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A hybrid iterative method without extrapolating step for solving mixed equilibrium problem

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ABSTRACT. In this paper, we introduce a hybrid iterative method without extrapolating step to approximate a solution of mixed equilibrium problem in real Hilbert space. We prove a strong convergence theorem for the sequences generated by the proposed iterative algorithm. The result presented in this paper is the extension and generalization of the previously known results in this area.

1. INTRODUCTION

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and induced norm $\| \cdot \|$. Let C be a nonempty, closed and convex subset of H; let $F : C \times C \to \mathbb{R}$, where $\mathbb{R}$ is a set of all real numbers, be a bifunction and let $A : C \to H$ be a nonlinear mapping. Then, we consider the following *mixed equilibrium problem* (in short, MEP): Find $x \in C$ such that

$$F(x, y) + \langle Ax, y - x \rangle \geq 0, \ \forall y \in C. \tag{1.1}$$

MEP (1.1) introduced and studied by Moudafi and Théra [12]. The solution set of MEP (1.1) is denoted by Sol (MEP(1.1)). If we set $A = 0$, MEP (1.1) reduces to the *equilibrium problem* (in short, EP): Find $x \in C$ such that

$$F(x, y) \geq 0, \ \forall y \in C, \tag{1.2}$$

which is introduced and studied by Blum and Oettli [1]. The set of solutions of EP (1.2) is denoted by Sol (EP(1.2)).

It is known that the equilibrium problem has a great impact and influence in the development of several topics of science and engineering. It turned out that the theories of many well known problems could be fitted into the theory of equilibrium problems. It has been shown that the theory of equilibrium problem provides a natural, novel and unified framework for several problems arising in nonlinear analysis, optimization, economics, finance, game theory, physics and engineering. The equilibrium problem includes many mathematical problems as particular cases, for example, mathematical programming problem, variational inclusion problem, variational inequality problem, complementary problem, saddle point problem, Nash equilibrium problem in noncooperative games, minimax inequality problem, minimization problem and fixed point problem, see [1, 13, 7].

For example, if we set $F(x, y) = \sup\limits_{\zeta \in Mx} \langle \zeta, y - x \rangle$ with $M : C \to 2^C$ a set-valued maximal monotone operator. Then MEP (1.1) reduces to the following basic class of variational

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inclusion peroblem: Find $x \in C$ such that

$$0 \in A(x) + M(x),\ \forall\, y \in C. \tag{1.3}$$

Set $F(x, y) = \psi(y) - \psi(x)$, where $\psi : C \to \mathbb{R}$ is a nonlinear function, then MEP (1.1) reduces to the following mixed variational inequality problem: Find $x \in C$ such that

$$\langle A(x), y - x\rangle + \psi(y) - \psi(x) \geq 0,\ \forall\, y \in C, \tag{1.4}$$

If we set $F = 0$, MEP (1.1) reduces to the classical *variational inequality problem* (in short, VIP): Find $x \in C$ such that

$$\langle Ax, y - x\rangle \geq 0,\ \forall y \in C, \tag{1.5}$$

which is introduced by Hartmann and Stampacchia [9]. The set of solutions of VIP (1.5) is denoted by Sol (VIP(1.5)).

In 1976, Korpelevich [10] introduced the following iterative method which is known as extragradient iterative method:

$$\begin{cases} x_0 & = x \in C, \\ u_n & = P_C(x_n - \lambda A x_n), \\ x_{n+1} & = P_C(x_n - \lambda A u_n), \end{cases} \tag{1.6}$$

where $\lambda > 0$, A is a monotone and Lipschitz continuous mapping and P_C is the metric projection of H onto C. He prove that that if Sol (VIP(1.5)) is nonempty then, under some suitable conditions, the sequence generated by (1.6) converges to a solution of VIP(1.5).

In 2006, Nadezkhina and Takahashi [14] introduced a hybrid extragradient method to approximate a common solution of VIP(1.5) and a fixed point problem for a nonexpansive mapping in real Hilbert space. A lot of efficient generalizations and modifications of iterative method given by [14] exist at this moment, for instance, see [3, 4, 5, 15, 2] and references therein. Very recently, Malitsky and Semenov [11] introduced a new hybrid iterative method without extrapolating step for solving VIP(1.5), which generalizes the methods given by [10, 14]. They proved that a strong convergence theorem in Hilbert space.

In this paper, we suggest and analyze an iterative method based on hybrid iterative method without extrapolating step for solving MEP (1.1). Further, we obtain a strong convergence theorem for the sequences generated by the proposed iterative algorithm. The result and method presented in this paper extend and generalize some known results and iterative methods, see for instance [11].

2. Preliminaries

We recall some concepts and results needed in the sequel. Let symbols $\to$ and $\rightharpoonup$ denote strong and weak convergence, respectively.

In a real Hilbert space H, it is well known that

$$\|\lambda x + (1-\lambda)y\|^2 = \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)\|x - y\|^2, \tag{2.7}$$

for all $x, y \in H$ and $\lambda \in [0, 1]$.

Furthe, any Hilbert space H has the Kadec-Klee property [8], that is, if $\{x_n\}$ be a sequence in H which satisfies $x_n \rightharpoonup x$ and $\|x_n\| \to \|x\|$ as $n \to \infty$, then $\|x_n - x\| \to 0$ as $n \to \infty$.

It is well known that P_C is nonexpansive and satisfies

$$\langle x-y, P_Cx-P_Cy\rangle \geq \|P_Cx-P_Cy\|^2,\ \forall x\in H. \tag{2.8}$$

Moreover, P_Cx is characterized by the fact $P_Cx\in C$ and

$$\langle x-P_Cx, y-P_Cx\rangle \leq 0, \tag{2.9}$$

and

$$\|x-y\|^2 \geq \|x-P_Cx\|^2+\|y-P_Cx\|^2,\ \forall x\in H,\ y\in C. \tag{2.10}$$

Definition 2.1. A mapping $T:H\to H$ is said to be

(i) *monotone*, if

$$\langle Tx-Ty, x-y\rangle \geq 0,\ \forall x,y\in H;$$

(ii) *α-inverse strongly monotone*, if there exists a constant $\alpha>0$ such that

$$\langle Tx-Ty, x-y\rangle\ \geq \alpha\|Tx-Ty\|^2,\ \forall x,y\in H;$$

(iii) *β-Lipschitz continuous*, if there exists a constant $\beta>0$ such that

$$\|Tx-Ty\|\leq \beta\|x-y\|,\ \forall x,y\in H.$$

We note that if T is α-inverse strongly monotone mapping, then T is monotone and $\frac{1}{\alpha}$-Lipschitz continuous.

Assumption 1. *The bifunction $F:C\times C\longrightarrow\mathbb{R}$ satisfies the following assumptions:*

(i) $F(x,x)=0,\ \forall x\in C$;

(ii) F is monotone, i.e., $F(x,y)+F(y,x)\leq 0,\ \forall x\in C$;

(iii) For each $x,y,z\in C$, $\limsup\limits_{t\to 0}F(tz+(1-t)x,y)\leq F(x,y)$;

(iv) For each $x\in C$, $y\to F(x,y)$ is convex and lower semicontinuous.

Assumption 2. *The bifunction $F:C\times C\to\mathbb{R}$ holds the following relation:*

$$F(x,y)+F(y,z)+F(z,x)\leq 0,\ \forall x,y,z\in C. \tag{2.11}$$

We easily observe that, for $y=z$, Assumption 1(i) and Assumption 2 implies Assumption 1 (ii).

Now, we have the following lemma.

Lemma 2.1. [6] *Let C be a nonempty closed convex subset of H. Assume that $F:C\times C\longrightarrow\mathbb{R}$ satisfying Assumption* 1*. For $r>0$ and for all $x\in H$, define a mapping $T_r:H\to C$ as follows:*

$$T_r(x)=\{z\in C: F(z,y)+\frac{1}{r}\langle y-z, z-x\rangle\geq 0,\ \forall y\in C\}. \tag{2.12}$$

Then the following results hold:

(i) For each $x\in H$, $T_r(x)\neq\emptyset$;

(ii) T_r is single-valued;

(iii) T_r is firmly nonexpansive, i.e.,

$$\|T_rx-T_ry\|^2\leq\langle T_rx-T_ry, x-y\rangle,\ \forall x,y\in H; \tag{2.13}$$

(iv) $\mathrm{Fix}(T_r)=\mathrm{Sol}(\mathrm{EP}(1.2))$;

(v) $\mathrm{Sol}(\mathrm{EP}(1.2))$ *is closed and convex, where* $\mathrm{Fix}(T_r)$ *denotes the set of fixed points of T_r.*

Remark 2.1. It follows from Lemma 2.1 (i)-(ii) that

$$rF(T_rx, y) + \langle T_rx - x, y - T_rx\rangle \geq 0,\ \forall y \in C,\ x \in H. \tag{2.14}$$

Further Lemma 2.1 (iii) implies the nonexpansivity of T_r, i.e.,

$$\|T_rx - T_ry \leq \|x - y\|,\ \forall x, y \in H. \tag{2.15}$$

Furthermore (2.14) implies the following inequality

$$\|T_rx - y\|^2 \leq \|x - y\|^2 - \|T_rx - x\|^2 + 2rF(T_rx, y),\ \forall y \in C, x \in H. \tag{2.16}$$

Lemma 2.2. [11] *Let $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ are nonnegative real sequences, $\alpha, \beta \in \mathbb{R}$ and for all $n \in \mathbb{N}$ the following inequality holds*

$$a_n \leq b_n - \alpha c_{n+1} + \beta c_n.$$

If $\sum\limits_{n=1}^{\infty} b_n < +\infty$ and $\alpha > \beta \geq 0$ then $\lim\limits_{n\to\infty} a_n = 0$

3. Hybrid iterative method

Theorem 3.1. *Let H be a real Hilbert space and $C \subseteq H$ be a nonempty, closed and convex subset. Let $F : C \times C \to \mathbb{R}$ be a bifunction satisfying Assumption 1 ((i),(iii) and (iv)), and Assumption 2, and let $A : C \to H$ be a σ-inverse strongly monotone such that* $\mathrm{Sol}(\mathrm{MEP}(1.1)) \neq \emptyset$. *Let the iterative sequences $\{x_n\}$ and $\{z_n\}$ be generated by the following iterative algorithm:*

$$\begin{cases} x_0, z_0 & \in C, \\ z_{n+1} & = T_{r_n}(x_n - r_nAz_n), \\ C_n & = \Big\{z \in C : \|z_{n+1} - z\|^2 \leq \|x_n - z\|^2 + k\|x_n - x_{n-1}\|^2 \\ & \quad -\big(1 - \frac{1}{k} - \frac{r_n}{\sigma}\big)\|z_{n+1} - z_n\|^2 + \frac{r_n}{\sigma}\|z_n - z_{n-1}\|^2\Big\}, \\ Q_n & = \{z \in C : \langle x_n - z, x - x_n\rangle \geq 0\}, \\ x_{n+1} & = P_{C_n\cap Q_n}x, \end{cases} \tag{3.17}$$

for $n = 1, 2, \ldots$, where $\{r_n\} \subset [a, b]$ for some $a, b \in (0, \frac{\sigma}{2})$ and $k > \frac{\sigma}{\sigma - 2r_n}$. Then the sequences $\{x_n\}$ and $\{z_n\}$ converge strongly to $z = P_{\mathrm{Sol}(\mathrm{MEP}(1.1))}x$.

Proof. Let $\bar{x} \in \mathrm{Sol}(\mathrm{MEP}(1.1))$. From iterative algorithm (3.17), we have

$$z_{n+1} = T_{r_n}(x_n - r_nAz_n),\ \forall n. \tag{3.18}$$

Now, applying (2.16) with $x_n - r_nAz_n$ and $\bar{x}$, we have

$$\begin{aligned} \|z_{n+1} - \bar{x}\|^2 &\leq \|x_n - r_nAz_n - \bar{x}\|^2 - \|z_{n+1} - (x_n - r_nAz_n)\|^2 + 2r_nF(z_{n+1}, \bar{x}) \\ &= \|x_n - \bar{x}\|^2 - \|z_{n+1} - x_n\|^2 + 2r_n\langle Az_n, \bar{x} - z_{n+1}\rangle + 2r_nF(z_{n+1}, \bar{x}) \\ &= \|x_n - \bar{x}\|^2 - \|z_{n+1} - x_n\|^2 + 2r_n\big[\langle Az_n - A\bar{x}, \bar{x} - z_n\rangle \\ &\quad + \langle A\bar{x}, \bar{x} - z_n\rangle - \langle Az_n, z_{n+1} - z_n\rangle\big] + 2r_nF(z_{n+1}, \bar{x}). \end{aligned} \tag{3.19}$$

Since A is σ-inverse strongly monotone, then A is monotone and $\frac{1}{\sigma}$-Lipschitz continuous. Further since $\bar{x} \in \mathrm{Sol}(\mathrm{MEP}(1.1))$ and $z_n \in C$, then

$$F(\bar{x}, z_n) + \langle A\bar{x}, z_n - \bar{x}\rangle \geq 0,\ \forall z_n \in C. \tag{3.20}$$

Using (3.20) and monotonicity of A in (3.19), we have

$$\begin{aligned}
\|z_{n+1}-\bar{x}\|^2 &\le \|x_n-\bar{x}\|^2-\|z_{n+1}-x_n\|^2+2r_n\langle Az_n, z_n-z_{n+1}\rangle \\
&\quad + 2r_n\big[F(\bar{x},z_n)+F(z_{n+1},\bar{x})\big] \\
&= \|x_n-\bar{x}\|^2-\|x_n-x_{n-1}\|^2-\|x_{n-1}-z_{n+1}\|^2-2\langle x_n-x_{n-1},x_{n-1}-z_{n+1}\rangle \\
&\quad + 2r_n\langle Az_n, z_n-z_{n+1}\rangle+2r_n\big[F(\bar{x},z_n)+F(z_{n+1},\bar{x})\big] \\
&= \|x_n-\bar{x}\|^2-\|x_n-x_{n-1}\|^2-2\langle x_n-x_{n-1},x_{n-1}-z_{n+1}\rangle-\|x_{n-1}-z_n\|^2 \\
&\quad -\|z_n-z_{n+1}\|^2-2\langle x_{n-1}-z_n,z_n-z_{n+1}\rangle-2r_n\langle Az_n-Az_{n-1},z_{n+1}-z_n\rangle \\
&\quad -2r_n\langle Az_{n-1},z_{n+1}-z_n\rangle+2r_n\big[F(\bar{x},z_n)+F(z_{n+1},\bar{x})\big] \\
&= \|x_n-\bar{x}\|^2-\|x_n-x_{n-1}\|^2-2\langle x_n-x_{n-1},x_{n-1}-z_{n+1}\rangle-\|x_{n-1}-z_n\|^2 \\
&\quad -\|z_n-z_{n+1}\|^2-2r_n\langle Az_n-Az_{n-1},z_{n+1}-z_n\rangle \\
&\quad +2\langle x_{n-1}-r_nAz_{n-1}-z_n,z_{n+1}-z_n\rangle+2r_n\big[F(\bar{x},z_n)+F(z_{n+1},\bar{x})\big].
\end{aligned} \tag{3.21}$$

As $z_n=T_{r_n}(x_{n-1}-r_nAz_{n-1})$ and $z_{n+1}\in C$, we have from (2.14)

$$\langle x_{n-1}-r_nAz_{n-1}-z_n,z_{n+1}-z_n\rangle\le r_nF(z_n,z_{n+1})$$

This implies that

$$\begin{aligned}
\|z_{n+1}-\bar{x}\|^2 \;=\;& \|x_n-\bar{x}\|^2-\|x_n-x_{n-1}\|^2-2\langle x_n-x_{n-1},x_{n-1}-z_{n+1}\rangle-\|x_{n-1}-z_n\|^2 \\
&-\|z_n-z_{n+1}\|^2-2r_n\langle Az_n-Az_{n-1},z_{n+1}-z_n\rangle \\
&+2r_n\big[F(\bar{x},z_n)+F(z_n,z_{n+1})+F(z_{n+1},\bar{x})\big].
\end{aligned} \tag{3.22}$$

Now, using the triangle, the Cauchy-Schwarz, and the Cauchy inequalities, we get

$$\begin{aligned}
-2\langle x_n-x_{n-1},x_{n-1}-z_{n+1}\rangle \;\le\;& 2\|x_n-x_{n-1}\|\|x_{n-1}-z_n\|+2\|x_n-x_{n-1}\|\|z_n-z_{n+1}\| \\
\le\;& \|x_n-x_{n-1}\|^2+\|x_{n-1}-z_n\|^2+k\|x_n-x_{n-1}\|^2 \\
&+\frac{1}{k}\|z_{n+1}-z_n\|^2.
\end{aligned} \tag{3.23}$$

Since A is σ-inverse strongly monotone, we get

$$\begin{aligned}
-2r_n\langle Az_n-Az_{n-1},z_{n+1}-z_n\rangle \;\le\;& 2r_n\frac{1}{\sigma}\|z_n-z_{n-1}\|\|z_{n+1}-z_n\| \\
\le\;& \frac{r_n}{\sigma}(\|z_{n+1}-z_n\|^2+\|z_n-z_{n-1}\|^2).
\end{aligned} \tag{3.24}$$

Combining inequalities (3.22)-(3.24) and using Assumption 2, we get

$$\begin{aligned}
\|z_{n+1}-\bar{x}\|^2 \;\le\;& \|x_n-\bar{x}\|^2+k\|x_n-x_{n-1}\|^2-\big(1-\frac{1}{k}-\frac{r_n}{\sigma}\big)\|z_{n+1}-z_n\|^2 \\
&+\frac{r_n}{\sigma}\|z_n-z_{n-1}\|^2,
\end{aligned} \tag{3.25}$$

which implies that $\bar{x}\in C_n$ and hence $\mathrm{Sol}(\mathrm{MEP}(1.1))\subseteq C_n,\ \forall n$. Further, it is easily observed that the sets C_n and Q_n is closed and convex for each $n=0,1,2,\dots$. Next, by mathematical induction method, we show that $\mathrm{Sol}(\mathrm{MEP}(1.1))\subseteq Q_n,\ \forall n$. For $n=0$, evidently $\mathrm{Sol}(\mathrm{MEP}(1.1))\subseteq C_0$ and $\mathrm{Sol}(\mathrm{MEP}(1.1))\subseteq Q_0=H$, it follows that $\mathrm{Sol}(\mathrm{MEP}(1.1))\subseteq C_0\cap Q_0$ and hence $C_0\cap Q_0$ is nonempty, closed and convex set. Therefore $x_1=P_{C_0\cap Q_0}x$ is well defined. Now, we suppose that $\mathrm{Sol}(\mathrm{MEP}(1.1))\subseteq C_{n-1}\cap Q_{n-1}$, for some $n>1$. Let $x_n=P_{C_n\cap Q_n}x$. Since $\mathrm{Sol}(\mathrm{MEP}(1.1))\subseteq C_n$ and for any $\bar{x}\in\mathrm{Sol}(\mathrm{MEP}(1.1))$, it follows from

(2.9) that $\langle x - x_n, x_n - \bar{x}\rangle = \langle x - P_{C_{n-1}\cap Q_{n-1}}x, P_{C_{n-1}\cap Q_{n-1}}x - \bar{x}\rangle \geq 0$, and hence $\bar{x} \in Q_n$. Therefore $\mathrm{Sol}(\mathrm{MEP}(1.1)) \subseteq C_n \cap Q_n$ for every $n = 0, 1, 2, \dots$ and hence $x_{n+1} = P_{C_n\cap Q_n}x$ is well defined for every $n = 0, 1, 2, \dots$. Thus the sequence $\{x_n\}$ is well defined.

Since A is σ-inverse strongly monotone and Sol(MEP(1.1))$\neq \emptyset$ then $T_{r_n}(I - r_nA)$ is nonexpansive and hence $\mathrm{Sol}(\mathrm{MEP}(1.1)) = \mathrm{Fix}(T_{r_n}(I - r_nA))$ is closed and convex where I denotes the identity operator on H.

Let $w = P_{\mathrm{Sol}(\mathrm{MEP}(1.1))}x$. From $x_{n+1} = P_{C_n\cap Q_n}x$ and $w \in \mathrm{Sol}(\mathrm{MEP}(1.1)) \subset C_n \cap Q_n$, we have

$$\|x_{n+1} - x\| \leq \|w - x\|, \tag{3.26}$$

for every $n = 0, 1, 2, \dots$. Therefore $\{x_n\}$ is bounded. From (3.17), we have respectively $x_{n+1} \in C_n \cap Q_n$ and $x_n = P_{Q_n}x$, and hence, we have

$$\|x_n - x\| \leq \|x_{n+1} - x\|, \tag{3.27}$$

for every $n = 0, 1, 2, \dots$. It follows from (3.26) and (3.27) that the sequence $\{\|x_n - x\|\}$ is monotonically increasing and bounded, and hence convergent. Therefore $\lim\limits_{n\to\infty} \|x_n - x\|$ exists.

Since $x_n = P_{Q_n}x$ and $x_{n+1} \in Q_n$, using (2.10), we have

$$\|x_{n+1} - x_n\|^2 \leq \|x_{n+1} - x\|^2 - \|x_n - x\|^2, \tag{3.28}$$

for every $n = 0, 1, 2, \dots$. Hence, it follows from (3.28) that

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{3.29}$$

Since $x_{n+1} \in C_n$, we obtain

$$\begin{aligned} \|z_{n+1} - x_{n+1}\|^2 \quad \leq \quad & \|x_{n+1} - x_n\|^2 + k\|x_n - x_{n-1}\|^2 - \big(1 - \frac{1}{k} - \frac{r_n}{\sigma}\big)\|z_{n+1} - z_n\|^2 \\ & + \frac{r_n}{\sigma}\|z_n - z_{n-1}\|^2. \end{aligned} \tag{3.30}$$

Set $a_n = \|z_{n+1} - x_{n+1}\|^2$, $b_n = \|x_{n+1} - x_n\|^2 + k\|x_n - x_{n-1}\|^2$, $c_n = \|z_n - z_{n-1}\|^2$, $\alpha = \big(1 - \frac{1}{k} - \frac{r_n}{\sigma}\big)$, $\beta = \frac{r_n}{\sigma}$.

Since $\sum\limits_{n=1}^{\infty} b_n < +\infty$ and $\alpha > \beta$, it follows from Lemma 2.2 that

$$\lim_{n\to\infty} \|z_n - x_n\| = 0. \tag{3.31}$$

Since $\{x_n\}$ is bounded sequence in C, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \rightharpoonup \hat{x}$, say, and $\hat{x} \in C$. Further, it follows from (3.31) that the sequences $\{x_n\}$ and $\{z_n\}$ both have the same asymptotic behavior. Therefore, there exists a subsequence $\{z_{n_k}\}$ of $\{z_n\}$ such that $z_{n_k} \rightharpoonup \hat{x}$. Next, we show that $\hat{x} \in \mathrm{Sol}(\mathrm{MEP}(1.1))$. It follows from (2.14) and (3.17) that

$$F(z_{n_k+1}, y) + \frac{1}{r_{n_k}}\langle z_{n_k+1} - (x_{n_k} - r_{n_k}Az_{n_k}), y - z_{n_k+1}\rangle \geq 0, \ \forall y \in C,$$

which implies

$$\left\langle \frac{z_{n_k+1} - x_{n_k}}{r_{n_k}}, y - z_{n_k+1} \right\rangle \geq F(y, z_{n_k+1}) - \langle Az_{n_k}, y - z_{n_k+1}\rangle, \ \forall y \in C, \tag{3.32}$$

using monotonicity of F.
For t with $0 < t \leq 1$, let $y_t = ty + (1-t)\hat{x} \in C$. So, from (3.32), we have

$$\begin{aligned}
\langle Ay_t, y_t - z_{n_k+1}\rangle &\geq \langle Ay_t, y_t - z_{n_k+1}\rangle - \langle Az_{n_k}, y_t - z_{n_k+1}\rangle \\
&\quad - \left\langle \frac{z_{n_k+1} - x_{n_k}}{r_{n_k}}, y_t - z_{n_k+1}\right\rangle + F(y_t, z_{n_k+1}) \\
&= \langle Ay_t - Az_{n_k+1}, y_t - z_{n_k+1}\rangle + \langle Az_{n_k+1} - Az_{n_k}, y_t - z_{n_k+1}\rangle \\
&\quad - \left\langle \frac{z_{n_k+1} - x_{n_k}}{r_{n_k}}, y_t - z_{n_k+1}\right\rangle + F(y_t, z_{n_k+1}).
\end{aligned}$$

Since A is Lipschitz continuous, we have $\lim_{k\to\infty} \|Az_{n_k+1} - Az_{n_k}\| = 0$. Further, from the monotonicity of A and the convexity and lower semicontinuity of F, $\frac{z_{n_k+1} - x_{n_k}}{r_{n_k}} \to 0$ and $z_{n_k+1} \rightharpoonup \hat{x}$, we have

$$\langle Ay_t, y_t - \hat{x}\rangle \geq F(y_t, \hat{x}), \tag{3.33}$$

as $k \to \infty$. Further, we have

$$\begin{aligned}
0 &\leq F(y_t, y_t) \\
&\leq tF(y_t, y) + (1-t)F(y_t, \hat{x}) \\
&\leq tF(y_t, y) + (1-t)\langle Ay_t, y_t - \hat{x}\rangle \\
&= tF(y_t, y) + (1-t)t\langle Ay_t, y - \hat{x}\rangle
\end{aligned}$$

and hence

$$0 \leq F(y_t, y) + (1-t)\langle Ay_t, y - \hat{x}\rangle.$$

Letting $t \to 0_+$, we have, for each $y \in C$,

$$F(\hat{x}, y) + \langle A\hat{x}, y - \hat{x}\rangle \geq 0.$$

This implies that $\hat{x} \in \mathrm{Sol}(\mathrm{MEP}(1.1))$. From $w = P_{\mathrm{Sol}(\mathrm{MEP}(1.1))}x$ and (3.26), we have

$$\|w - x\| \leq \|\hat{x} - x\| \leq \lim\inf_{k\to\infty} \|x_{n_k} - x\| \leq \lim\sup_{k\to\infty} \|x_{n_k} - x\| \leq \|w - x\|.$$

Thus, we have

$$\lim_{k\to\infty} \|x_{n_k} - x\| = \|\hat{x} - x\|.$$

Since $x_{n_k} - x \rightharpoonup \hat{x} - x$ and from Kadec-Klee property of Hilbert space, we have $x_{n_k} - x \to \hat{x} - x$ and hence $x_{n_k} \to \hat{x}$. Since by definition of Q_n, we obtain $x_n = P_{Q_n}x$ and $w \in \mathrm{Sol}(\mathrm{MEP}(1.1)) \subset C_n \cap Q_n$, we have

$$-\|w - x_{n_k}\|^2 = \langle w - x_{n_k}, x_{n_k} - x\rangle + \langle w - x_{n_k}, x - w\rangle \geq \langle w - x_{n_k}, x - w\rangle.$$

As $k \to \infty$, we obtain $-\|w - \hat{x}\|^2 \geq \langle w - \hat{x}, x - w\rangle \geq 0$ by $w = P_{\mathrm{Sol}(\mathrm{MEP}(1.1))}x$ and $\hat{x} \in \mathrm{Sol}(\mathrm{MEP}(1.1))$. Hence we have $\hat{x} = w$. This implies that $x_n \to w$. It is easy to see that $z_n \to w$. This completes the proof. □

Remark 3.2. If we set $F = 0$ then $T_{r_n} = P_C$ and thus Theorem 3.1 is reduced to Theorem 1 given by Malitsky and Semenov [11]. Further, the method presented in this paper can be extended to the Mixed equilibrium problem for set-valued mappings.

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