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Connectedness via b - open sets

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ABSTRACT. In this paper we define a new type of connectedness by using b - open sets and discuss the relationship between this connectedness and various types of connectedness already defined in topological spaces.

1. INTRODUCTION

The idea of b - open sets has been introduced by Andrijević [1] in 1996, but the same is defined by El-Atik [6] under the term of γ - open sets. Formally a set A in a topological space X is said to be b - open if $A \subset Cl(Int(A)) \cup Int(Cl(A))$, where $'Cl'$ and $'Int'$ denote the closure and interior operator, respectively, in the space X. The collection of all b - open sets in a space X is denoted as $BO(X)$. The complement of a b - open set is called a b - closed set. The b - closure of a set A, denoted by $bCl(A)$, is the intersection of all b - closed sets containing A. $bCl(A)$ is the smallest b - closed set containing A. The b - interior of a set A, denoted by $bInt(A)$, is the union of all b - open sets contained in A. $bInt(A)$ is the largest b - open set contained in A.

In the theory of b - open sets, connectedness and disconnectedness [7] have already been defined. In this paper, we define and investigate the notions of $bCl - bCl$ - separated sets and $bCl - bCl$ - connected sets with the help of b - open sets in a topological space.

2. $bCl - bCl$ - SEPARATED SETS

Definition 2.1. Two subsets A and B in a space X are said to be $bCl{-}bCl$ - separated (resp. half b - separated [10], b - separated [5], $Cl - Cl$ - separated [8]) if $bCl(A) \cap bCl(B) = \emptyset$ (resp. $A \cap bCl(B) = \emptyset$ or $bCl(A) \cap B = \emptyset$, $A \cap bCl(B) = \emptyset = bCl(A) \cap B$, $Cl(A) \cap Cl(B) = \emptyset$).

Definition 2.2. (i). [7] A subset S of a space X is said to be b - connected relative to X if there are no two b - separated subsets A and B relative to X with $S = A \cup B$.

(ii). A subset A of a space X is said to be half b - connected (resp. $Cl - Cl$ - connected [8]) if A is not the union of two nonempty half - b - separated (resp. $Cl - Cl$ - separated) sets in X.

Remark 2.1. From the above definitions, we have the following implications. However, converses are not always true as shown in the following examples.

$$\begin{array}{ccccc} Cl - Cl - separated & \Longrightarrow & bCl - bCl - separated & & \\ \Downarrow & & \Downarrow & & \\ separated & \Longrightarrow & b - separated & \Longrightarrow & half - b - separated \end{array}$$

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Example 2.1. Let $X = \{p, q, r, s\}$ with a topology $\tau = \{X, \emptyset, \{p\}, \{q, r\}, \{p, q, r\}\}$. $BO(X) = \{X, \emptyset, \{p\}, \{q\}, \{r\}, \{p, q\}, \{p, r\}, \{q, r\}, \{p, q, r\}, \{p, q, s\}, \{p, r, s\}, \{q, r, s\}\}$. b - closed sets are $X, \emptyset, \{q, r, s\}, \{p, r, s\}, \{p, q, s\}, \{r, s\}, \{q, s\}, \{p, s\}, \{s\}, \{r\}, \{q\}$ and $\{p\}$. Here $\{p, q\}$ and $\{r, s\}$ are $half$ - b - $separated$ as $(\{p, q\}) \cap bCl(\{r, s\}) = \emptyset$. Since $bCl(\{p, q\}) \cap (\{r, s\}) \neq \emptyset$, so they are not b - separated.

From the fact that $bCl(A) \subset Cl(A)$ for every subset A of X, every $Cl - Cl$ - separated sets are $bCl - bCl - separated$. But the converse may not be true as shown in the following example.

Example 2.2. *From the above example $\{r\}$ and $\{s\}$ are $bCl - bCl$ - separated but they are not $Cl - Cl$ - separated.*

Example 2.3. [8] In $\mathbb{R}$ with the usual topology on $\mathbb{R}$, the sets $A = (0, 1)$ and $B = (1, 2)$ are separated sets but they are not $bCl - bCl$ - separated sets.

Example 2.4. [7] Let $X = \{p, q, r, s\}$ with a topology $\tau = \{X, \emptyset, \{p\}, \{q\}, \{p, q\}\}$. The subsets $\{p\}$, $\{r, s\}$ are b - separated but they are not separated.

Theorem 2.1. *Let A and B be $bCl - bCl$ - separated in a space X. If $C \subset A$ and $D \subset B$, then C and D are also $bCl - bCl$ - separated.*

Proof. This proof is obvious. □

Definition 2.3. A subset A of a space X is said to be $bCl - bCl$ - connected if A cannot be expressed as the union of two $bCl - bCl$ - separated sets in X.

It is obvious that every b - connected space is a $bCl - bCl$ - connected space but the converse need not hold in general.

Example 2.5. Let $\mathbb{R}$ be the real line with the usual topology on $\mathbb{R}$. Let $X = (0, 1) \cup (1, 2)$. Consider $A = (0, 1) \cap X$ and $B = (1, 2) \cap X$, then X is not a b - connected set in $\mathbb{R}$, since $X = A \cup B,\ A \cap bCl(B) = \emptyset = bCl(A) \cap B$. But the set X is a $bCl - bCl$ - connected set.

Definition 2.4. [11] A subset A of X is $V - \theta$ - connected if it cannot be expressed as the union of nonempty subsets with disjoint closed neighbourhoods in X, i.e., if there are no disjoint nonempty sets B_1 and B_2 and no open sets U and V such that $A = B_1 \cup B_2,\ B_1 \subset U, B_2 \subset V$ and $Cl(U) \cap Cl(V) = \emptyset$.

Theorem 2.2. *Every $V - \theta$ - connected space is a $bCl - bCl$ - connected space.*

Proof. Proof is obvious from the fact that $bCl(A) \subseteq Cl(A)$ for any subset A of X. □

Theorem 2.3. *A space X is $bCl - bCl$ - connected if and only if it cannot be expressed as the disjoint union of two nonempty b - clopen sets.*

Proof. Let X be a $bCl - bCl$ - connected space. If possible suppose that $X = W_1 \cup W_2$, where $W_1 \cap W_2 = \emptyset$, W_1 and W_2 are nonempty b - clopen sets of X. Since W_1 and W_2 are b - clopen sets in X, then $bCl(W_1) \cap bCl(W_2) = \emptyset$. Therefore X is not a $bCl - bCl$ - connected space. This is a contradiction.

Conversely suppose that $X \neq W_1 \cup W_2$ and $W_1 \cap W_2 = \emptyset$, where W_1 and W_2 are nonempty b - clopen sets of X. We shall prove that X is a $bCl - bCl$ - connected space.

If possible suppose that X is not a $bCl - bCl$ - connected space, then there exist $bCl - bCl$ - separated sets A and B such that $X = A \cup B$. Then, $X = bCl(A) \cup bCl(B)$ and $bCl(A) \cap bCl(B) = \emptyset$. Set $W_1 = bCl(A)$ and $W_2 = bCl(B)$. Then, W_1 and W_2 are nonempty b - clopen sets.

Moreover, we have $W_1 \cup W_2 = X$ and $W_1 \cap W_2 = \emptyset$. This is a contradiction. So, X is a $bCl - bCl$ - connected space. □

Theorem 2.4. *Let X be a space. If A is a $bCl - bCl$ - connected subset of X and H, G are $bCl - bCl$ - separated subsets of X with $A \subset H \cup G$, then either $A \subset H$ or $A \subset G$.*

Proof. Let A be a $bCl - bCl$ - connected set. Let $A \subset H \cup G$. Since $A = (A \cap H) \cup (A \cap G)$, then $bCl(A \cap G) \cap bCl(A \cap H) \subset bCl(G) \cap bCl(H) = \emptyset$. Suppose $A \cap H$ and $A \cap G$ are nonempty. Then A is not $bCl - bCl$ - connected. This is a contradiction. Thus, either $A \cap H = \emptyset$ or $A \cap G = \emptyset$. This implies that $A \subset H$ or $A \subset G$. □

Theorem 2.5. *If A and B are $bCl - bCl$ - connected sets of a space X and A and B are not $bCl - bCl$ - separated, then $A \cup B$ is $bCl - bCl$ - connected.*

Proof. Let A and B be $bCl - bCl$ - connected sets in X. Suppose $A \cup B$ is not $bCl - bCl$ - connected. Then, there exist two nonempty disjoint $bCl - bCl$ - separated sets G and H such that $A \cup B = G \cup H$. Suppose that $bCl(G) \cap bCl(H) = \emptyset$. Since A and B are $bCl - bCl$ - connected, by Theorem 2.4, either $A \subset G$ and $B \subset H$ or $B \subset G$ and $A \subset H$.

Case (i). If $A \subset G$ and $B \subset H$, then $A \cap H = B \cap G = \emptyset$. Therefore, $(A \cup B) \cap G = (A \cap G) \cup (B \cap G) = (A \cap G) \cup \emptyset = A \cap G = A$. Also, $(A \cup B) \cap H = (A \cap H) \cup (B \cap H) = B \cap H = B$. Now, $bCl(A) \cap bCl(B) = bCl((A \cup B) \cap G) \cap bCl((A \cup B) \cap H) \subset bCl(H) \cap bCl(G) = \emptyset$. Thus, A and B are $bCl - bCl$ - separated, which is a contradiction. Hence, $A \cup B$ is $bCl - bCl$ - connected.

Case (ii). If $B \subset G$ and $A \subset H$, then $B \cap H = A \cap G = \emptyset$. Therefore $(A \cup B) \cap H = (A \cap H) \cup (B \cap H) = (A \cap H) \cup \emptyset = A$. Also, $(A \cup B) \cap G = (A \cap G) \cup (B \cap G) = B \cap G = B$. Now, $bCl(A) \cap bCl(B) = bCl((A \cup B) \cap H) \cap bCl((A \cup B) \cap G) \subset bCl(H) \cap bCl(G) = \emptyset$. Thus, A and B are $bCl - bCl$ - separated, which is a contradiction. Hence, $A \cup B$ is $bCl - bCl$ - connected. □

Theorem 2.6. *If $\{M_i : i \in I\}$ is a nonempty family of $bCl - bCl$ - connected sets of a space X, with $\cap_{i \in I} M_i \neq \emptyset$, then $\cup_{i \in I} M_i$ is $bCl - bCl$ - connected.*

Proof. Suppose $\cup_{i \in I} M_i$ is not $bCl - bCl$ - connected. Then we have $\cup_{i \in I} M_i = H \cup G$, where H and G are $bCl - bCl$ - separated sets in X. Since $\cap_{i \in I} M_i \neq \emptyset$, we have a point $x \in \cap_{i \in I} M_i$. Since $x \in \cup_{i \in I} M_i$, either $x \in H$ or $x \in G$. Suppose that $x \in H$. Since $x \in M_i$ for each $i \in I$, then M_i and H intersect for each $i \in I$. By Theorem 2.4, $M_i \subset H$ or $M_i \subset G$. Since H and G are disjoint, $M_i \subset H$ for all $i \in I$ and hence $\cup_{i \in I} M_i \subset H$. This implies that G is empty. This is a contradiction. Suppose that $x \in G$. By the similar way, we have that H is empty. This is a contradiction. Thus, $\cup_{i \in I} M_i$ is $bCl - bCl$ - connected. □

Theorem 2.7. *Let X be a space, $\{A_\alpha : \alpha \in \triangle\}$ be a family of $bCl - bCl$ - connected sets and A be a $bCl - bCl$ - connected set. If $A \cap A_\alpha \neq \emptyset$ for every $\alpha \in \triangle$, then $A \cup (\cup_{\alpha \in \triangle} A_\alpha)$ is $bCl - bCl$ - connected.*

Proof. Since $A \cap A_\alpha \neq \emptyset$ for each $\alpha \in \triangle$, by Theorem 2.6, $A \cup A_\alpha$ is $bCl - bCl$ - connected for each $\alpha \in \triangle$. Moreover, $A \cup (\cup A_\alpha) = \cup(A \cup A_\alpha)$ and $\cap(A \cup A_\alpha) \supset A \neq \emptyset$. Thus by Theorem 2.6, $A \cup (\cup A_\alpha)$ is $bCl - bCl$ - connected. □

Definition 2.5. [2] A function $f : X \to Y$ is said to be b - irresolute if for each point $x \in X$ and each b - open set V of Y containing $f(x)$, there exists a b - open set U of X containing x such that $f(U) \subset V$.

Theorem 2.8. *The b - irresolute image of a $bCl - bCl$ - connected space is a $bCl - bCl$ - connected space.*

Proof. Let $f : X \to Y$ be a b - irresolute function and X be a $bCl - bCl$ - connected space. If possible suppose that $f(X)$ is not a $bCl - bCl$ - connected subset of Y. Then, there exist nonempty $bCl - bCl$ - separated sets A and B such that $f(X) = A \cup B$. Since f is b - irresolute and $bCl(A) \cap bCl(B) = \emptyset$, $bCl(f^{-1}(A)) \cap bCl(f^{-1}(B)) \subset f^{-1}(bCl(A)) \cap f^{-1}(bCl(B)) = f^{-1}(bCl(A) \cap bCl(B)) = \emptyset$. Since A and B is nonempty, $f^{-1}(A)$ and $f^{-1}(B)$ are nonempty. Therefore, $f^{-1}(A)$ and $f^{-1}(B)$ are $bCl - bCl$ - separated and $X = f^{-1}(A) \cup f^{-1}(B)$. This contradicts that X is $bCl - bCl$ - connected. Therefore, $f(X)$ is $bCl - bCl$ - connected. □

Lemma 2.1. [9] *Let (X, τ) be a topological space and $A \subset X$. Then the topologies τ, τ_s and τ_θ have the same family of open and closed sets, i.e., CO(τ_θ) = CO(τ_s) = CO(τ).*

Corollary 2.1. *The $bCl - bCl$ - connection of θ - topology, semi-regularization topology and original topology are same concept.*

Conclusion: The closure operator which is as like complement operator of the interior operator is an important part in the study of topological spaces. Generalizations are also a part of the study of topology. We study these two concepts together through this paper and define a connectedness. We have also tried to solve the question "Which relations between this connectedness and coneectedness in other literatures are there?"

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