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Some inequalities for geometrically-arithmetically h-convex functions

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ABSTRACT. In this paper, we consider a class of geometrically convex function which is called geometrically-arithmetically h-convex function. Some inequalities of Hermite-Hadamard type for geometrically-arithmetically h-convex functions are derived. Several special cases are discussed.

1. INTRODUCTION AND PRELIMINARIES

Recently much attention has been devoted to theory of convex sets and convex functions by generalizing and extending these concepts in different dimensions using innovative ideas. For useful details readers are referred to [1, 2, 3, 4, 5, 8, 9, 10, 11, 12, 13, 14, 15, 16, 18]. Ozdemir et al. [12] and Zhang et al. [18] investigated the concepts of geometrically h-convex functions.

Inspired by this ongoing research we in this paper introduce the notion of geometrically-arithmetically (GA) h-convex functions. We derive some inequalities of Hermite-Hadamard type for this new class of geometrically convex functions. This is the main motivation of this paper.

Definition 1.1. Let $h : [0,1] \to [0,\infty)$ be a non-negative function. A function $f : I \subseteq (0,\infty) \to [0,\infty)$ is said to be geometrically-arithmetically (GA) h-convex function, if

$$f\left(x^t y^{1-t}\right) \leq h(t)f(x) + h(1-t)f(y), \quad \forall x, y \in I, t \in (0,1). \tag{1.1}$$

We note that for $t = \frac{1}{2}$, we have the definition of Jensen type of geometrically-arithmetically h-convex functions, that is

$$f\left(\sqrt{xy}\right) \leq h(\frac{1}{2})[f(x) + f(y)], \quad \forall x, y \in I. \tag{1.2}$$

Note that, for $h(t) = t^s$ our definition of geometrically-arithmetically h-convex functions reduces to the definition of geometrically-arithmetically s-convex functions, which appears to be a new one.

Definition 1.2. A function $f : I \subseteq (0,\infty) \to [0,\infty)$ is said to be geometrically-arithmetically s-convex function, if

$$f\left(x^t y^{1-t}\right) \leq t^s f(x) + (1-t)^s f(y), \quad \forall x, y \in I, t \in [0,1], s \in (0,1]. \tag{1.3}$$

For $h(t) = t^{-1}$ our definition of geometrically-arithmetically h-convex functions reduces to a new definition.

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Definition 1.3. A function $f : I \subseteq (0,\infty) \to [0,\infty)$ is said to be geometrically-arithmetically Q-function, if

$$f\left(x^t y^{1-t}\right) \leq t^{-1}f(x) + (1-t)^{-1}f(y), \quad \forall x,y \in I, t \in (0,1). \tag{1.4}$$

For $h(t) = 1$ our definition of geometrically-arithmetically h-convex functions reduces to the definition of geometrically-arithmetically P-functions, which also appears to be new class.

Definition 1.4. A function $f : I \subseteq (0,\infty) \to [0,\infty)$ is said to be geometrically-arithmetically P-function, if

$$f\left(x^t y^{1-t}\right) \leq f(x) + f(y), \quad \forall x,y \in I, t \in [0,1]. \tag{1.5}$$

Definition 1.5 ([14]). A function $\mathcal{G} : I \subseteq \mathbb{R} \to \mathbb{R}$ is super-additive if

$$\mathcal{G}(x+y) \geq \mathcal{G}(x) + \mathcal{G}(y), \quad \forall x,y \in I, \; when \; x+y \in I.$$

Theorem 1.1. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function on I where $a,b \in I$ with $a < b$. Then*

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2},$$

The above double inequality is known as Hermite-Hadamard inequality in the literature. Hermite-Hadamard inequality can be considered as necessary and sufficient condition for function to be convex.
For various generalizations and extensions of Hermite-Hadamard type of inequalities, and their applications in pure and applied sciences, interested readers are referred to [1, 2, 4, 5, 6, 7, 9, 10, 11, 13, 16, 17] and the references therein.
Now we recall a result which is mainly due to Jiang et al. [7].

Lemma 1.1 ([7]). *If $f^{(n)}$ for $n \in \mathbb{N}$ exists and is integrable on $[a,b]$, then*

$$\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx - \sum_{k=2}^{n-1}\frac{(k-1)(b-a)^k}{2(k+1)!}f^{(k)}(a)$$
$$= \frac{(b-a)^n}{2n!}\int_0^1 t^{n-1}(n-2t)f^{(n)}(ta+(1-t)b)dt.$$

Due to AM-GM inequality following result is very much obvious.

Lemma 1.2. *For $a,b \in I \subseteq \mathbb{R}$ and $t \in [0,1]$, if $a < b$, then*

$$a^t b^{1-t} \leq ta + (1-t)b.$$

2. Main results

In this section, we prove our main results.

Theorem 2.2. *Let $f : I \subseteq (0,\infty) \to [0,\infty)$ be GA h-convex function on I, $a,b \in I$ with $a < b$ and $h(\frac{1}{2}) \neq 0$. Then*

$$\frac{1}{2h(\frac{1}{2})}f\left(\sqrt{ab}\right) \leq \frac{1}{\ln b - \ln a}\int_a^b \frac{f(x)}{x}dx \leq [f(a)+f(b)]\int_0^1 h(t)dt.$$

Proof. Let f be GA h-convex function. Then

$$f\left(\sqrt{xy}\right) \leq h(\frac{1}{2})[f(x) + f(y)].$$

Let $x = a^t b^{1-t}$ and $y = a^{1-t} b^t$. Then, we have

$$f\left(\sqrt{ab}\right) \leq h(\frac{1}{2})[f(a^t b^{1-t}) + f(a^{1-t} b^t)].$$

Integrating above inequality with respect to t on $[0,1]$, we have

$$\frac{1}{2h(\frac{1}{2})} f\left(\sqrt{ab}\right) \leq \frac{1}{\ln b - \ln a} \int_a^b \frac{f(x)}{x} dx. \tag{2.6}$$

Also

$$f\left(x^t y^{1-t}\right) \leq h(t) f(x) + h(1-t) f(y), \quad \forall x, y \in I, t \in (0,1).$$

Integrating above inequality with respect to t on $[0,1]$, we have

$$\frac{1}{\ln b - \ln a} \int_a^b \frac{f(x)}{x} dx \leq [f(a) + f(b)] \int_0^1 h(t) dt. \tag{2.7}$$

Combining (2.6) and (2.7), completes the proof. $\square$

I. For $h(t) = t^s$ where $s \in (0,1]$, we have

Corollary 2.1. *Let $f : I \subseteq (0,\infty) \to [0,\infty)$ be GA s-convex function on I, $a, b \in I$ with $a < b$. Then*

$$2^{s-1} f\left(\sqrt{ab}\right) \leq \frac{1}{\ln b - \ln a} \int_a^b \frac{f(x)}{x} dx \leq \frac{f(a) + f(b)}{s+1}.$$

II. For $h(t) = 1$, we have

Corollary 2.2. *Let $f : I \subseteq (0,\infty) \to [0,\infty)$ be GA P-convex function on I, $a, b \in I$ with $a < b$. Then*

$$\frac{1}{2} f\left(\sqrt{ab}\right) \leq \frac{1}{\ln b - \ln a} \int_a^b \frac{f(x)}{x} dx \leq f(a) + f(b).$$

Theorem 2.3. *Let $f : I \subseteq (0,\infty) \to [0,\infty)$ be n-times differentiable and integrable and decreasing on I, $a, b \in I$ with $a < b$. If $|f^{(n)}|$ is GA h-convex function, then*

$$\begin{aligned}
&\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx - \sum_{k=2}^{n-1} \frac{(k-1)(b-a)^k}{2(k+1)!} f^{(k)}(a) \right| \\
&\leq \frac{(b-a)^n}{2n!} \left[\mathcal{M}_1 |f^{(n)}(a)| + \mathcal{M}_2 |f^{(n)}(b)| \right],
\end{aligned}$$

where $\mathcal{M}_1 = \int_0^1 t^{n-1}(n-2t) h(t) dt$, and $\mathcal{M}_2 = \int_0^1 t^{n-1}(n-2t) h(1-t) dt$, respectively.

Proof. Using Lemma 1.1 and the fact that $|f^{(n)}|$ is decreasing and GA h-convex function, we have

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx-\sum_{k=2}^{n-1}\frac{(k-1)(b-a)^k}{2(k+1)!}f^{(k)}(a)\right|\\
&=\left|\frac{(b-a)^n}{2n!}\int_0^1 t^{n-1}(n-2t)f^{(n)}(ta+(1-t)b)dt\right|\\
&\leq \frac{(b-a)^n}{2n!}\int_0^1 t^{n-1}(n-2t)|f^{(n)}(ta+(1-t)b)|dt\\
&\leq \frac{(b-a)^n}{2n!}\int_0^1 t^{n-1}(n-2t)|f^{(n)}(a^tb^{1-t})|dt\\
&\leq \frac{(b-a)^n}{2n!}\int_0^1 t^{n-1}(n-2t)[h(t)|f^{(n)}(a)|+h(1-t)|f^{(n)}(b)|]dt\\
&=\frac{(b-a)^n}{2n!}\left[\mathcal{M}_1|f^{(n)}(a)|+\mathcal{M}_2|f^{(n)}(b)|\right].
\end{aligned}$$

This completes the proof. □

Now we discuss some special cases of Theorem 2.3.

I. For $h(t)=t^s$, Theorem 2.3 reduces to result for GA s-convex functions

Corollary 2.3. *Let $f: I\subseteq(0,\infty)\to[0,\infty)$ be n-times differentiable and integrable on I, $a,b\in I$ with $a<b$. If $|f^{(n)}|$ is GA s-function, then*

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx-\sum_{k=2}^{n-1}\frac{(k-1)(b-a)^k}{2(k+1)!}f^{(k)}(a)\right|\\
&\leq \frac{(b-a)^n}{2n!}\left[\frac{n(n-1)+s(n-2)}{(n+s)(n+s+1)}|f^{(n)}(a)|+\{n\beta(n,s+1)-2\beta(n+1,s+1)\}|f^{(n)}(b)|\right],
\end{aligned}$$

where $\beta(.,.)$ is the well known beta function.

II. For $h(t)=1$, Theorem 2.3 reduces to the following result for GA P-functions

Corollary 2.4. *Let $f: I\subseteq(0,\infty)\to[0,\infty)$ be n-times differentiable and integrable on I, $a,b\in I$ with $a<b$. If $|f^{(n)}|$ is GA P-function, then following inequality holds*

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx-\sum_{k=2}^{n-1}\frac{(k-1)(b-a)^k}{2(k+1)!}f^{(k)}(a)\right|\\
&\leq \frac{(b-a)^n}{2n!}\left(\frac{n-1}{n+1}\right)\left[|f^{(n)}(a)|+|f^{(n)}(b)|\right].
\end{aligned}$$

Theorem 2.4. *Let $f : I \subseteq (0,\infty) \to [0,\infty)$ be n-times differentiable and integrable on I $a, b \in I$ with $a < b$. If $|f^{(n)}|$ is GA h-convex function, then*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx - \sum_{k=2}^{n-1}\frac{(k-1)(b-a)^k}{2(k+1)!}f^{(k)}(a)\right|$$
$$\leq \frac{(b-a)^n}{2n!}\left(\frac{n-1}{n+1}\right)^{1-\frac{1}{q}}\left[\mathcal{M}_1|f^{(n)}(a)|^q + \mathcal{M}_2|f^{(n)}(b)|^q\right]^{\frac{1}{q}},$$

where $\mathcal{M}_1 = \int_0^1 t^{n-1}(n-2t)h(t)dt$, and $\mathcal{M}_2 = \int_0^1 t^{n-1}(n-2t)h(1-t)dt$, respectively.

Proof. Using Lemma 1.1, well known power-mean inequality and the fact that $|f^{(n)}|^q$ is decreasing and GA h-convex function, we have

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx - \sum_{k=2}^{n-1}\frac{(k-1)(b-a)^k}{2(k+1)!}f^{(k)}(a)\right|$$
$$= \left|\frac{(b-a)^n}{2n!}\int_0^1 t^{n-1}(n-2t)f^{(n)}(ta+(1-t)b)dt\right|$$
$$\leq \frac{(b-a)^n}{2n!}\left(\int_0^1 t^{n-1}(n-2t)dt\right)^{1-\frac{1}{q}}\left(\int_0^1 t^{n-1}(n-2t)|f^{(n)}(ta+(1-t)b)|^q dt\right)^{\frac{1}{q}}$$
$$\leq \frac{(b-a)^n}{2n!}\left(\frac{n-1}{n+1}\right)^{1-\frac{1}{q}}\left(\int_0^1 t^{n-1}(n-2t)|f^{(n)}(a^t b^{1-t})|^q dt\right)^{\frac{1}{q}}$$
$$\leq \frac{(b-a)^n}{2n!}\left(\frac{n-1}{n+1}\right)^{1-\frac{1}{q}}\left(\int_0^1 t^{n-1}(n-2t)[h(t)|f^{(n)}(a)|^q + h(1-t)|f^{(n)}(b)|^q]dt\right)^{\frac{1}{q}}$$
$$= \frac{(b-a)^n}{2n!}\left(\frac{n-1}{n+1}\right)^{1-\frac{1}{q}}\left[\mathcal{M}_1|f^{(n)}(a)|^q + \mathcal{M}_2|f^{(n)}(b)|^q\right]^{\frac{1}{q}}.$$

This completes the proof. □

Now we discuss some special cases of Theorem 2.4.
I. For $h(t) = t^s$, Theorem 2.4 reduces to result for GA s-convex functions

Corollary 2.5. *Let $f : I \subseteq (0,\infty) \to [0,\infty)$ be n-times differentiable and integrable on I, $a, b \in I$ with $a < b$. If $|f^{(n)}|$ is GA s-convex function, then*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx - \sum_{k=2}^{n-1}\frac{(k-1)(b-a)^k}{2(k+1)!}f^{(k)}(a)\right| \leq \frac{(b-a)^n}{2n!}\left(\frac{n-1}{n+1}\right)^{1-\frac{1}{q}}$$
$$\left[\frac{n(n-1)+s(n-2)}{(n+s)(n+s+1)}|f^{(n)}(a)|^q + \{n\beta(n,s+1) - 2\beta(n+1,s+1)\}|f^{(n)}(b)|^q\right]^{\frac{1}{q}}.$$

II. For $h(t) = 1$, Theorem 2.4 reduces to the following result for GA P-functions

Corollary 2.6. *Let $f : I \subseteq (0,\infty) \to [0,\infty)$ be n-times differentiable and integrable on I, $a, b \in I$ with $a < b$. If $|f^{(n)}|$ is GA P-function, then*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx - \sum_{k=2}^{n-1}\frac{(k-1)(b-a)^k}{2(k+1)!}f^{(k)}(a)\right| \leq \frac{(b-a)^n}{2n!}\left(\frac{n-1}{n+1}\right)\left[|f^{(n)}(a)|^q + |f^{(n)}(b)|^q\right]^{\frac{1}{q}}.$$

Theorem 2.5. *Under the assumptions of Theorem 2.4, if $|f^{(n)}| \leq M$ and h is super-additive function. Then, we have*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx - \sum_{k=2}^{n-1}\frac{(k-1)(b-a)^k}{2(k+1)!}f^{(k)}(a)\right| \leq \frac{(b-a)^n M h^{\frac{1}{q}}(1)}{2n!}\left(\frac{n-1}{n+1}\right).$$

Proof. Using Lemma 1.1, well known power-mean inequality and the fact that $|f^{(n)}|^q$ is decreasing and GA h-convex function, we have

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx - \sum_{k=2}^{n-1}\frac{(k-1)(b-a)^k}{2(k+1)!}f^{(k)}(a)\right| \\
&= \left|\frac{(b-a)^n}{2n!}\int_0^1 t^{n-1}(n-2t)f^{(n)}(ta+(1-t)b)dt\right| \\
&\leq \frac{(b-a)^n}{2n!}\left(\int_0^1 t^{n-1}(n-2t)dt\right)^{1-\frac{1}{q}}\left(\int_0^1 t^{n-1}(n-2t)|f^{(n)}(ta+(1-t)b)|^q dt\right)^{\frac{1}{q}} \\
&\leq \frac{(b-a)^n}{2n!}\left(\frac{n-1}{n+1}\right)^{1-\frac{1}{q}}\left(\int_0^1 t^{n-1}(n-2t)|f^{(n)}(a^t b^{1-t})|^q dt\right)^{\frac{1}{q}} \\
&\leq \frac{(b-a)^n}{2n!}\left(\frac{n-1}{n+1}\right)^{1-\frac{1}{q}}\left(\int_0^1 t^{n-1}(n-2t)[h(t)|f^{(n)}(a)|^q + h(1-t)|f^{(n)}(b)|^q]dt\right)^{\frac{1}{q}} \\
&\leq \frac{(b-a)^n M h^{\frac{1}{q}}(1)}{2n!}\left(\frac{n-1}{n+1}\right).
\end{aligned}$$

This completes the proof. □

Definition 2.6 ([12, 18])**.** Let $h : [0,1] \to [0,\infty)$ be a non-negative function. A function $f : I \subset (0,\infty) \to (0,\infty)$ is said to be geometrically h-convex function, if

$$f\left(x^t y^{1-t}\right) \leq (f(x))^{h(t)}(f(y))^{h(1-t)}, \quad \forall x, y \in I, t \in (0,1). \tag{2.8}$$

We note that for $t = \frac{1}{2}$, we have the definition of Jensen type of geometrically h-convex functions, that is

$$f\left(\sqrt{xy}\right) \leq (f(x)f(y))^{h(\frac{1}{2})}, \quad \forall x, y \in I. \tag{2.9}$$

Theorem 2.6. *Let $f : I \subseteq (0,\infty) \to (0,\infty)$ be n-times differentiable and integrable on I, $a, b \in I$ with $a < b$. If $|f^{(n)}|$ is GG h-convex function and $|f^{(n)}| \leq M$, then*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx - \sum_{k=2}^{n-1}\frac{(k-1)(b-a)^k}{2(k+1)!}f^{(k)}(a)\right| \leq \frac{(b-a)^n M^{qh(1)}}{2n!}\left(\frac{n-1}{n+1}\right).$$

Proof. Using Lemma 1.1, well known power-mean inequality and the fact that $|f^{(n)}|^q$ is decreasing and GG h-convex function, we have

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx - \sum_{k=2}^{n-1}\frac{(k-1)(b-a)^k}{2(k+1)!}f^{(k)}(a)\right| \\
&= \left|\frac{(b-a)^n}{2n!}\int_0^1 t^{n-1}(n-2t)f^{(n)}(ta+(1-t)b)dt\right| \\
&\leq \frac{(b-a)^n}{2n!}\left(\int_0^1 t^{n-1}(n-2t)dt\right)^{1-\frac{1}{q}}\left(\int_0^1 t^{n-1}(n-2t)|f^{(n)}(ta+(1-t)b)|^q dt\right)^{\frac{1}{q}} \\
&\leq \frac{(b-a)^n}{2n!}\left(\frac{n-1}{n+1}\right)^{1-\frac{1}{q}}\left(\int_0^1 t^{n-1}(n-2t)|f^{(n)}(a^t b^{1-t})|^q dt\right)^{\frac{1}{q}} \\
&\leq \frac{(b-a)^n}{2n!}\left(\frac{n-1}{n+1}\right)^{1-\frac{1}{q}}\left(\int_0^1 t^{n-1}(n-2t)[|(f^{(n)}(a))^{h(t)}(f^{(n)}(b))^{h(1-t)}|^q]dt\right)^{\frac{1}{q}} \\
&\leq \frac{(b-a)^n M^{qh(1)}}{2n!}\left(\frac{n-1}{n+1}\right).
\end{aligned}$$

This completes the proof. $\square$

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