where equality holds if and only if $a = b = c$.

According to the relations

$$x + z = a+b+c-y, \quad x^k + z^k = a^k + b^k + c^k - y^k,$$

we may consider x and z as functions of y. From

$$x' + z' = -1, \quad x^{k-1}x' + z^{k-1}z' = -y^{k-1},$$

we get

$$x' = \frac{y^{k-1} - z^{k-1}}{z^{k-1} - x^{k-1}}, \quad z' = \frac{y^{k-1} - x^{k-1}}{x^{k-1} - z^{k-1}}. \tag{3.5}$$

The two-sided inequality

$$x(y) \le y \le z(y)$$

is equivalent to the inequalities $f_1(y) \le 0$ and $f_2(y) \ge 0$, where

$$f_1(y) = x(y) - y, \quad f_2(y) = z(y) - y.$$

Using (3.5), we get

$$f_1'(y) = \frac{y^{k-1} - z^{k-1}}{z^{k-1} - x^{k-1}} - 1$$

and

$$f_2'(y) = \frac{y^{k-1} - x^{k-1}}{x^{k-1} - z^{k-1}} - 1.$$

Since $f_1'(y) \le -1$ and $f_2'(y) \le -1$, f_1 and f_2 are strictly decreasing. Thus, the inequality $f_1(y) \le 0$ involves $y \ge y_1$, where y_1 is the root of the equation $x(y) = y$, while the inequality $f_2(y) \ge 0$ involves $y \le y_2$, where y_2 is the root of the equation $z(y) = y$. Moreover, $y = y_1$ if and only if $x = y$, and $y = y_2$ if and only if $y = z$.

Using now Lemma 3.1, we can prove the following proposition.

Proposition 3.3. *Let a, b, c be fixed real numbers, let x, y, z be real numbers satisfying*

$$x \le y \le z, \quad x+y+z=a+b+c, \quad x^k+y^k+z^k=a^k+b^k+c^k,$$

where k is an even positive integer, and let f be a differentiable function on $\mathbb{R}$ such that the associated function $g : \mathbb{R} \to \mathbb{R}$ defined by

$$g(x) = f'\left(\sqrt[k-1]{x}\right)$$

is strictly convex on $\mathbb{R}$. Then, the sum

$$S = f(x) + f(y) + f(z)$$

is minimum if and only if $y = z$, and is maximum if and only if $x = y$.

Proof. If $a = b = c$, then

$$a=b=c \;\Rightarrow\; a^k+b^k+c^k = 3\left(\frac{a+b+c}{3}\right)^k$$
$$\Rightarrow\; x^k+y^k+z^k = 3\left(\frac{x+y+z}{3}\right)^k \;\Rightarrow\; x=y=z.$$

Consider further that a, b, c are not all equal. As it is shown in the proof of Lemma 3.1, we have $x < z$. According to the relations

$$x+z=a+b+c-y, \quad x^k+z^k=a^k+b^k+c^k-y^k,$$

we may consider x and z as functions of y. Thus, we have

$$S = f(x(y)) + f(y) + f(z(y)) := F(y).$$

According to Lemma 3.1, it suffices to show that F is maximum for $y = y_1$ and is minimum for $y = y_2$. Using (3.5), we have

$$\begin{aligned} F'(y) &= x'f'(x) + f'(y) + z'f'(z) \\ &= \frac{y^{k-1}-z^{k-1}}{z^{k-1}-x^{k-1}} g(x^{k-1}) + g(y^{k-1}) + \frac{y^{k-1}-x^{k-1}}{x^{k-1}-z^{k-1}} g(z^{k-1}), \end{aligned}$$

which, for $x < y < z$, is equivalent to

$$\begin{aligned} \frac{F'(y)}{(y^{k-1}-x^{k-1})(y^{k-1}-z^{k-1})} &= \frac{g(x^{k-1})}{(x^{k-1}-y^{k-1})(x^{k-1}-z^{k-1})} \\ &\quad + \frac{g(y^{k-1})}{(y^{k-1}-z^{k-1})(y^{k-1}-x^{k-1})} + \frac{g(z^{k-1})}{(z^{k-1}-x^{k-1})(z^{k-1}-y^{k-1})}. \end{aligned}$$

Since g is strictly convex, the right hand side is positive. Moreover, since

$$(y^{k-1}-x^{k-1})(y^{k-1}-z^{k-1}) < 0,$$

we have $F'(y) < 0$ for $y \in (y_1, y_2)$, hence F is strictly decreasing on $[y_1, y_2]$. Therefore, F is maximum for $y = y_1$ and is minimum for $y = y_2$.

Proof of Theorem 2.3.

For $n = 3$, Theorem 2.3 follows immediately from Proposition 3.3. Consider next that $n \ge 4$. Since $X = (x_1, x_2, \ldots, x_n)$ is defined in Theorem 2.3 as a compact set in $\mathbb{R}^n$, S_n attains its minimum and maximum values. Using this property and Proposition 3.3, we can prove Theorem 2.3 via contradiction. Thus, for the sake of contradiction, assume that

S_n attains its maximum at $(b_1, b_2, \dots, b_n)$, where $b_1 \le b_2 \le \dots \le b_n$ and $b_1 < b_{n-1}$. Let x_1, x_{n-1} and x_n be real numbers such that

$$x_1 \le x_{n-1} \le x_n, \quad x_1 + x_{n-1} + x_n = b_1 + b_{n-1} + b_n, \quad x_1^k + x_{n-1}^k + x_n^k = b_1^k + b_{n-1}^k + b_n^k.$$

According to Proposition 3.3, the sum $f(x_1)+f(x_{n-1})+f(x_n)$ is maximum for $x_1 = x_{n-1}$, when

$$f(x_1) + f(x_{n-1}) + f(x_n) > f(b_1) + f(b_{n-1}) + f(b_n).$$

This result contradicts the assumption that S_n attains its maximum value at $(b_1, b_2, \dots, b_n)$ with $b_1 < b_{n-1}$. Similarly, we can prove that S_n is minimum for $x_2 = x_3 = \dots = x_n$.

4. Proof of proposition 2.1

In order to prove Proposition 2.1, we use the following lemma.

Lemma 4.2. *Let a, b, c be fixed real numbers, and let x, y, z be real numbers such that*

$$x + y + z = a + b + c, \quad x^2 + y^2 + z^2 = a^2 + b^2 + c^2.$$

The power sum

$$S = x^4 + y^4 + z^4$$

is minimum and maximum when two of x, y, z are equal; more precisely, S is constant for $a + b + c = 0$, while for $a + b + c \neq 0$, S is minimum and maximum if and only if two of x, y, z are equal.

Proof. The proof is based on Lemma 3.1. Without loss of generality, assume that $x \le y \le z$. For the nontrivial case when a, b, c are not all equal (which involves $x < z$), consider the function of y

$$F(y) = x^4(y) + y^4 + z^4(y).$$

According to (3.5), we have

$$F'(y) = 4x^3x' + 4y^3 + 4z^3z' = 4x^3\frac{y-z}{z-x} + 4y^3 + 4z^3\frac{y-x}{x-z}$$
$$= 4(x+y+z)(y-x)(y-z) = 4(a+b+c)(y-x)(y-z).$$

There are three cases to consider.

Case 1: $a + b + c < 0$. Since $F'(y) > 0$ for $x < y < z$, F is strictly increasing on $[y_1, y_2]$.
Case 2: $a + b + c > 0$. Since $F'(y) < 0$ for $x < y < z$, F is strictly decreasing on $[y_1, y_2]$.

Case 3: $a + b + c = 0$. Since $F'(y) = 0$, F is constant on $[y_1, y_2]$.

In all cases, F is monotonic on $[y_1, y_2]$. Therefore, F is minimum and maximum for $y = y_1$ or $y = y_2$; that is, when $x = y$ or $y = z$ (see Lemma 3.1). Notice that for $a + b + c \neq 0$, F is strictly monotonic on $[y_1, y_2]$, hence F is minimum and maximum if and only if $y = y_1$ or $y = y_2$; that is, if and only if $x = y$ or $y = z$.

Proof of Proposition 2.1.

For $n = 3$, Proposition 2.1 follows from Lemma 4.2. In order to prove Proposition 2.1 for any $n \ge 4$, we will use the contradiction method. For the sake of contradiction, assume that $(b_1, b_2, \dots, b_n)$ is an extremal point having at least three distinct components; let us say $b_1 < b_2 < b_3$. Let x_1, x_2 and x_3 be real numbers such that

$$x_1 \le x_2 \le x_3, \quad x_1 + x_2 + x_3 = b_1 + b_2 + b_3 \quad x_1^2 + x_2^2 + x_3^2 = b_1^2 + b_2^2 + b_3^2.$$

We need to consider two cases.

Case 1: $b_1 + b_2 + b_3 \neq 0$. According to Lemma 4.2, the sum $x_1^4 + x_2^4 + x_3^4$ is extremal only when two of x_1, x_2, x_3 are equal, which contradicts the assumption that the sum $x_1^4 + x_2^4 + \dots + x_n^4$ attains its extremal at $(b_1, b_2, \dots, b_n)$ with $b_1 < b_2 < b_3$.

Case 2: $b_1+b_2+b_3=0$. There exist three real numbers x_1, x_2, x_3 such that $x_1=x_2$ and

$$x_1+x_2+x_3=b_1+b_2+b_3=0, \quad x_1^2+x_2^2+x_3^2=b_1^2+b_2^2+b_3^2.$$

Letting $x_1=x_2:=x$ and $x_3:=y$, we have $2x+y=0$, $x\neq y$. According to Lemma 4.2, the sum $x_1^4+x_2^4+x_3^4$ is constant (equal to $b_1^4+b_2^4+b_3^4$). Thus, $(x,x,y,b_4,\ldots,b_n)$ is also an extremal point. According to our hypothesis, this extremal point has at least three distinct components. Therefore, among the numbers $b_4,\ldots,b_n$ there is one, let us say b_4, such that x, y and b_4 are distinct. Since

$$x+y+b_4=-x+b_4\neq 0,$$

we have a case similar to Case 1, which leads to a contradiction.

5. Proof of proposition 2.2

Using Lemma 5.3 below and the contradiction method, we can prove Proposition 2.2 in a similar way as the proof of Theorem 2.3.

Lemma 5.3. *Let a,b,c be fixed real numbers, let x,y,z be real numbers such that*

$$x+y+z=a+b+c, \quad x^2+y^2+z^2=a^2+b^2+c^2.$$

For $m\in\{6,8\}$, the power sum

$$S_m=x^m+y^m+z^m$$

is maximum if and only if two of x,y,z are equal.

Proof. Consider the nontrivial case where a,b,c are not all equal. Let $p=a+b+c$, $q=ab+bc+ca$ and $r=xyz$. Since $x+y+z=p$ and $xy+yz+zx=q$, from

$$(x-y)^2(y-z)^2(z-x)^2\geq 0,$$

which is equivalent to

$$27r^2+2(2p^3-9pq)r-p^2q^2+4q^3\leq 0,$$

we get $r\in[r_1,r_2]$, where

$$r_1=\frac{9pq-2p^3-2(p^2-3q)\sqrt{p^2-3q}}{27}, \quad r_2=\frac{9pq-2p^3+2(p^2-3q)\sqrt{p^2-3q}}{27}.$$

Obviously, the product $r=xyz$ attains its minimum value r_1 and its maximum value r_2 only when two of x,y,z are equal. For fixed p and q, we have

$$S_6(x,y,z)=3r^2+f_6(p,q)r+h_6(p,q):=g_6(r),$$

$$S_8(x,y,z)=4(3p^2-2q)r^2+f_8(p,q)r+h_8(p,q):=g_8(r).$$

Since

$$3p^2-2q=\frac{7}{3}p^2+\frac{2}{3}(p^2-3q)>0,$$

the functions g_6 and g_8 are strictly convex, hence are maximum only for $r=r_1$ or $r=r_2$; that is, only when two of x,y,z are equal.

6. Conclusions

This paper deals with constrained optimization for real variables in a framework initiated by author for positive variables in [2]. The main extension of EV-Theorem to real variables is given by Theorem 2.3 for a function f differentiable on $\mathbb{R}$ such that the associated function $g : \mathbb{R} \to \mathbb{R}$, defined by

$$g(x) = f'\left(\sqrt[k-1]{x}\right),$$

where k is an even positive integer, is strictly convex on $\mathbb{R}$. An extension of the EV-Theorem for real variables to other functions f than those in Theorem 2.3 is an interesting open problem. Two such extensions are given by Proposition 2.1 for $f(t) = t^4$, and by Proposition 2.2 for $f(t) = t^m$, where $m \in \{6, 8\}$. We conjecture that Proposition 2.2 is valid for any integer $m \geq 3$.

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