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# Kannan contractions and strongly demicontractive mappings

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ABSTRACT. Inclusion relations between strongly demicontractive mappings, quasi $(L, m)$-contractions, and Kannan contractions are established. As corollaries, $T$-stability and strong convergence of Picard and Mann iterations for strongly demicontractive mappings are obtained.

## 1. INTRODUCTION

Let $(X, d)$ be a complete metric space. Kannan ([9], 1968) introduced a class of mappings $T : X \to X$ satisfying the following contractive condition:

$$d(Tx, Ty) \leq \delta[d(x, Tx) + d(y, Ty)], \ \forall x, y \in X, \tag{1.1}$$

where $\delta \in (0, 1/2)$.

The *Kannan fixed point theorem* or *Kannan principle* can be briefly stated as follows: in a complete metric space $X$, a self mapping $T$ that satisfies (1.1) has a unique fixed point $p \in X$ and the Picard iteration converges to $p$.

Note that (1.1) does not imply the continuity of $T$, as the well known Banach contraction does, and that the fixed point theorem of Kannan characterizes the metric completeness of the underlying space, see [24].

In the period 1973-1977, see [7, 12, 13], the authors have considered the so-called *demicontractive* class of mappings $T : \mathcal{H} \to \mathcal{H}$, defined on a real Hilbert space $\mathcal{H}$, by the conditions $Fix(T) \neq \emptyset$ and

$$\|Tx - p\|^2 \leq \|x - p\|^2 + K\|x - Tx\|^2, \ \forall x \in \mathcal{H}, \ p \in Fix(T), \tag{1.2}$$

where $K \geq 0$.

If $T$ is demicontractive and satisfies some weak smoothness condition, for instance, if $T$ is demiclosed at zero, then the Mann iteration with suitable control sequence converges weakly to some fixed point of $T$.

Actually, in [13], this class is defined by the following accretive condition: there exists $\lambda > 0$ such that

$$\langle x - Tx, x - p\rangle \geq \lambda\|Tx - p\|^2.$$

It is very easy to show that this condition is equivalent to (1.2) (with $\lambda = (1 - K)/2$). Note that, usually, the inequality (1.2) is required to be satisfied only on a closed convex subset of $\mathcal{H}$; for simplicity, we shall assume that $T$ is defined on $\mathcal{H}$ and that (1.2) holds on the whole space.

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The inequality (1.2) may be strengthened in the following way:

$$\|Tx - p\|^2 \le \alpha\|x - p\|^2 + K\|x - Tx\|^2,\ \forall x \in \mathcal{H},\ p \in Fix(T), \tag{1.3}$$

where $\alpha \in (0,1)$ and $K \ge 0$. In this case, we will say that $T$ is *strongly demicontractive*.

If $T$ is strongly demicontractive, then the fixed point is unique. Indeed, if $q$ would be another fixed point of $T$, then

$$\|q - p\|^2 = \|Tq - p\|^2 \le \alpha\|q - p\|^2 + K\|Tq - q\|^2 = \alpha\|q - p\|^2,$$

which implies $\alpha \ge 1$, contrary to the hypothesis. In this case, the condition that (1.3) should be satisfied *for all* $p$ in $Fix(T)$ is superfluous.

But even the strongly demicontractivity condition does not ensure the convergence of Mann iteration for any $t \in (0,1)$.

For example, the function $f : [-1,1] \to [-1,1]$ given by $f(x) = 0.5(x^3 - 3x)$ is strongly demicontractive for any pair $\alpha,\ L$ such that $0 < \alpha < 1,\ L > 0$ and $0.16\alpha + L > 0.36$ but the Mann iteration fails to converge if $t \ge 0.8$ (thus including the Picard iteration, which is a particular case of Mann iteration).

**Remark 1.1.** The condition (1.3) was considered also in [25] and it was proposed the name *firmly pseudo-demicontractive* for a mapping satisfying (1.3). *Strongly demicontractive* seems to be a more appropriate term.

Various types contractive mappings, similar to those defined by (1.2) or (1.3), were considered by several authors.

For example, Osilike ([16], 1996) considered a class of mappings in complete metric spaces satisfying the following contractive type condition

$$d(Tx, Ty) \le Ld(x, Tx) + md(x, y),\ \forall x, y \in X, \tag{1.4}$$

where $L \ge 0$ and $m \in [0,1)$.

He proved the *T-stability* of Mann, Ishikawa and Kirk iteration procedures for this class of mappings. Note that the Osilike contractive condition is not sufficient to guarantee the existence of fixed points: the real function $f : [-1,1] \to [-1,1]$, given by $f(x) = x + 1,\ x \le 0$ and $f(x) = x - 1,\ x > 0$, is an example for this negative assertion.

Berinde ([2], 2004) has introduced the so-called *almost contractive* mappings defined by a condition similar to Osilike's above inequality (1.4), namely

$$d(Tx, Ty) \le Ld(y, Tx) + md(x, y),\ \forall x, y \in X,$$

where $L \ge 0,\ m \in [0,1)$.

Surprisingly, with this very small modification, the almost contractive condition ensures the existence of fixed points (without uniqueness) and the convergence of Picard iteration to a fixed point. A priori and a posteriori error estimations are also given in [2].

Qing and Rhoades ([18], 2008) considered a particular case of (1.4) requiring that this inequality should be satisfied only for $y \in Fix(T)$. More precisely, the two conditions in [18] are: $Fix(T) \ne \emptyset$ and

$$d(Tx, p) \le Ld(x, Tx) + md(x, p),\ \forall (x, p) \in X \times Fix(T), \tag{1.5}$$

where $L \ge 0$ and $m \in [0,1)$. The T-stability of Picard iteration for this class of mappings is proved in [18].

Note that any Kannan contraction verifies the Osilike's condition (1.4) and Berinde's almost contractive condition [17] (Lemma 1.3.7) and also satisfies (1.5).

**Remark 1.2.** In [6] it is proposed the terminology "$T$ satisfies the $(L, m)$-property" for a mapping satisfying (1.5). In the sequel we will use the following terminology: $(L, m)$-*contraction* for a mapping satisfying (1.4) and *quasi* $(L, m)$-*contraction* for a mapping satisfying (1.5).

There exist several connections between the known contractive type mappings [19, 6] and between the completeness of a metric space and the existence of fixed points of different contractive mappings, see [10, 11, 22, 23]. Among the three classes of mappings mentioned above, Kannan contractions, quasi (L, m)-contractions and strongly demicontractive mappings, there exist also some inclusion relations.

The main purpose of our study is to highlight these relationships and, on this base, to establish some results concerning the convergence and $T$-stability of the Picard and Mann iterations in the framework of a real Hilbert space. As usually, $\langle \cdot, \cdot \rangle$, $\| \cdot \|$ will denote the inner product and the corresponding norm on a Hilbert space $\mathcal{H}$.

## 2. Picard iteration

M. De la Sen [6] has established some inclusion relations between Kannan contractions and quasi (L, m)-contractions. More precisely, he proved the following facts (Theorem 9, (i), (iii) from [6]):

(a) If $T : X \to X$ is quasi (L, m)-contractive with $0 \leq m < 1/3$ and $0 \leq L < (1 - 3m)/2$, then $T$ is a $\delta$-Kannan contraction with $\delta = (L + m)/(1 - m)$;

(b) If $T : X \to X$ is $\delta$-Kannan contraction, then $T$ is quasi $(L, m)$-contractive with $L = \delta + 2/(1 - \delta)$ and $m \in (0, 1)$.

Similarly to (a) and (b), the theorems below establish some inclusion relations between strongly demicontractive mappings, quasi (L, m)-contractions, and Kannan contractions.

**Theorem 2.1.** *The class of strongly demicontractive mappings coincides with the class of quasi* $(L, m)$*-contractions.*

*Proof.* Let $T$ be a strongly demicontractive mapping. Using (1.3) we have

$$\begin{aligned} \|Tx - p\|^2 &\leq \alpha \|x - p\|^2 + K\|x - Tx\|^2 \\ &= (\sqrt{\alpha}\|x - p\| + \sqrt{K}\|x - Tx\|)^2 - 2\sqrt{\alpha K}\|x - p\|\|x - Tx\| \\ &\leq (\sqrt{\alpha}\|x - p\| + \sqrt{K}\|x - Tx\|)^2, \end{aligned}$$

and $T$ is quasi $(\sqrt{K}, \sqrt{\alpha})$-contractive.

Now, let $T$ be a quasi $(L, m)$-contraction and let $\alpha,\ K$ be real numbers such that

$$m^2 < \alpha < 1,\ K = \frac{m^2 L^2}{\alpha - m^2} + L^2. \tag{2.6}$$

We have

$$\begin{aligned} &\left( \sqrt{\alpha - m^2}\|x - p\| - \frac{mL}{\sqrt{\alpha - m^2}}\|x - Tx\| \right)^2 \\ &\qquad = (\alpha - m^2)\|x - p\|^2 + \frac{m^2 L^2}{\alpha - m^2}\|x - Tx\|^2 - 2mL\|x - p\|\|x - Tx\| \geq 0. \end{aligned}$$

Using (2.6) and arranging the terms, we obtain

$$\begin{aligned} \alpha\|x - p\|^2 + K\|x - Tx\|^2 &\geq m^2\|x - p\|^2 + L^2\|x - Tx\|^2 + 2mL\|x - p\|\|x - Tx\| \\ &= (m\|x - p\| + L\|x - Tx\|)^2 \geq \|Tx - p\|^2, \end{aligned}$$

and $T$ is strongly demicontractive. □

**Remark 2.3.** From (a) and Theorem 2.1, we can obtain conditions under which a strongly demicontractive mapping is a Kannan contraction. The two constants $(\sqrt{K}, \sqrt{\alpha})$ must verify the conditions from (a), that is, $\sqrt{\alpha} < 1/3$ and $\sqrt{K} < (1 - 3\sqrt{\alpha})/2$, which yield

$$0 < \alpha < \frac{1}{9},\ 0 \le K < \frac{1 + 9\alpha - 6\sqrt{\alpha}}{4}.$$

Note that when $\alpha \in (0, 1/9)$, the corresponding values of $K$ belong to $(0, 1/4)$.

Using Theorem 1 of [18] we can formulate the following result for strongly demicontractive mappings.

**Corollary 2.1.** *If $T$ is a strongly demicontractive mapping, then the Picard iteration is $T$-stable.*

The conditions under which a strongly demicontractive mapping is a Kannan contraction, can also be obtained directly, without using (a).

**Theorem 2.2.** *Suppose $T : \mathcal{H} \to \mathcal{H}$ is a strongly demicontractive mapping and that the constants $\alpha,\ K$ satisfy the following conditions*

$$0 \le K < \frac{1}{4},\ 0 < \alpha < \frac{1 - 4K}{9}. \tag{2.7}$$

*Then $T$ is $\delta$-Kannan contractive with $\delta = \alpha + \frac{\sqrt{\alpha + K - \alpha K}}{1-\alpha}$.*

*Proof.* From the strongly demicontractive definition, we have

$$\begin{aligned}\|Tx - p\|^2 &\le \alpha\|x - p\|^2 + K\|x - Tx\|^2 \\ &= \alpha\|(x - Tx) + (Tx - p)\|^2 + K\|x - Tx\|^2 \\ &\le \alpha\|x - Tx\|^2 + \alpha\|Tx - p\|^2 + 2\alpha\|x - Tx\|\|Tx - p\| + K\|x - Tx\|^2,\end{aligned}$$

and

$$(1 - \alpha)\|Tx - p\|^2 - 2\alpha\|x - Tx\|\|Tx - p\| - (\alpha + K)\|x - Tx\|^2 \le 0.$$

Because $\alpha + K - \alpha K > 0$ for any $\alpha \in (0, 1)$ and $K \ge 0$, it results

$$\|Tx - p\| \le \frac{\alpha + \sqrt{\alpha + K - \alpha K}}{1 - \alpha}\|x - Tx\|.$$

We have

$$\begin{aligned}\|Tx - Ty\| &\le \|Tx - p\| + \|Ty - p\| \\ &\le \frac{\alpha + \sqrt{\alpha + K - \alpha K}}{1 - \alpha}(\|x - Tx\| + \|y - Ty\|).\end{aligned}$$

The proof is complete since $(\alpha + \sqrt{\alpha + K - \alpha K})/(1 - \alpha) < 1/2$ holds in accordance with conditions (2.7). □

**Remark 2.4.** The conditions (2.7) can be formulated in the following equivalent form

$$0 < \alpha < \frac{1}{9},\ 0 \le K < \frac{1 - 9\alpha}{4}.$$

In comparison with conditions from Remark 2.3, these conditions are somewhat weaker, because $1 + 9\alpha - 6\sqrt{\alpha} < 1 - 9\alpha,\ \forall\alpha \in (0, 1/9)$. This means that the limits imposed by (2.7) are larger then those imposed by Theorem 2.1, underlined in Remark 2.3.

Using the Kannan fixed point theorem, we obtain the following result concerning the strong convergence of Picard iteration for strongly demicontractive mappings.

**Corollary 2.2.** *Suppose that $T$ is strongly demicontractive mapping and that $\alpha,\ K$ satisfy conditions (2.7). Then $T$ has a unique fixed point $p$ and the Picard iteration converges to $p$.*

## 3. Mann iteration

As demicontractivity or even strongly demicontractivity, ensures only weak convergence of the Mann iteration, to get strong convergence some additional conditions are needed (see, for example, [5])

The problem of additional conditions for strong convergence was discussed in several papers, including the papers in which the concept of demicontractivity was introduced [7, 13]. For example in [7] it is required, in addition, that $I - T$ maps closed bounded subsets of $C$ into closed subsets of $C$; in particular, this is satisfied if $T$ is demicompact. In [13] the existence of a nonzero solution $h \in \mathcal{H},\ h \neq 0$, of the variational inequality $\langle x - Tx, h\rangle \leq 0,\ \forall x \in \mathcal{C}$ is required as an additional condition for strong convergence. It is obvious that the existence of a nonzero solution of this variational inequality occurs only in very particular cases; an example for linear equations is given in [13].

In [3] it is required (as the main additional condition) that the mapping $T$ should be *demicompact* (Corollary 3.3). Note that this result was proved in [4] for a strictly pseudocontractive mapping (such mappings are more restrictive than the demicontractive ones). The same type of additional conditions ($T$ is demicompact or $\mathcal{C}$ is a compact subset of $\mathcal{H}$) appear in [8]. In [14] the concept of $\alpha$-demicontractive mappings is introduced and it is proved that strong demicontractivity together with $\alpha$-demicontractivity ensure the strong convergence. A recent additional condition is given in [1], namely,

$$\langle Tx, x\rangle \geq \|x\|^2 - \lambda\|x - Tx\|^2,\ \forall x \in C,$$

where $\lambda$ is the constant that appears in the (A) condition [13].

In this section we prove the strong convergence of the Mann iteration without any genuine additional condition, but with some restrictions on the constants $\alpha, K$ and on the control sequence.

To simplify the exposition, we consider the particular case of the Mann iteration with constant control sequence, $t_n = t,\ n = 0, 1, ...$, that is, $x_{n+1} = T_t(x_n) = (1-t)x_n + tTx_n$, commonly known as Krasnoselskij method.

Usually, it is required that $0 < t < 1$, but this is a restriction of convenience only, there exist cases when $t > 1$.

For example, the function $T : [0,1] \times [0,1] \to [0,1] \times [0,1]$, given by $T(x,y) = (x/2 + 0.1y^2, y/2 + 0.1x^2)^T$ is strongly demicontractive with $\alpha = 0.2$ and $K = 1$, and the Mann iteration converges for any $t \in (0, 4)$.

A non-artificial example in which the Mann iteration converges for $t > 1$ is the projection method for solving the convex feasibility problem.

The theorem below gives conditions which ensure that the averaged mapping $T_t = (1-t)I + tT$ is a Kannan contraction.

**Theorem 3.3.** *Suppose that $T$ is strongly demicontractive and $\alpha, K, t$ satisfy the condition*

$$\frac{1}{1-\alpha} \leq t < \frac{4(1-K)}{3}. \tag{3.8}$$

*Then $T_t$ is a Kannan contraction.*

*Proof.* The inequality (1.3) is equivalent to

$$\langle x - Tx, x - p\rangle \geq \frac{1-\alpha}{2}\|x-p\|^2 + \frac{1-K}{2}\|x - Tx\|^2.$$

We have

$$\begin{aligned}\|T_tx-p\|^2 &= \|x-p-t(x-Tx)\|^2\\ &= \|x-p\|^2-2t\langle x-Tx,x-p\rangle+t^2\|x-Tx\|^2\\ &\le \|x-p\|^2-t(1-\alpha)\|x-p\|^2-t(1-K)\|x-Tx\|^2+t^2\|x-Tx\|^2\\ &= (1-t+t\alpha)\|x-p\|^2+(t^2-t+tK)\|x-Tx\|^2.\end{aligned}$$

Using (3.8) it results that $1-t+t\alpha\le 0$ and because $\|x-Tx\|=\|x-T_tx\|/t$, we have

$$\|T_tx-p\|\le\frac{\sqrt{t^2-t+tK}}{t}\|x-T_tx\|.$$

Now, as in Theorem 2.2, it results

$$\|T_tx-T_ty\|\le\frac{\sqrt{t^2-t+tK}}{t}(\|x-T_tx\|+\|y-T_ty\|).$$

Due to the conditions (3.8) we have that $\sqrt{t^2-t+tK}/t<1/2$ and so $T_t$ is a Kannan contraction. □

**Remark 3.5.** The conditions (3.8) imply certain restrictions on the strongly demicontractive constants $\alpha, K$. Mote precisely, these constants must satisfy

$$0\le K<\frac{1}{4},\ 0<\alpha\le\frac{1-4K}{4(1-K)}.$$

From Kannan fixed point theorem we have the following convergence result of Mann iteration for strongly demicontractive mappings without additional conditions.

**Corollary 3.3.** *Suppose that $T$ is strongly demicontractive and that $t,\alpha,K$ satisfy the restrictions given by* (3.8). *Then the Mann iteration with constant control sequence $t_n=t$ converges to the unique fixed point of $T$.*

The condition on $t$, $1-t+t\alpha\le 0$, is the key of the proof of Theorem **??**. This condition imposes a value greater than 1 for $t$, which is not currently used in Mann iteration (if $\mathcal{C}$ is a closed convex subset and $T:\mathcal{C}\to\mathcal{C}$, then if $t>1$ there is no assurance that $(1-t)x+tTx\in\mathcal{C}$).

In the following we obtain conditions under which a strongly demicontractive mapping satisfies a Kannan contraction inequality without this restriction.

Let $T:\mathcal{C}\to\mathcal{C}$ be a mapping having a unique fixed point $p\in\mathcal{C}$. We will say that $T$ is *quasi-expansive* if there exists a closed convex subset $\mathcal{D}\subseteq\mathcal{C}$, containing $p$, such that

$$\|x-p\|\le\beta\|x-Tx\|,\ x\in D, \tag{3.9}$$

where $\beta>0$. It is easy to see that (3.9) implies $\|Tx-p\|\ge\frac{1-\beta}{\beta}\|x-p\|$, which motivates the terminology of *quasi-expansive*. Note that the strongly demicontractivity and the quasi-expansivity are not contradictory.

For example, the function $f:[-0.5,0.5]$, $f(x)=2x^3-1.2x$, is strongly demicontractive with constants $\alpha=0.5,\ K=0.2$ and quasi-expansive with $\beta=0.589$ on $[-0.5,0.5]$; if $\beta=0.5$ then $f$ is quasi-expansive on $[-0.316,0.316]$ and the minimum value of $\beta$ for which $f$ is still quasi-expansive is $0.4546...$

**Theorem 3.4.** *Suppose that $T:\mathcal{C}\to\mathcal{C}$ is strongly demicontractive with $\alpha,\ K\in[0.1,1)$ and quasi-expansive with $\beta=(1-K)/1.8$ on some closed convex subset $\mathcal{D}\subseteq\mathcal{C}$. Then*

$$\|T_tx-T_ty\|\le\delta(\|x-T_tx\|-\|y-T_ty\|),\ \forall x,y\in\mathcal{D},$$

*where* $\delta = 0.43588985...$, $t \in (t_1, t_2) \cap (0, 1)$ *and* $t_1, t_2$ *are the roots of the polynomial*

$$P(t) = (1 - \delta^2)t^2 - [1 - K + (1 - \alpha)\beta^2]t + \beta^2. \tag{3.10}$$

*Proof.* Let $f1, f2$ be two real functions of two variable each, defined by

$$f1(\alpha, K, \delta) = \frac{\sqrt{1 - \delta^2} - \sqrt{\Delta}}{1 - \alpha} - \frac{1 - K}{1.8}, \; f2(\alpha, K, \delta) = \sqrt{\frac{1 + K - 2\delta^2}{1 - \alpha}} - \frac{1 - K}{1.8},$$

where $\Delta = 1 - \delta^2 - (1 - \alpha)(1 - K)$. Consider now the following two constrained optimization problem:

$$\begin{cases} \min f1, \; \min f2, \\ 0.1 \le \alpha < 1, \; 0.1 \le K < 1. \end{cases}$$

Observe first that for any $\alpha, K, \delta$ satisfying these constrains, it results $\Delta > 0$. Then it is easily to get $\min f1 = 0$ and $\min f2 = 0.394427...$ and that $0.4355...$ is the lowest value of $\delta$ for which $\min f1 = 0$.

Consider the polynomial $Q(y) := (1 - \alpha)y^2 - 2\sqrt{1 - \delta^2}y + 1 - K$. Because $\Delta = 1 - \delta^2 - (1 - \alpha)(1 - k) > 0$, $Q$ has two real roots, $y_1, y_2$, and $y_1 = (\sqrt{1 - \delta^2} - \sqrt{\Delta})/(1 - \alpha)$. As $\beta < y_1$ it follows that $Q(\beta) > 0$. Using the notation $d := 1 - K + (1 - \alpha)\beta^2$, we have $Q(\beta) = (1 - \alpha)\beta^2 - 2\sqrt{1 - \delta^2}\beta + 1 - K = d - 2\sqrt{1 - \delta^2}\beta > 0$ and $d^2 - 4(1 - \delta^2)\beta^2 > 0$. Thus $P$ has also two real roots

$$t_{1,2} = \frac{d \pm \sqrt{d^2 - 4(1 - \delta^2)\beta^2}}{2(1 - \delta^2)}.$$

Obvious, $t_2 > 0$. The lowest root $t_1$ is less than 1, $t_1 < 1$. Indeed, from $\min f2 > 0$ we have

$$\beta < \sqrt{\frac{1 + K - 2\delta^2}{1 - \alpha}}$$

and

$$(1 - \alpha)\beta^2 < 1 + K - 2\varrho^2 = 2(1 - \delta^2) - 1 + K.$$

Thus $d - 2(1 - \delta^2) < 0$, from which it follows that $d - 2(1 - \delta^2) < \sqrt{d^2 - 4(1 - \delta^2)\beta^2}$ and $d - \sqrt{d^2 - 4(1 - \delta^2)\beta^2} < 2(1 - \delta^2)$. Therefore, $0 < t_1 < 1$ and $(t_1, t_2) \cap (0, 1) \neq \emptyset$. For $t \in (t_1, t_2) \cap (0, 1)$ we have that $P(t) < 0$ which means that

$$\frac{(1 - t + t\alpha)\beta^2 + t^2 - t + tK}{t^2} < \delta^2.$$

As in the proof of Theorem 3.3 we have

$$\|T_t x - p\|^2 \le (1 - t + t\alpha)\|x - p\|^2 + (t^2 - t + tK)\|x - Tx\|^2,$$

then, taking into account that $\|x - p\| \le \beta\|x - Tx\|$ and $\|x - Tx\| = \|x - T_t x\|/t$ we obtain

$$\|T_t x - p\| \le \frac{(1 - t + t\alpha)\beta^2 + t^2 - t + tK}{t^2}\|x - T_t x\|^2.$$

Therefore, for $t \in (t_1, t_2) \cap (0, 1)$ it results

$$\|T_t x - p\| \le \delta\|x - T_t x\|, \; \forall x \in \mathcal{D}.$$

Finally we have

$$\|T_t x - T_t y\| \le \|T_t x - p\| + \|T_t y - p\| \le \delta(\|x - T_t x\| + \|y - T_t y\|), \; \forall x, y \in \mathcal{D}.$$

$\square$

**Remark 3.6.** For the sake of simplicity, we formulated the Theorem 3.4 in a particular form concerning the values of $\beta$ and $\delta$. For a more general form of this theorem, we can take certain indeterminate values of quasi-expansive constant, for instance, we can of thake $\beta = (1-K)/c$, where $c > 0$, and then the main condition would be: for this value of $\beta$ there exists $0 < \delta < 0.5$ such that $\min f1 \geq 0,\ \min f2 \geq 0$.

The real function in the example below fulfils the conditions of Theorem 3.4.

**Example 3.1.** Let $f$ be a real function $f : [0.75, 1.25] \to [0.75, 1.25]$ given by

$$f(x) = \begin{cases} -x + 2 - (x-1)^4 \ if\ x \geq 1, \\ -x + 2 - (x-1)^3 \ if\ x < 1. \end{cases}$$

The function $f$ is strongly demicontractive with $p = 1$ and $\alpha = 0.9,\ K = 0.05$ and quasi-expansive with $\beta = (1-K)/1.8 = 0.52777...$ on the same interval. The two roots of $P$ given by (3.3) are $t_1 = 0.461...$ and $t_2 = 0.746....$ Therefore $f_t$ with $t \in [t_1, t_2]$ satisfies the Kannan condition with $\delta = 0.4355...$ Note also that, because in this case $\mathcal{D} = \mathcal{C}$, $f_t$ is a Kannan contraction.

## 4. The case of a metric space

Let $(X, d)$ be a metric space (not necessarily complete). We consider the following classes of operators $T : X \to X$:

(1) quasi l-contractions, i.e., $0 \leq l < 1$, $Fix(T) \neq \emptyset$ and

$$d(Tx, p) \leq l d(x, p),\ \forall x \in X,\ p \in Fix(T);$$

(2) quasi (L,m)-contractions, i.e., $L \geq 0,\ 0 \leq m < 1$, $Fix(T) \neq \emptyset$ and

$$d(Tx, p) \leq L d(x, Tx) + m d(x, p),\ \forall x \in X,\ p \in Fix(T);$$

(3) strongly $(\alpha, K)$-demicontractive mapings, i.e., $0 \leq \alpha < 1,\ K \geq 0$, $Fix(T) \neq \emptyset$ and

$$(d(Tx, p))^2 \leq \alpha (d(x, p))^2 + K (d(x, Tx))^2,\ \forall x \in X,\ p \in Fix(T).$$

First of all we remark that:

(a) In (1), (2) and (3) we have that $Fix(T) = \{p\}$.

(b) (2), with $m + 2L < 1$ implies (1) with $l = \dfrac{m+L}{1-L}$.

(c) (3), with $\sqrt{\alpha} + 2\sqrt{K} < 1$, implies (1) with $l = \frac{\sqrt{\alpha}+\sqrt{K}}{1-\sqrt{K}}$.

On the other hand we have for the class (1) the following result:

**Theorem 4.5.** *Let $(X, d)$ be a metric space and $T : X \to X$ be a quasi l-contraction. We have*

*(i) Fix(T)={ p}.*

*(ii) $T^n x \to p$ as $n \to \infty$, i.e., $T$ is a Picard mapping.*

*iii $d(x, p) \leq \frac{1}{1-l} d(x, Tx),\ \forall x \in X$, i.e., $T$ is $\frac{1}{1-l}$-Picard mapping (see* [20]*).*

*(iv) $y_n \in X,\ n \in \mathbb{N}$, with $d(y_{n+1}, Ty_n) \to 0$ as $n \to \infty$ imply that, $y_n \to p$ as $n \to \infty$, i.e., the Picard iteration of $T$ is Ostrowski stable, i.e., the mapping $T$ has the limit shadowing property (see* [15] *and* [21]*)*

*(v) Let $S : X \to X$ be such that there exists $\eta > 0$ with*

$$d(Tx, Sx) \leq \eta,\ \forall x \in X.$$

*Then,*

$$d(p,q) \leq \frac{\eta}{1-l},\ \forall q \in Fix(S).$$

*(vi) Let $S_n : X \to X,\ n \in \mathbb{N}$ be such that:*

$\cdot\ Fix(S_n) \neq \emptyset,\ \forall n \in \mathbb{N};$

$\cdot\ \{S_n\}$ *converges uniformly to $T$.*

*Then:*

*(a) $q_n \in Fix(S_n),\ n \in \mathbb{N},\ \Rightarrow\ q_n \to p$ as $n \to \infty$;*

*(b) the sequence $\{y_n\}$ defined by $y_{n+1} = S_n y_n$, converges to $p$. Moreover if $\{y_n\}$ is a sequence in $X$ such that*

$$d(y_{n+1}, S_n y_n) \to 0,\ as\ n \to \infty,$$

*then $y_n \to p$ as $n \to \infty$.*

*Proof.* (iii) $d(x,p) \leq d(x,Tx) + d(Tx,p) \leq d(x,Tx) + ld(x,p)$.

(iv) We have

$$\begin{aligned} d(y_{n+1},p) &\leq d(y_{n+1},Ty_n) + d(Ty_n,p) \leq d(y_{n+1},Ty_n) + ld(Ty_n,p) \\ &\leq ... \leq \sum_{k=0}^{n+1} l^k d(y_{n+1-k},Ty_{n-k}) \to 0\ as\ n \to \infty. \end{aligned}$$

(v) From (iii), if $Fix(S) \neq \emptyset$, we have for $q \in Fix(S)$,

$$d(q,p) \leq \frac{1}{1-l} d(q,Tq) \leq \frac{1}{1-l} d(Sq,Tq) \leq \frac{\eta}{1-l}.$$

(vi) (a) Since $\{S_n\}$ converges uniformly to $T$, then exists $\eta_n > 0,\ \eta_n \to 0$ such that

$$d(S_n x, T_n x) \leq \eta_n,\ \forall x \in X,\ n \in \mathbb{N}.$$

Now the proof follows from (v).

(b) We remark that

$$d(y_{n+1},Ty_n) \leq d(y_{n+1},S_n y_n) + d(S_n y_n,Ty_n) \to 0,\ as\ n \to \infty.$$

The proof follows from (iv). □

**Remark 4.7.** For the case of contractions see [15], pp. 393-395.

**Remark 4.8.** For the Picard mapping theory see [20].

**Remark 4.9.** In the case of a Banach space, one can use the above results to study Krasnoselskij and Mann iterations, as in Section 3 of the present paper.

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