CREAT. MATH. INFORM. **24** (2015), No. 2, 213 - 219
Online version at http://creative-mathematics.ubm.ro/
Print Edition: ISSN 1584 - 286X Online Edition: ISSN 1843 - 441X

Some univalence criteria for a family of integral operators

VIRGIL PESCAR and LAURA STANCIU

ABSTRACT. The main objective of this paper is to obtain sufficient conditions for a family of integral operators to be univalent in the open unit disk $\mathcal{U}$, using new results on univalence of analytic functions. These integral operators were considered in a recent work, see [Stanciu, L., *The univalence conditions of some integral operators*, Abstr. Appl. Anal., **2012**, Art. ID 924645, 9 pp.].

1. INTRODUCTION

Let $\mathcal{A}$ denote the class of all functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n,$$

which are analytic in the open unit disk

$$\mathcal{U} = \{z : z \in \mathbb{C} \quad \text{and} \quad |z| < 1\}$$

and satisfy the following usual normalization condition

$$f(0) = f'(0) - 1 = 0,$$

$\mathbb{C}$ being the set of complex numbers.
Also, let $\mathcal{S}$ denote the subclass of $\mathcal{A}$ consisting of functions f which are univalent in $\mathcal{U}$.
In [5], Pescar gave the following univalence criteria for an integral operator.

Theorem 1.1. [5] *Let α be a complex number, $\mathrm{Re}\alpha > 0$ and c be a complex number, $|c| \leq 1$, $c \neq -1$ and $f \in \mathcal{A}$, $f(z) = z + a_2 z^2 + \ldots$. If*

$$|c| |z|^{2\mathrm{Re}\alpha} + \frac{1 - |z|^{2\mathrm{Re}\alpha}}{\mathrm{Re}\alpha} \left| \frac{z f''(z)}{f'(z)} \right| \leq 1,$$

for all $z \in \mathcal{U}$, then the function

$$F_\alpha(z) = \left(\alpha \int_0^z t^{\alpha - 1} f'(t) dt \right)^{\frac{1}{\alpha}} = z + \ldots$$

is regular and univalent in $\mathcal{U}$.

Theorem 1.2. [5] *Let α be a complex number, $\mathrm{Re}\alpha > 0$ and c be a complex number, $|c| \leq 1$, $c \neq -1$ and $f \in \mathcal{A}$. If*

$$|c| |z|^{2\mathrm{Re}\alpha} + \frac{1 - |z|^{2\mathrm{Re}\alpha}}{\mathrm{Re}\alpha} \left| \frac{z f''(z)}{f'(z)} \right| \leq 1,$$

Received: 25.05.2015. In revised form: 09.10.2015. Accepted: 16.10.2015
2010 *Mathematics Subject Classification.* 30C45, 30C75.
Key words and phrases. *Analytic functions, integral operators, univalence conditions, general Schwarz Lemma.*
Corresponding author: Virgil Pescar; virgilpescar@unitbv.ro

for all $z \in \mathcal{U}$, *then for any complex number* β, $\mathrm{Re}\beta \geq \mathrm{Re}\alpha$, *the function*

$$F_\beta(z) = \left(\beta \int_0^z t^{\beta-1} f'(t) dt\right)^{\frac{1}{\beta}}$$

is in the class $\mathcal{S}$.

In [9] it is defined the class $\mathcal{S}(p)$, which, for $0 < p \leq 2$, includes the functions $f \in \mathcal{A}$ which satisfy the conditions:

$$f(z) \neq 0, \quad 0 < |z| < 1$$

and

$$\left|\left(\frac{z}{f(z)}\right)''\right| \leq p, \quad z \in \mathcal{U}.$$

Theorem 1.3. [7] *If* $f \in \mathcal{S}(p)$, *then the following inequality is true*

$$\left|\frac{z^2 f'(z)}{[f(z)]^2} - 1\right| \leq p\,|z|^2, \quad z \in \mathcal{U}. \tag{1.1}$$

In this paper, we obtain new univalence conditions for the following integral operators:

$$F_1(f,g)(z) = \left(\alpha \int_0^z \left(f(t) e^{g(t)}\right)^{\alpha-1} dt\right)^{\frac{1}{\alpha}}, \quad f,g \in \mathcal{A}; \quad \alpha \in \mathbb{C}. \tag{1.2}$$

$$G_1(f,g)(z) = \left(\alpha \int_0^z \left(t f'(t) e^{g(t)}\right)^{\alpha-1} dt\right)^{\frac{1}{\alpha}}, \quad f,g \in \mathcal{A}; \quad \alpha \in \mathbb{C}. \tag{1.3}$$

$$H_1(f,g)(z) = \left(\beta \int_0^z t^{\beta-1} \left(\frac{f(t)}{t} e^{g(t)}\right)^{\frac{1}{\alpha}} dt\right)^{\frac{1}{\beta}}, \quad f,g \in \mathcal{A}; \quad \alpha,\beta \in \mathbb{C} - \{0\}. \tag{1.4}$$

In order to derive our main results, we need the General Schwarz Lemma (see, for details [3]).

Theorem 1.4. (General Schwarz Lemma) [3] *Let the function* f *be regular in the disk* $\mathcal{U}_R = \{z \in \mathbb{C} : |z| < R\}$, *with* $|f(z)| < M$ *for fixed* M. *If* f *has one zero with multiplicity order bigger than* m *for* $z = 0$, *then*

$$|f(z)| \leq \frac{M}{R^m} |z|^m, \quad z \in \mathcal{U}_R.$$

The equality can hold only if

$$f(z) = e^{i\theta} \frac{M}{R^m} z^m,$$

where θ *is constant.*

2. Main results

Theorem 2.5. *Let* $f, g \in \mathcal{A}$, *where* g *be in the class* $\mathcal{S}(p)$, $0 < p \leq 2$, M_1, M_2 *are real positive numbers and* α, c *are complex numbers,*

$$\mathrm{Re}\alpha > |\alpha - 1| \left(\mathrm{M}_1 + \mathrm{M}_2^2 (\mathrm{p}+1) + 1\right), \quad |\mathrm{c}| \leq 1, \mathrm{c} \neq -1.$$

If

$$\left|\frac{z f'(z)}{f(z)}\right| \leq M_1, \quad z \in \mathcal{U}, \quad |g(z)| < M_2, \quad z \in \mathcal{U}$$

and

$$|c| \leq 1 - \frac{|\alpha - 1|}{\text{Re}\alpha} \left(M_1 + M_2^2 (p+1) + 1 \right), \tag{2.5}$$

then the function $F_1(f,g)(z)$ defined by (1.2) is in the class $\mathcal{S}$.

Proof. We begin by observing that the function $F_1(f,g)(z)$ in (1.2) can be rewritten as follows:

$$F_1(f,g)(z) = \left(\alpha \int_0^z t^{\alpha-1} \left(\frac{f(t)}{t} e^{g(t)} \right)^{\alpha-1} dt \right)^{\frac{1}{\alpha}}.$$

Let us define the function $h(z)$ by

$$h(z) = \int_0^z \left(\frac{f(t)}{t} e^{g(t)} \right)^{\alpha-1} dt.$$

The function h is regular in $\mathcal{U}$ and satisfies the following normalization condition $h(0) = h'(0) - 1 = 0$. Now, calculating the derivatives of $h(z)$ of the first and second orders, we readily obtain

$$h'(z) = \left(\frac{f(z)}{z} e^{g(z)} \right)^{\alpha-1} \tag{2.6}$$

and

$$h''(z) = (\alpha - 1) \left(\frac{f(z)}{z} e^{g(z)} \right)^{\alpha-2} \left(\frac{zf'(z) - f(z)}{z^2} e^{g(z)} + \frac{f(z)}{z} g'(z) e^{g(z)} \right). \tag{2.7}$$

We easily find from (2.6) and (2.7) that

$$\frac{zh''(z)}{h'(z)} = (\alpha - 1) \left[\left(\frac{zf'(z)}{f(z)} - 1 \right) + zg'(z) \right],$$

which readily shows that

$$\begin{aligned}
&|c|\,|z|^{2\text{Re}\alpha} + \frac{1 - |z|^{2\text{Re}\alpha}}{\text{Re}\alpha} \left| \frac{zh''(z)}{h'(z)} \right| \\
&= |c|\,|z|^{2\text{Re}\alpha} + \frac{1 - |z|^{2\text{Re}\alpha}}{\text{Re}\alpha} \left| (\alpha - 1) \left(\left(\frac{zf'(z)}{f(z)} - 1 \right) + zg'(z) \right) \right| \\
&\leq |c| + \frac{|\alpha - 1|}{\text{Re}\alpha} \left(\left(\left| \frac{zf'(z)}{f(z)} \right| + 1 \right) + \left| \frac{z^2 g'(z)}{[g(z)]^2} \right| \left| \frac{[g(z)]^2}{z} \right| \right), \quad z \in \mathcal{U}.
\end{aligned} \tag{2.8}$$

From the hypothesis of Theorem 2.5, we have

$$\left| \frac{zf'(z)}{f(z)} \right| \leq M_1, \quad z \in \mathcal{U}, \quad |g(z)| < M_2, \quad z \in \mathcal{U},$$

then by *General Schwarz Lemma* for the function g, we obtain

$$|g(z)| \leq M_2 |z|, \quad z \in \mathcal{U}.$$

Using the inequality (2.8), we have

$$\begin{aligned}
&|c|\,|z|^{2\text{Re}\alpha} + \frac{1 - |z|^{2\text{Re}\alpha}}{\text{Re}\alpha} \left| \frac{zh''(z)}{h'(z)} \right| \\
&\leq |c| + \frac{|\alpha - 1|}{\text{Re}\alpha} \left(M_1 + 1 + \left(\left| \frac{z^2 g'(z)}{[g(z)]^2} - 1 \right| + 1 \right) M_2^2 \right).
\end{aligned} \tag{2.9}$$

Since $g \in \mathcal{S}(p)$, $0 < p \leq 2$, using (2.5), from (2.9), we obtain

$$|c|\,|z|^{2\mathrm{Re}\alpha} + \frac{1-|z|^{2\mathrm{Re}\alpha}}{\mathrm{Re}\alpha}\left|\frac{zh''(z)}{h'(z)}\right| \leq |c| + \frac{|\alpha-1|}{\mathrm{Re}\alpha}\left(M_1 + M_2^2\,(p+1)+1\right) \leq 1, \quad z \in \mathcal{U}.$$

Finally, by applying Theorem 1.1, we conclude that the integral operator $F_1(f,g)(z)$ defined by (1.2) is in the class $\mathcal{S}$. □

Setting $M_1 = 1$ and $M_2 = 1$ in Theorem 2.5, we immediately arrive at the following application of Theorem 2.5.

Corollary 2.1. *Let $f, g \in \mathcal{A}$, where g be in the class $\mathcal{S}(p)$, $0 < p \leq 2$ and α, c are complex numbers,* $\mathrm{Re}\alpha > |\alpha - 1|\,(\mathrm{p}+3)\,,\ |c| < 1$.

If

$$\left|\frac{zf'(z)}{f(z)}\right| \leq 1, \quad z \in \mathcal{U}, \quad |g(z)| < 1, \quad z \in \mathcal{U}$$

and

$$|c| \leq 1 - \frac{|\alpha-1|}{\mathrm{Re}\alpha}\,(p+3)\,,$$

then the function $F_1(f,g)(z)$ defined by (1.2) is in the class $\mathcal{S}$.

Theorem 2.6. *Let $f, g \in \mathcal{A}$, where g be in the class $\mathcal{S}(p)$, $0 < p \leq 2$, M_1 is a real positive number and α, c are complex numbers,*

$$\mathrm{Re}\alpha > |\alpha-1|\left(M_1^2\,(p+1)+1\right), \quad |c| < 1.$$

If

$$\left|\frac{f''(z)}{f'(z)}\right| \leq 1, \quad z \in \mathcal{U}, \quad |g(z)| < M1, \quad z \in \mathcal{U} \tag{2.10}$$

and

$$|c| \leq 1 - \frac{|\alpha-1|}{\mathrm{Re}\alpha}\left(M_1^2\,(p+1)+1\right), \tag{2.11}$$

then the integral operator $G_1(f,g)(z)$ defined by (1.3) is in the class $\mathcal{S}$.

Proof. We observe that

$$G_1(f,g)(z) = \left(\alpha \int_0^z t^{\alpha-1}\left(f'(t)e^{g(t)}\right)^{\alpha-1} dt\right)^{\frac{1}{\alpha}}.$$

Let us consider the function

$$h(z) = \int_0^z \left(f'(t)e^{g(t)}\right)^{\alpha-1} dt. \tag{2.12}$$

The function h is regular in $\mathcal{U}$. From (2.12), we have

$$h'(z) = \left(f'(z)e^{g(z)}\right)^{\alpha-1}$$

and

$$\frac{zh''(z)}{h'(z)} = (\alpha-1)\left(\frac{zf''(z)}{f'(z)} + zg'(z)\right),$$

which readily shows that

$$\begin{aligned}
&|c|\,|z|^{2\mathrm{Re}\alpha} + \frac{1-|z|^{2\mathrm{Re}\alpha}}{\mathrm{Re}\alpha}\left|\frac{zh''(z)}{h'(z)}\right| \\
&\le |c| + \frac{|\alpha-1|}{\mathrm{Re}\alpha}\left(\left|\frac{zf''(z)}{f'(z)}\right| + \left|\frac{z^2g'(z)}{[g(z)]^2}\right|\left|\frac{[g(z)]^2}{z}\right|\right) \\
&\le |c| + \frac{|\alpha-1|}{\mathrm{Re}\alpha}\left(\left|\frac{zf''(z)}{f'(z)}\right| + \left(\left|\frac{z^2g'(z)}{[g(z)]^2}-1\right|+1\right)\left|\frac{[g(z)]^2}{z}\right|\right).
\end{aligned}$$

From (2.10), (2.11) and applying General Schwarz Lemma for the function $g(z)$, we obtain $|g(z)| \le M_1\,|z|\,,\, z \in \mathcal{U}$, we obtain

$$\begin{aligned}
|c|\,|z|^{2\mathrm{Re}\alpha} + \frac{1-|z|^{2\mathrm{Re}\alpha}}{\mathrm{Re}\alpha}\left|\frac{zh''(z)}{h'(z)}\right| &\le |c| + \frac{|\alpha-1|}{\mathrm{Re}\alpha}\left(1+M_1^2\,(p+1)\right) \\
&\le 1, \quad z \in \mathcal{U}.
\end{aligned}$$

Applying Theorem 1.1, we conclude that the integral operator $G_1(f,g)(z)$ defined by (1.3) is in the class $\mathcal{S}$. □

Setting $M_1 = 1$ in Theorem 2.6, we obtain the following consequence of Theorem 2.6.

Corollary 2.2. *Let $f,g \in \mathcal{A}$, where g be in the class $\mathcal{S}(p)$, $0 < p \le 2$ and α, c are complex numbers,* $\mathrm{Re}\alpha > |\alpha-1|\,(p+2)\,,\, |c| \le 1,\, c \ne -1.$

If

$$\left|\frac{f''(z)}{f'(z)}\right| \le 1, \quad z \in \mathcal{U}, \quad |g(z)| < 1, \quad z \in \mathcal{U}$$

and

$$|c| \le 1 - \frac{|\alpha-1|}{\mathrm{Re}\alpha}\,(p+2)\,,$$

then the integral operator $G_1(f,g)(z)$ defined by (1.3) is in the class $\mathcal{S}$.

Theorem 2.7. *Let $f,g \in \mathcal{A}$, where g be in the class $\mathcal{S}(p)$, $0 < p \le 2$, M_1, M_2 are real positive numbers, α, c are complex numbers, $|c| \le 1$, $c \ne -1$ and*

$$\mathrm{Re}\alpha > \frac{\mathrm{M_1 + M_2^2\,(p+1)+1}}{|\alpha|}. \tag{2.13}$$

If

$$\left|\frac{zf'(z)}{f(z)}\right| \le M_1, \quad z \in \mathcal{U}, \quad |g(z)| < M_2, \quad z \in \mathcal{U}$$

and

$$|c| \le 1 - \frac{M_1 + M_2^2\,(p+1)+1}{|\alpha|\,\mathrm{Re}\alpha},$$

then for any complex number β, $\mathrm{Re}\beta \ge \mathrm{Re}\alpha$ the integral operator $H_1(f,g)(z)$ defined by (1.4) is in the class $\mathcal{S}$.

Proof. Let us consider the function

$$h(z) = \int_0^z \left(\frac{f(t)}{t}e^{g(t)}\right)^{\frac{1}{\alpha}} dt. \tag{2.14}$$

The function h is regular in $\mathcal{U}$. From (2.14), we have

$$h'(z) = \left(\frac{f(z)}{z} e^{g(z)}\right)^{\frac{1}{\alpha}}$$

and

$$\frac{zh''(z)}{h'(z)} = \frac{1}{\alpha}\left(\left(\frac{zf'(z)}{f(z)} - 1\right) + zg'(z)\right),$$

which readily shows that

$$|c|\,|z|^{2\mathrm{Re}\alpha} + \frac{1-|z|^{2\mathrm{Re}\alpha}}{\mathrm{Re}\alpha}\left|\frac{zh''(z)}{h'(z)}\right|$$
$$\le |c| + \frac{1}{|\alpha|\,\mathrm{Re}\alpha}\left(\left|\frac{zf'(z)}{f(z)}\right| + 1 + \left(\left|\frac{z^2 g'(z)}{[g(z)]^2} - 1\right| + 1\right)\left|\frac{[g(z)]^2}{z}\right|\right). \tag{2.15}$$

By the general Schwarz Lemma for the function $g(z)$, we obtain $|g(z)| \le M_2\,|z|$, $z \in \mathcal{U}$, since $g \in \mathcal{S}(p)$, $0 < p \le 2$ and using the inequality (2.15), we have

$$|c|\,|z|^{2\mathrm{Re}\alpha} + \frac{1-|z|^{2\mathrm{Re}\alpha}}{\mathrm{Re}\alpha}\left|\frac{zh''(z)}{h'(z)}\right|$$
$$\le |c| + \frac{1}{|\alpha|\,\mathrm{Re}\alpha}\left(M_1 + 1 + M_2^2\,(p+1)\right), \quad z \in \mathcal{U}. \tag{2.16}$$

From (2.13), we have

$$\frac{1}{|\alpha|\,\mathrm{Re}\alpha}\left(M_1 + 1 + M_2^2\,(p+1)\right) \le 1 - |c|$$

and using (2.16), we obtain

$$|c|\,|z|^{2\mathrm{Re}\alpha} + \frac{1-|z|^{2\mathrm{Re}\alpha}}{\mathrm{Re}\alpha}\left|\frac{zh''(z)}{h'(z)}\right| \le 1, \quad z \in \mathcal{U}.$$

Applying Theorem 1.2, we conclude that the integral operator $H_1(f,g)(z)$ defined by (1.4) is in the class $\mathcal{S}$. □

Setting $M_1 = 1$ and $M_2 = 1$ in Theorem 2.7, we obtain

Corollary 2.3. *Let $f, g \in \mathcal{A}$, where g be in the class $\mathcal{S}(p)$, $0 < p \le 2$, α, c are complex numbers, $|c| \le 1$, $c \ne -1$ and $\mathrm{Re}\alpha > \frac{3+\mathrm{p}}{|\alpha|}$.*

If

$$\left|\frac{zf'(z)}{f(z)}\right| \le 1, \quad z \in \mathcal{U}, \quad |g(z)| < 1, \quad z \in \mathbb{U}$$

and

$$|c| \le 1 - \frac{3+p}{|\alpha|\,\mathrm{Re}\alpha},$$

then for any complex number β, $\mathrm{Re}\beta \ge \mathrm{Re}\alpha$, the integral operator $H_1(f,g)(z)$ defined by (1.4) is in the class $\mathcal{S}$.

REFERENCES

[1] Blezu, D., *On univalence criteria*, Gen. Math., **14** (2006), 77–84
[2] Blezu, D. and Pascu, R. N., *Univalence criteria for integral operators*, Glas. Mat., **36** (2001), 241–245
[3] Nehari, Z., Conformal Mapping, Dover, New York, NY, USA, 1975
[4] Pescar, V., *On the univalence of some integral operators*, Gen. Math., **14** (2006), No. 2, 77–84
[5] Pescar, V., *New generalizations of Ahlfors's, Becker's and Pascu's univalence criterions*, Acta Univ. Apul., No. **34** (2013), 173–178
[6] Pescar, V., *Univalence of certain integral operators*, Acta Univ. Apulensis Math. Inform., No. 12, (2006), 43–48
[7] Singh, V., *On a class of univalent functions*, Int. J. Math. Math. Sci., **23** (2000), 855–857
[8] Stanciu, L., *The univalence conditions of some integral operators*, Abstr. Appl. Anal., **2012**, Art. ID 924645, 9 pp.
[9] Yang, D. and Liu, J., *On a class of univalent functions*, Int. J. Math. Math. Sci. **22** (1999), No. 3, 605–610

DEPARTMENT OF MATHEMATICS
"TRANSILVANIA" UNIVERSITY OF BRAŞOV
FACULTY OF MATHEMATICS AND COMPUTER SCIENCE
BRAŞOV, ROMANIA
E-mail address: virgilpescar@unitbv.ro

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF PITEŞTI
PITEŞTI, ROMANIA
E-mail address: laura_stanciu_30@yahoo.com