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Some conjugate WP-Bailey pairs and transformation formulas for q-series

H. M. SRIVASTAVA [1,2], S. N. SINGH [3], S. P. SINGH[3] and VIJAY YADAV[4]

ABSTRACT. In this paper, the authors prove several theorems involving q-series identities by applying a certain family of conjugate WP-Bailey pairs. Making use of these theorems in conjunction with some WP-Bailey pairs, various transformation formulas for q-series are also established.

1. INTRODUCTION, NOTATIONS AND DEFINITIONS

For $q, \lambda, \mu \in \mathbb{C}$ $(|q| < 1)$, the basic (or q-) shifted factorial $(\lambda; q)_\mu$ is defined by (see, for example, [2], [9] and [10]; see also the recent works [3], [4], [5] and [8] dealing with the q-analysis)

$$(\lambda; q)_\mu := \prod_{j=0}^{\infty} \left(\frac{1 - \lambda q^j}{1 - \lambda q^{\mu+j}} \right) \qquad (|q| < 1;\ \lambda, \mu \in \mathbb{C}), \tag{1.1}$$

so that

$$(\lambda; q)_n := \begin{cases} 1 & (n = 0) \\ \prod_{j=0}^{n-1} \left(1 - \lambda q^j\right) & (n \in \mathbb{N}) \end{cases} \tag{1.2}$$

and

$$(\lambda; q)_\infty := \prod_{j=0}^{\infty} \left(1 - \lambda\, q^j\right) \qquad (|q| < 1;\ \lambda \in \mathbb{C}), \tag{1.3}$$

where, as usual, $\mathbb{C}$ denotes the set of complex numbers and $\mathbb{N}$ denotes the set of positive integers (*with* $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$). For convenience, we write

$$(a_1, \cdots, a_r; q)_n = (a_1; q)_n \cdots (a_r; q)_n \tag{1.4}$$

and

$$(a_1, \cdots, a_r; q)_\infty = (a_1; q)_\infty \cdots (a_r; q)_\infty. \tag{1.5}$$

In our investigation, we shall also make use of the basic (or q-) hypergeometric function ${}_r\Phi_s$ with r numerator and s denominator parameters, which is defined by (see, for

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Corresponding author: H. M. Srivastava; harimsri@math.uvic.ca

example, [6] and [10, p. 347, Eq. 9.4 (272)])

$$
{}_r\Phi_s\left[\begin{array}{c} a_1,\cdots,a_r; \\ \\ b_1,\cdots,b_s; \end{array} q,z\right] := \sum_{k=0}^{\infty}(-1)^{(1-r+s)k}\, q^{(1-r+s)\binom{k}{2}} \cdot \frac{(a_1;q)_k\cdots(a_r;q)_k}{(b_1;q)_k\cdots(b_s;q)_k}\,\frac{z^k}{(q;q)_k}, \tag{1.6}
$$

provided that the *generalized basic* (or q-) *hypergeometric series* in (1.6) converges.

In his study of identities of the Rogers-Ramanujan type, Bailey [2] was led to the following simple, but remarkably important, observation.

Theorem 1.1. [Bailey's Transform] (see Bailey [2]). *If*

$$
\beta_n = \sum_{r=0}^{n} \alpha_r u_{n-r} v_{n+r} \tag{1.7}
$$

and

$$
\gamma_n = \sum_{r=0}^{\infty} \delta_{r+n} u_r v_{r+2n}, \tag{1.8}
$$

then, *under suitable convergence conditions*,

$$
\sum_{n=0}^{\infty} \alpha_n \gamma_n = \sum_{n=0}^{\infty} \beta_n \delta_n, \tag{1.9}
$$

where $\{u_n\}_{n\in\mathbb{N}_0}$, $\{v_n\}_{n\in\mathbb{N}_0}$, $\{\alpha_n\}_{n\in\mathbb{N}_0}$ *and* $\{\delta_n\}_{n\in\mathbb{N}_0}$ *are arbitrarily chosen sequences.*

The proof of Theorem 1.1 is straightforward and merely requires interchanging of the order of the sums in double series.

In an application of Theorem 1.1, Bailey [2] chose

$$
u_r = \frac{1}{(q;q)_r} \qquad \text{and} \qquad v_r = \frac{1}{(aq;q)_r} \qquad (r \in \mathbb{N}_0)
$$

and derived the following result.

Theorem 1.2. (see Bailey [2]). *If*

$$
\beta_n = \sum_{r=0}^{n} \frac{\alpha_r}{(q;q)_{n-r}(aq;q)_{n+r}} \tag{1.10}
$$

and

$$
\gamma_n = \sum_{r=0}^{\infty} \frac{\delta_{r+n}}{(q;q)_r(aq;q)_{r+2n}}, \tag{1.11}
$$

then, *under suitable convergence conditions*,

$$
\sum_{n=0}^{\infty} \alpha_n \gamma_n = \sum_{n=0}^{\infty} \beta_n \delta_n, \tag{1.12}
$$

where $\{\alpha_n\}_{n\in\mathbb{N}_0}$ *and* $\{\delta_n\}_{n\in\mathbb{N}_0}$ *are arbitrarily chosen sequences.*

By definition, a Bailey pair is a pair of sequences $\langle \alpha_n, \beta_n \rangle$ that satisfy (1.10). On the other hand, a pair of sequences $\langle \gamma_n, \delta_n \rangle$ satisfying (1.11) is called a *conjugate* Bailey pair with respect to the parameter a.

Following Andrews's work on the WP-Bailey lemma (see, for details, [1]; see also [7]), a WP-Bailey pair relative to the parameter a is a pair of sequences $\langle \alpha_n(a,k), \beta_n(a,k) \rangle$ which are constrained by

$$\begin{aligned}\beta_n(a,k) &= \sum_{r=0}^{n} \frac{\left(\frac{k}{a};q\right)_{n-r}(k;q)_{n+r}}{(q;q)_{n-r}(aq;q)_{n+r}}\ \alpha_r(a,k) \\ &= \frac{\left(k,\frac{k}{a};q\right)_n}{(q,aq;q)_n} \sum_{r=0}^{n} \frac{(kq^n, q^{-n};q)_r}{\left(\frac{aq^{1-n}}{k}, aq^{1+n};q\right)_r} \left(\frac{aq}{k}\right)^r\ \alpha_r(a,k). \end{aligned} \tag{1.13}$$

Indeed, in its limit case when $k \to 0$, a WP-Bailey pair reduces to the classical Bailey pair given in (1.10).

Bailey's definition of a conjugate Bailey pair can now be extended to define a conjugate WP-Bailey pair relative to the parameter a to be a pair of sequences $\langle \gamma_n(a,k), \delta_n(a,k) \rangle$ such that

$$\gamma_n(a,k) = \sum_{r=0}^{\infty} \frac{\left(\frac{k}{a};q\right)_r (k;q)_{r+2n}}{(q;q)_r(aq;q)_{r+2n}}\ \delta_{r+n}(a,k). \tag{1.14}$$

Thus, analogous to the Bailey transform involving (1.10), (1.11) and (1.12), we have the following result.

Theorem 1.3. *Let $\langle \alpha_n(a,k), \beta_n(a,k) \rangle$ be a WP-Bailey pair. Suppose also that $\langle \gamma_n(a,k), \delta_n(a,k) \rangle$ is a conjugate WP-Bailey pair. Then, under suitable convergence conditions,*

$$\sum_{n=0}^{\infty} \alpha_n(a,k)\gamma_n(a,k) = \sum_{n=0}^{\infty} \beta_n(a,k)\delta_n(a,k). \tag{1.15}$$

In our present study of the Bailey and WP-Bailey pairs, together with their above-defined conjugates, We shall make use of following known summation formulas (see [6], [10], [11] and [12]):

$${}_2\Phi_1\left[\begin{matrix} a, b; & \\ & q, \dfrac{c}{ab} \\ c; & \end{matrix}\right] = \frac{\left(\frac{c}{a}, \frac{c}{b};q\right)_\infty}{\left(c, \frac{c}{ab};q\right)_\infty} \qquad \left(\left|\frac{c}{ab}\right| < 1\right). \tag{1.16}$$

(see [6, Appendix II, Entry (II.8)] and [10, p. 348, Eq. 9.4 (277)])

$${}_2\Phi_1\left[\begin{matrix} a, b; & \\ & q, \dfrac{c}{ab} \\ cq; & \end{matrix}\right] = \frac{\left(\frac{cq}{a}, \frac{cq}{b};q\right)_\infty}{\left(cq, \frac{cq}{ab};q\right)_\infty} \left(\frac{ab(1+c)-c(a+b)}{ab-c}\right) \tag{1.17}$$

$$\left(\left|\frac{c}{ab}\right| < 1\right).$$

(see [11, p. 771, Eq. (1.4)])

$$
{}_8\Phi_7\left[\begin{array}{c} a, q\sqrt{a}, -q\sqrt{a}, b, c, d, e, q^{-n}; \\ \sqrt{a}, -\sqrt{a}, \frac{aq}{b}, \frac{aq}{c}, \frac{aq}{d}, \frac{aq}{e}, aq^{1+n}; \end{array} q, q\right]
= \frac{\left(aq, \frac{aq}{bc}, \frac{aq}{bd}, \frac{aq}{cd}; q\right)_n}{\left(\frac{aq}{b}, \frac{aq}{c}, \frac{aq}{d}, \frac{aq}{bcd}; q\right)_n} \qquad \left(a^2 q = bcdeq^{-n}\right). \tag{1.18}
$$

(see [6, Appendix II, Entry (II.22)])

$$
{}_2\Phi_1\left[\begin{array}{c} a^2, b; \\ \frac{a^2 q}{b}; \end{array} q, \frac{q^{1/2}}{b}\right] = \frac{1}{2}\,\frac{(a^2, q^{1/2}; q)_\infty}{\left(\frac{a^2 q}{b}, \frac{q^{1/2}}{b}; q\right)_\infty}
\cdot \left(\frac{\left(\frac{aq^{1/2}}{b}; q^{1/2}\right)_\infty}{(a; q^{1/2})_\infty} + \frac{\left(-\frac{aq^{1/2}}{b}; q^{1/2}\right)_\infty}{(-a; q^{1/2})_\infty}\right) \qquad \left(\left|\frac{q^{1/2}}{b}\right| < 1\right). \tag{1.19}
$$

(see [12, p. 74, Eq. (3.5)])

$$
{}_2\Phi_1\left[\begin{array}{c} a^2, b; \\ \frac{a^2 q}{b}; \end{array} q, \frac{q^{3/2}}{b}\right] = \frac{1}{2a}\,\frac{(a^2, q^{1/2}; q)_\infty}{\left(\frac{a^2 q}{b}, \frac{q^{1/2}}{b}; q\right)_\infty}
\cdot \left(\frac{\left(\frac{aq^{1/2}}{b}; q^{1/2}\right)_\infty}{(a; q^{1/2})_\infty} - \frac{\left(-\frac{aq^{1/2}}{b}; q^{1/2}\right)_\infty}{(-a; q^{1/2})_\infty}\right) \qquad \left(\left|\frac{q^{3/2}}{b}\right| < 1\right). \tag{1.20}
$$

(see [12, p. 75, Eq. (3.6)])

$$
{}_4\Phi_3\left[\begin{array}{c} a, c, \frac{a}{c}\, q^{\frac{1}{2}+m}, q^{-m}; \\ \frac{aq}{c}, cq^{\frac{1}{2}-m}, aq^{1+m}; \end{array} q, q\right] = \frac{(a; q)_{m+1}(q^{1/2}; q)_m \left(\frac{\sqrt{aq}}{c}; q^{1/2}\right)_{2m}}{2\left(\frac{aq}{c}; q\right)_m \left(\frac{q^{1/2}}{c}; q\right)_m (a^{1/2}; q^{1/2})_{2m+1}}
+ \frac{(a; q)_{m+1}(q^{1/2}; q)_m \left(-\frac{\sqrt{aq}}{c}; q^{1/2}\right)_{2m}}{2\left(\frac{aq}{c}; q\right)_m (q^{1/2}/c; q)_m (-a^{1/2}; q^{1/2})_{2m+1}}. \tag{1.21}
$$

(see [12, p. 71, Eq. (1.3)])

$$
{}_4\Phi_3\left[\begin{array}{c} a, c, \frac{a}{c}\, q^{\frac{1}{2}+m}, q^{-m}; \\ \frac{aq}{c}, cq^{\frac{1}{2}-m}, aq^{1+m}; \end{array} q, q^2\right] = \frac{(a; q)_{m+1}(q^{1/2}; q)_m \left(\frac{\sqrt{aq}}{c}; q^{1/2}\right)_{2m}}{2\sqrt{a}\left(\frac{aq}{c}; q\right)_m \left(\frac{q^{1/2}}{c}; q\right)_m (a^{1/2}; q^{1/2})_{2m+1}}
- \frac{(a; q)_{m+1}(q^{1/2}; q)_m \left(-\frac{\sqrt{aq}}{c}; q^{1/2}\right)_{2m}}{2\sqrt{a}\left(\frac{aq}{c}; q\right)_m \left(\frac{q^{1/2}}{c}; q\right)_m (-a^{1/2}; q^{1/2})_{2m+1}}. \tag{1.22}
$$

(see [12, p. 77, Eq. (4.4)])

2. Conjugate WP-Bailey pairs and related results

(i) Equation (1.14) can be put in the following form:

$$\gamma_n(a,k) = \frac{(k;q)_{2n}}{(aq;q)_{2n}} \sum_{r=0}^{\infty} \frac{(k/a;q)_r (kq^{2n};q)_r}{(q;q)_r (aq^{1+2n};q)_r}\, \delta_{r+n}(a,k), \tag{2.23}$$

where $\langle \gamma_n(a,k), \delta_n(a,k)\rangle$ is a conjugate Bailey pair. Upon setting

$$\delta_r(a,k) = \left(\frac{a^2 q}{k^2}\right)^r$$

in (2.23), if we evaluate the resulting series by using (1.16), we get

$$\gamma_n(a,k) = \frac{(k;q)_{2n} \left(\frac{aq}{k};q\right)_\infty \left(\frac{a^2q}{k};q\right)_\infty}{\left(\frac{a^2q}{k};q\right)_{2n} (aq;q)_\infty \left(\frac{a^2q}{k^2};q\right)_\infty} \left(\frac{a^2q}{k^2}\right)^n. \tag{2.24}$$

Here, as indicated above, $\gamma_n(a,k)$ given by (2.24) and $\delta_n(a,k)$ form a conjugate Bailey pair. We are thus led from (2.24) to the following result.

Theorem 2.4. *For the WP-Bailey pair $\langle \alpha_n(a,k), \beta_n(a,k)\rangle$ satisfying* (1.13),

$$\frac{\left(\frac{aq}{k}, \frac{a^2q}{k};q\right)_\infty}{\left(aq, \frac{a^2q}{k^2};q\right)_\infty} \sum_{n=0}^{\infty} \frac{(k;q)_{2n}}{\left(\frac{a^2q}{k};q\right)_{2n}} \left(\frac{a^2q}{k^2}\right)^n \alpha_n(a,k)$$
$$= \sum_{n=0}^{\infty} \beta_n(a,k) \left(\frac{a^2q}{k^2}\right)^n \qquad \left(\left|\frac{a^2q}{k^2}\right| < 1\right). \tag{2.25}$$

(ii) *Putting*

$$\delta_r(a,k) = \left(\frac{a^2}{k^2}\right)^r$$

in (2.23) *and applying the summation formula* (1.17), *we get*

$$\gamma_n(a,k) = \frac{\left(\frac{a^2q}{k}, \frac{aq}{k};q\right)_\infty}{\left(aq, \frac{a^2q}{k^2};q\right)_\infty} \left(\frac{k}{k+a}\right) \frac{(k;q)_{2n}}{\left(\frac{a^2q}{k};q\right)_{2n}} \left(\frac{a^2}{k^2}\right)^n (1+aq^{2n}), \tag{2.26}$$

which leads us to the following result.

Theorem 2.5. *For the WP-Bailey pair $\langle \alpha_n(a,k), \beta_n(a,k)\rangle$ satisfying* (1.13),

$$\left(\frac{k}{k+a}\right) \frac{\left(\frac{a^2q}{k}, \frac{aq}{k};q\right)_\infty}{\left(aq, \frac{a^2q}{k^2};q\right)_\infty} \sum_{n=0}^{\infty} \frac{(k;q)_{2n}}{\left(\frac{a^2q}{k};q\right)_{2n}} \left(\frac{a}{k}\right)^{2n} (1+aq^{2n})\alpha_n(a,k)$$
$$= \sum_{n=0}^{\infty} \beta_n(a,k) \left(\frac{a}{k}\right)^{2n} \qquad \left(\left|\frac{a}{k}\right| < 1\right). \tag{2.27}$$

(iii) *If we first take*

$$\delta_r(a,k) = \left(\frac{a}{k}\, q^{1/2}\right)^r$$

in (2.23) *and then apply the summation formula* (1.19), *we find that*

$$\gamma_n(a,k) = \frac{1}{2}\,\frac{(k, q^{1/2};q)_\infty}{\left(aq, \frac{aq^{1/2}}{k};q\right)_\infty}\left(\frac{a}{k}\,q^{1/2}\right)^r \cdot \left(\frac{\left(\frac{aq^{1/2}}{k^{1/2}};q^{1/2}\right)_\infty (k^{1/2}q^{1/2};q^{1/2})_n}{(k^{1/2}q^{1/2};q^{1/2})_\infty \left(\frac{aq^{1/2}}{k^{1/2}};q^{1/2}\right)_n} + \frac{\left(-\frac{aq^{1/2}}{k^{1/2}};q^{1/2}\right)_\infty (-k^{1/2}q^{1/2};q^{1/2})_n}{(-k^{1/2}q^{1/2};q^{1/2})_\infty \left(-\frac{aq^{1/2}}{k^{1/2}};q^{1/2}\right)_n}\right), \tag{2.28}$$

which yields the following result.

Theorem 2.6. *For the WP-Bailey pair* $\langle \alpha_n(a,k), \beta_n(a,k)\rangle$ *satisfying* (1.13),

$$\begin{aligned}&\frac{1}{2}\,\frac{(k, q^{1/2};q)_\infty}{\left(aq, \frac{aq^{1/2}}{k};q\right)_\infty}\,\frac{\left(\frac{aq^{1/2}}{k^{1/2}};q^{1/2}\right)_\infty}{(k^{1/2}q^{1/2};q^{1/2})_\infty} \cdot \sum_{n=0}^\infty \frac{(k^{1/2}q^{1/2};q^{1/2})_n}{\left(\frac{aq^{1/2}}{k^{1/2}};q^{1/2}\right)_n}\left(\frac{a}{k}\,q^{1/2}\right)^n \alpha_n(a,k)\\ &+\frac{1}{2}\,\frac{(k, q^{1/2};q)_\infty}{\left(aq, \frac{aq^{1/2}}{k};q\right)_\infty}\,\frac{\left(-\frac{aq^{1/2}}{k^{1/2}};q^{1/2}\right)_\infty}{(-k^{1/2}q^{1/2};q^{1/2})_\infty} \cdot \sum_{n=0}^\infty \frac{(-k^{1/2}q^{1/2};q^{1/2})_n}{\left(-\frac{aq^{1/2}}{k^{1/2}};q^{1/2}\right)_n}\left(\frac{a}{k}\,q^{1/2}\right)^n \alpha_n(a,k)\\ &= \sum_{n=0}^\infty \beta_n(a,k)\left(\frac{a}{k}\,q^{1/2}\right)^n \qquad \left(\left|\frac{aq}{k^2}\right|<1\right).\end{aligned} \tag{2.29}$$

(iv) *By setting*

$$\delta_r(a,k) = \left(\frac{a}{k}\,q^{3/2}\right)^r$$

in (2.23) *and applying the summation formula* (1.20), *we obtain*

$$\gamma_n(a,k) = \frac{1}{2k^{1/2}}\,\frac{(k, q^{1/2};q)_\infty}{\left(aq, \frac{aq^{1/2}}{k};q\right)_\infty}\left(\frac{a}{k}\,q^{1/2}\right)^n \cdot \left(\frac{\left(\frac{aq^{1/2}}{k^{1/2}};q^{1/2}\right)_\infty (k^{1/2};q^{1/2})_n}{\left(\frac{aq^{1/2}}{k^{1/2}};q^{1/2}\right)_n (k^{1/2};q^{1/2})_\infty} - \frac{\left(-\frac{aq^{1/2}}{k^{1/2}};q^{1/2}\right)_\infty (-k^{1/2};q^{1/2})_n}{\left(-\frac{aq^{1/2}}{k^{1/2}};q^{1/2}\right)_n (-k^{1/2};q^{1/2})_\infty}\right). \tag{2.30}$$

Theorem 2.7 below is a consequence of (2.30).

Theorem 2.7. *For the WP-Bailey pair* $\langle \alpha_n(a,k), \beta_n(a,k)\rangle$ *satisfying* (1.13),

$$\begin{aligned}
&\frac{1}{2k^{1/2}} \frac{(k, q^{1/2}; q)_\infty}{\left(aq, \frac{aq^{1/2}}{k}; q\right)_\infty} \frac{\left(\frac{aq^{1/2}}{k^{1/2}}; q^{1/2}\right)_\infty}{(k^{1/2}; q^{1/2})_\infty} \\
&\qquad \cdot \sum_{n=0}^{\infty} \frac{(k^{1/2}; q^{1/2})_n}{(aq^{1/2}/k^{1/2}; q^{1/2})_n} \left(\frac{a}{k}\, q^{1/2}\right)^n \alpha_n(a,k) \\
&\quad - \frac{1}{2k^{1/2}} \frac{(k, q^{1/2}; q)_\infty}{\left(aq, \frac{aq^{1/2}}{k}; q\right)_\infty} \frac{\left(-\frac{aq^{1/2}}{k^{1/2}}; q^{1/2}\right)_\infty}{(-k^{1/2}; q^{1/2})_\infty} \\
&\qquad \cdot \sum_{n=0}^{\infty} \frac{(-k^{1/2}; q^{1/2})_n}{\left(-\frac{aq^{1/2}}{k^{1/2}}; q^{1/2}\right)_n} \left(\frac{a}{k}\, q^{1/2}\right)^n \alpha_n(a,k) \\
&= \sum_{n=0}^{\infty} \beta_n(a,k) \left(\frac{a}{k}\, q^{3/2}\right)^n \qquad \left(\left|\frac{aq^{3/2}}{k}\right| < 1\right).
\end{aligned} \tag{2.31}$$

3. Families of WP-Bailey pairs

In this section, we shall investigate some families of WP-Bailey pairs.

(i) In the summation formula (1.14), if we put

$$d = kq^n \qquad \text{and} \qquad e = \frac{a^2 q}{bck},$$

then it takes the following form:

$${}_8\Phi_7\left[\begin{array}{c} a, q\sqrt{a}, -q\sqrt{a}, b, c, \frac{a^2q}{bck}, kq^n, q^{-n}; \\ \sqrt{a}, -\sqrt{a}, \frac{aq}{b}, \frac{aq}{c}, \frac{bck}{a}, \frac{a}{k}\, q^{1-n}, aq^{1+n}; \end{array} q, q\right] = \frac{\left(aq, \frac{aq}{bc}, \frac{bk}{a}, \frac{ck}{a}; q\right)_n}{\left(\frac{aq}{b}, \frac{aq}{c}, \frac{k}{a}, \frac{bck}{a}; q\right)_n}. \tag{3.32}$$

Now, by putting

$$\alpha_n(a,k) = \frac{\left(a, q\sqrt{a}, -q\sqrt{a}, b, c, \frac{a^2q}{bck}; q\right)_n}{\left(q, \sqrt{a}, -\sqrt{a}, \frac{aq}{b}, \frac{aq}{c}, \frac{bck}{a}; q\right)_n} \left(\frac{k}{a}\right)^n$$

in (1.13), we find that

$$\beta_n(a,k) = \frac{\left(k, \frac{aq}{bc}, \frac{bk}{a}, \frac{ck}{a}; q\right)_n}{\left(q, \frac{aq}{b}, \frac{aq}{c}, \frac{bck}{a}; q\right)_n}. \tag{3.33}$$

Thus, clearly, $\langle \alpha_n(a,k), \beta_n(a,k)\rangle$ given in (3.33) forms a WP-Bailey pair.

(ii) If we set

$$c = \frac{a}{k}\, q^{1/2}$$

in (1.21), we get

$$
{}_4\Phi_3\left[\begin{array}{c} a, \frac{a}{k}\, q^{1/2}, kq^n, q^{-n}; \\ kq^{1/2}, \frac{a}{k}\, q^{1-n}, aq^{1+n}; \end{array} q, q\right] = \left(\frac{1+\sqrt{a}}{2}\right) \frac{(aq, \sqrt{q}; q)_n}{\left(kq^{1/2}, \frac{k}{a}; q\right)_n} \frac{\left(\frac{k}{\sqrt{a}}, k\sqrt{\frac{q}{a}}; q\right)_n}{(\sqrt{aq}, q\sqrt{a}; q)_n}
$$

$$
+ \left(\frac{1-\sqrt{a}}{2}\right) \frac{(aq, \sqrt{q}; q)_n}{\left(kq^{1/2}, \frac{k}{a}; q\right)_n} \frac{\left(-\frac{k}{\sqrt{a}}, -k\sqrt{\frac{q}{a}}; q\right)_n}{(-\sqrt{aq}, -q\sqrt{a}; q)_n}. \tag{3.34}
$$

Now, if we choose

$$
\alpha_n(a,k) = \frac{\left(a, \frac{aq^{1/2}}{k}; q\right)_n}{(q, kq^{1/2}; q)_n} \left(\frac{k}{a}\right)^n
$$

in (1.13), then we find that

$$
\beta_n(a,k) = \left(\frac{1+\sqrt{a}}{2}\right) \frac{\left(k, \sqrt{q}, \frac{k}{\sqrt{a}}, k\sqrt{\frac{q}{a}}; q\right)_n}{(q, kq^{1/2}, \sqrt{aq}, q\sqrt{a}; q)_n}
$$

$$
+ \left(\frac{1-\sqrt{a}}{2}\right) \frac{\left(k, \sqrt{q}, -\frac{k}{\sqrt{a}}, -k\sqrt{\frac{q}{a}}; q\right)_n}{(q, kq^{1/2}, -\sqrt{aq}, -q\sqrt{a}; q)_n}. \tag{3.35}
$$

Thus, clearly, $\langle \alpha_n(a,k), \beta_n(a,k) \rangle$ given in (3.35) forms a WP-Bailey pair.

(iii) Putting

$$
c = \frac{a}{k}\, q^{1/2}
$$

in (1.22), we obtain

$$
{}_4\Phi_3\left[\begin{array}{c} a, \frac{a}{k}\, q^{1/2}, kq^n, q^{-n}; \\ kq^{1/2}, \frac{a}{k}\, q^{1-n}, aq^{1+n}; \end{array} q, q^2\right] = \left(\frac{1+\sqrt{a}}{2\sqrt{a}}\right) \frac{(aq, \sqrt{q}; q)_n}{\left(kq^{1/2}, \frac{kq^{1/2}}{a}; q\right)_n} \frac{\left(\frac{k}{\sqrt{a}}, k\sqrt{\frac{q}{a}}; q\right)_n}{(\sqrt{aq}, q\sqrt{a}; q)_n}
$$

$$
- \left(\frac{1-\sqrt{a}}{2\sqrt{a}}\right) \frac{(aq, \sqrt{q}; q)_n}{\left(kq^{1/2}, \frac{kq^{1/2}}{a}; q\right)_n} \frac{\left(-\frac{k}{\sqrt{a}}, -k\sqrt{\frac{q}{a}}; q\right)_n}{(-\sqrt{aq}, -q\sqrt{a}; q)_n}. \tag{3.36}
$$

Thus, if we choose

$$
\alpha_n(a,k) = \frac{\left(a, \frac{aq^{1/2}}{k}; q\right)_n}{(q, kq^{1/2}; q)_n} \left(\frac{kq}{a}\right)^n
$$

in (1.13), we find that

$$
\beta_n(a,k) = \left(\frac{1+\sqrt{a}}{2\sqrt{a}}\right) \frac{\left(k, \sqrt{q}, \frac{k}{\sqrt{a}}, k\sqrt{\frac{q}{a}}; q\right)_n}{(q, kq^{1/2}, \sqrt{aq}, q\sqrt{a}; q)_n}
$$

$$
- \left(\frac{1-\sqrt{a}}{2\sqrt{a}}\right) \frac{\left(k, \sqrt{q}, -\frac{k}{\sqrt{a}}, -k\sqrt{\frac{q}{a}}; q\right)_n}{(q, kq^{1/2}, -\sqrt{aq}, -q\sqrt{a}; q)_n}. \tag{3.37}
$$

Clearly, therefore, $\langle \alpha_n(a,k), \beta_n(a,k) \rangle$ given in (3.37) forms a WP-Bailey pair.

4. TRANSFORMATION FORMULAS FOR q-SERIES

In this section, we propose to establish several transformation formulas for q-series.

(i) Upon substituting the WP-Bailey pair given by (3.33) into Theorem 2.4, we find that

$$\frac{\left(\frac{aq}{k},\frac{q^2q}{k};q\right)_\infty}{\left(aq,\frac{a^2q}{k^2};q\right)_\infty}\ {}_{10}\Phi_9\left[\begin{matrix}a,q\sqrt{a},-q\sqrt{a},b,c,\frac{a^2q}{bck},\sqrt{k},-\sqrt{k},\sqrt{kq},-\sqrt{kq};\\ \sqrt{a},-\sqrt{a},\frac{aq}{b},\frac{aq}{c},\frac{bck}{a},\frac{aq}{\sqrt{k}},-\frac{aq}{\sqrt{k}},a\sqrt{\frac{q}{k}},-a\sqrt{\frac{q}{k}};\end{matrix}\ q,\frac{aq}{k}\right]$$
$$=\ {}_4\Phi_3\left[\begin{matrix}k,\frac{bk}{a},\frac{ck}{a},\frac{aq}{bc};\\ \frac{aq}{b},\frac{aq}{c},\frac{bck}{a};\end{matrix}\ q,\frac{a^2q}{k^2}\right]\qquad\left(\left|\frac{a}{k}\right|<1\right). \tag{4.38}$$

(ii) Using the WP-Bailey pair given by (3.33) in Theorem 2.5, we get

$$\left(\frac{k}{k+a}\right)\frac{\left(\frac{a^2q}{k},\frac{aq}{k};q\right)_\infty}{\left(aq,\frac{a^2q}{k^2};q\right)_\infty}$$
$$\cdot{}_{10}\Phi_9\left[\begin{matrix}a,q\sqrt{a},-q\sqrt{a},b,c,\frac{a^2q}{bck},\sqrt{k},-\sqrt{k},\sqrt{kq},-\sqrt{kq};\\ \sqrt{a},-\sqrt{a},\frac{aq}{b},\frac{aq}{c},\frac{bck}{a},\frac{aq}{\sqrt{k}},-\frac{aq}{\sqrt{k}},a\sqrt{\frac{q}{k}},-a\sqrt{\frac{q}{k}};\end{matrix}\ q,\frac{a}{k}\right]$$
$$+\left(\frac{ka}{k+a}\right)\frac{\left(\frac{a^2q}{k},\frac{aq}{k};q\right)_\infty}{\left(aq,\frac{a^2q}{k^2};q\right)_\infty}$$
$$\cdot{}_{10}\Phi_9\left[\begin{matrix}a,q\sqrt{a},-q\sqrt{a},b,c,\frac{a^2q}{bck},\sqrt{k},-\sqrt{k},\sqrt{kq},-\sqrt{kq};\\ \sqrt{a},-\sqrt{a},\frac{aq}{b},\frac{aq}{c},\frac{bck}{a},\frac{aq}{\sqrt{k}},-\frac{aq}{\sqrt{k}},a\sqrt{\frac{q}{k}},-a\sqrt{\frac{q}{k}};\end{matrix}\ q,\frac{aq^2}{k}\right]$$
$$=\ {}_4\Phi_3\left[\begin{matrix}k,\frac{bk}{a},\frac{ck}{a},\frac{aq}{bc};\\ \frac{aq}{b},\frac{aq}{c},\frac{bck}{a};\end{matrix}\ q,\frac{a^2}{k^2}\right]\qquad\left(\left|\frac{a}{k}\right|<1\right). \tag{4.39}$$

(iii) Substituting the WP-Bailey pair given by (3.33) into Theorem 6, we find that

$$\frac{(k,q^{1/2};q)_\infty\left(\frac{aq^{1/2}}{k^{1/2}};q^{1/2}\right)_\infty}{(aq,aq^{1/2}/k;q)_\infty(k^{1/2}q^{1/2};q^{1/2})_\infty}$$
$$\cdot\sum_{n=0}^{\infty}\frac{(k^{1/2}q^{1/2};q^{1/2})_n}{\left(\frac{aq^{1/2}}{k^{1/2}};q^{1/2}\right)_n}\frac{\left(a,q\sqrt{a},-q\sqrt{a},b,c,\frac{a^2q}{bck};q\right)_n}{\left(q,\sqrt{a},-\sqrt{a},\frac{aq}{b},\frac{aq}{c},\frac{bck}{a};q\right)_n}\,q^{n/2}$$
$$+\frac{(k,q^{1/2};q)_\infty\left(-\frac{aq^{1/2}}{k^{1/2}};q^{1/2}\right)_\infty}{\left(aq,\frac{aq^{1/2}}{k};q\right)_\infty(-k^{1/2}q^{1/2};q^{1/2})_\infty}$$

$$\cdot \sum_{n=0}^{\infty} \frac{(-k^{1/2}q^{1/2};q^{1/2})_n}{\left(-\frac{aq^{1/2}}{k^{1/2}};q^{1/2}\right)_n} \frac{\left(a, q\sqrt{a}, -q\sqrt{a}, b, c, \frac{a^2q}{bck};q\right)_n}{\left(q, \sqrt{a}, -\sqrt{a}, \frac{aq}{b}, \frac{aq}{c}, \frac{bck}{a};q\right)_n} q^{n/2}$$

$$= 2\, {}_4\Phi_3 \left[\begin{array}{c} k, \frac{bk}{a}, \frac{ck}{a}, \frac{aq}{bc}; \\ \\ \frac{aq}{b}, \frac{aq}{c}, \frac{bck}{a}; \end{array} q, \frac{aq^{1/2}}{k} \right]. \tag{4.40}$$

(iv) Putting the WP-Bailey pair given by (3.33) in Theorem 7, we obtain

$$\frac{(k, q^{1/2};q)_\infty \left(\frac{aq^{1/2}}{k^{1/2}};q^{1/2}\right)_\infty}{\left(aq, \frac{aq^{1/2}}{k};q\right)_\infty (k^{1/2};q^{1/2})_\infty}$$

$$\cdot \sum_{n=0}^{\infty} \frac{(k^{1/2};q^{1/2})_n}{\left(\frac{aq^{1/2}}{k^{1/2}};q^{1/2}\right)_n} \frac{\left(a, q\sqrt{a}, -q\sqrt{a}, b, c, \frac{a^2q}{bck};q\right)_n}{\left(q, \sqrt{a}, -\sqrt{a}, \frac{aq}{b}, \frac{aq}{c}, \frac{bck}{a};q\right)_n} q^{n/2}$$

$$- \frac{(k, q^{1/2};q)_\infty \left(-\frac{aq^{1/2}}{k^{1/2}};q^{1/2}\right)_\infty}{\left(aq, \frac{aq^{1/2}}{k};q\right)_\infty (-k^{1/2};q^{1/2})_\infty}$$

$$\cdot \sum_{n=0}^{\infty} \frac{(-k^{1/2};q^{1/2})_n}{\left(-\frac{aq^{1/2}}{k^{1/2}};q^{1/2}\right)_n} \frac{\left(a, q\sqrt{a}, -q\sqrt{a}, b, c, \frac{a^2q}{bck};q\right)_n}{\left(q, \sqrt{a}, -\sqrt{a}, \frac{aq}{b}, \frac{aq}{c}, \frac{bck}{a};q\right)_n} q^{n/2}$$

$$= 2k^{1/2}\, {}_4\Phi_3 \left[\begin{array}{c} k, \frac{bk}{a}, \frac{ck}{a}, \frac{aq}{bc}; \\ \\ \frac{aq}{b}, \frac{aq}{c}, \frac{bck}{a}; \end{array} q, \frac{aq^{3/2}}{k} \right]. \tag{4.41}$$

Each of the WP-Bailey pairs given in (3.35) and (3.37), when substituted into Theorems 4 to 7, will similarly yield four (presumably new) transformation formulas analogous to the above results (4.38) to (4.41).

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[1]Department of Mathematics and Statistics
University of Victoria
Victoria, British Columbia V8W 3R4, Canada
[2] China Medical University
Taichung 40402, Taiwan, Republic of China
E-mail address: harimsri@math.uvic.ca

[3]Department of Mathematics
Tilak Dhari Post-Graduate College
Jaunpur 222002, Uttar Pradesh, India
E-mail address: snsp39@gmail.com, snsp39@yahoo.com

[4]Department of Mathematics and Statistics
S. P. D. T. College
Andheri (East), Mumbai 400059, Maharashtra, India
E-mail address: vijaychottu@yahoo.com