CREAT. MATH. INFORM. **Online version at** `http://creative-mathematics.ubm.ro/`
24 (2015), No. 2, 221 - 230 **Print Edition: ISSN 1584 - 286X Online Edition: ISSN 1843 - 441X**

Applying a utility based fuzzy probabilistic α-cut method to optimize a constrained multi objective model

MOHAMMADREZA TORKJAZI and HAMED FAZLOLLAHTABAR

ABSTRACT. This article is proposing an appropriate approach to solve a constrained multi objective model by using the theory of utility functions in fuzzy form. One of the approaches to optimize a multi objective mathematical model is to employ utility functions for the objectives. Recent studies on utility based multi objective optimization concentrate on considering just one utility function for each objective. But, in reality it is not reasonable to have a unique utility function corresponding to each objective function. Here, a constrained multi objective mathematical model is considered in which several utility functions are associated for each objective. All of these utility functions are uncertain and in fuzzy form, so a fuzzy probabilistic approach is incorporated to investigate the uncertainty of the utility functions for each objective and the total utility function of the problem will be a fuzzy nonlinear mathematical model. Since there are not any conventional approaches to solve such a model, a defuzzification method to change the total utility function to a crisp nonlinear model is employed. Meanwhile, α-cut method is applied to defuzzify the conditional utility functions. This action results in changing the total utility function to a crisp single objective nonlinear model and will simplify the optimization process of the total utility function. The effectiveness of the proposed approach is shown by solving a test problem.

1. INTRODUCTION

Multiple objectives often conflict with each other and require multi-objective approaches rather than a single objective approach [14]. There are three approaches to solve a multi-objective problem: A priori approaches, Interactive approaches, and a posteriori approaches [6]. From a decision maker perspective, the choice of a solution from all presented efficient solutions is called a posteriori approach. In the a priori approach, the decision maker expresses his preference relative to the objectives in one of two ways. The first one consists of attaching weights to each objective and combining them in a linear function [9, 10, 12]. In the second approach, objectives are ranked in decreasing order of importance; the problem is solved for the first objective, and then the second problem is solved for the second objective under the constraint that the optimal solution of the first objective does not change. This single-objective problem process is continued until the last objective [3, 4, 5, 11, 13]. In the interactive approach, the decision maker intervenes in the optimization process to guide it to the most suitable solutions [1, 2].

Research that considers uncertainty can be categorized according to the four primary approaches [8]: (1) Stochastic programming approach, (2) fuzzy programming approach, (3) stochastic dynamic programming approach, and (4) robust optimization approach. The second one seeks the solution considering some variables as fuzzy numbers [8].

In this paper, the Utility function method is applied to solve a multi-objective problem which is a method in the class of a priori approaches. The main question in this study

Received: 02.01.2015. In revised form: 09.09.2015. Accepted: 16.09.2015
2010 *Mathematics Subject Classification.* 90C29.
Key words and phrases. *Multi objective optimization, fuzzy probabilistic, utility function.*
Corresponding author: Hamed Fazlollahtabar; hfazl@iust.ac.ir

is how multiple objectives having several utility functions can be optimized. So, a multiple objective mathematical model having several utility functions is optimized under uncertainty. Fuzzy probabilistic programming is developed for multiple utility functions.

The organization of this paper is as follows. In Section 2, the utility function method accompanying with our proposed method are described. In Section 3 a numerical example of a multi-objective problem is solved by the proposed method. Finally in Section 4 our conclusions are given.

2. Proposed method

The Utility Function Method proposed by Keeney and Raiffa in 1976, is one of the approaches in solving a Multi-objective problem [7]. In this method, a conditional utility function to each objective should be assigned. For each objective, while the other objectives are fixed in their levels, we can check whether the objective is utility independent from other ones or not. Then the conditional utility functions should be combined and form the total utility function. The last step in the method is optimizing the total utility function which means maximizing the utility of the decision maker.

But there is a problem for using this method in real world, because this method does not assume all of the situations in the problem's environment. To simply explain these situations, consider a manufacturing environment in a company. Maximizing the profit is one of the most important goals to company managers. In this case, just one utility function for the given objective can be used which is maximizing the profit for an 8 hour shift. But sometimes there are also other shifts starting after the main shift for employees. Usually because of some factors such as employees' fatigue and environmental conditions or sometimes company's conditions, working in the main shift is more favorable to employees than working in the overtime shifts. Thus the same utility function for the overtime period cannot be used, while the objective function is maximizing the profit, another utility function for the overtime period has to be defined. So for better modeling of the problem in the real world and to have the nearest solutions to the ideal solution, different utility functions have to be defined for each objective based on different situations and different environments.

In this paper we assume that each objective function has several utility functions, while researchers in the past used to consider just one utility function for each objective.

Since companies plan for their future periods of time, they cannot use the crisp data in their planning. In other words, these plans are always considered under uncertainty. Assume that a plan is designated for a long time period of time for a company. To do this, different utility functions can be assigned to each one of the objectives based on the different situations in the company's environment. Because of the uncertainty conditions in the problem, each utility function corresponds to a fuzzy set and there is a probability for each utility function for occurring in the assumed period of the planning time. These functions can be called as Partial Utility Functions. So the first step in our method is calculating the probabilities of the partial utility functions. After that these partial utility functions should be combined to achieve the conditional utility function for each objective. So there will be only one conditional utility function for each objective. The next Step is combining conditional utility functions and creating the total utility function. At last by optimizing the total utility function, the utility of the decision maker can be maximized.

In this paper, the utility functions are in fuzzy environment and each of them is dependent to a fuzzy set which describes the uncertainty in the problem. So the crisp results cannot be achieved for their probabilities. To calculate the probability of these fuzzy functions, the Yager's method can be used [16]. The first step for calculating these probabilities is defining the set M for each partial utility functions. To do that, for every partial utility function, each variable should be selected from the crisp universe X with a degree of membership based on the conditions of the occurrence of the function. Then these variables and their degree of membership can be used to create the set M for the function.

Probability of a Fuzzy Event as a Fuzzy set

Yager [15] suggests a definition for the probability of a fuzzy event, which is derived as follows.

The truth of the proposition "the probability $\tilde{A}$ is at least w" is defined as the fuzzy set $P_y^*(\tilde{A})$ with the membership function $P_y^*(\tilde{A})\,(w) = \sup_{\alpha} \{ \alpha \mid P(A_\alpha) \geq w \}\ w \in [0,1]$. The complement of $\tilde{A}$ can be defined by $C\tilde{A} = \{(x, 1 - \mu_{\tilde{A}}(x)) | x \in X\}$ then $P_y^*(C\tilde{A})\,(w) = \sup_{\alpha} \{ \alpha | P(CA_\alpha) \geq w \}$ and $w \in [0,1]$ can be interpreted as the truth of the proposition "the probability of not $\tilde{A}$ is at least w."

On the other hand $\bar{P}_y^*(\tilde{A}) = 1 - P_y^*(C\tilde{A})$ can be interpreted as the truth of the proposition "probability of $\tilde{A}$ is at most w", Hence the following definition [16]:

Definition 2.1. The possibility distribution associated with the proposition "the probability of $\tilde{A}$ is exactly w" can be defined as

$$\bar{P}_y(\tilde{A})(w) = \min \left\{ P_y^*(\tilde{A})(w), \bar{P}_y^*(\tilde{A})(w) \right\} \tag{2.1}$$

Example 2.1. For example consider the following fuzzy set:

$\tilde{A} = \{(x_1, 1), (x_2, 0.7), (x_3, 0.6), (x_4, 0.2)\}$
$P_{x_1} = 0.1$, $P_{x_2} = 0.4$, $P_{x_3} = 0.3$, $P_{x_4} = 0.2$

α	A_α	P(A_α)	w	$P_y^*(\tilde{A})(w)$
[0,0.2]	[x_1,x_2,x_3,x_4]	1	[0.8,1]	0.2
[0.2,0.6]	[x_1,x_2,x_3]	0.8	[0.5,0.8]	0.6
[0.6,0.7]	[x_1,x_2]	0.5	[0.1,0.5]	0.7
[0.7,1]	[x_1]	0.1	[0,0.1]	1

α	(CA)$_\alpha$	P(CA)$_\alpha$	w	$P_y^*(C\tilde{A})(w)$	$\bar{P}_y^*(\tilde{A})(w)$
0	[x_1,x_2, x_3,x_4]	1	[0.9, 1]	0	1
[0,0.3]	[x_2,x_3, x_4]	0.9	[0.5,0.9]	0.3	0.7
[0.3,0.4]	[x_3,x_4]	0.5	[0.2,0.5]	0.4	0.6
[0.4,0.8]	[x_4]	0.2	[0, 0.2]	0.8	0.2
[0.8,1]	0	0	0	1	0

The probability $\bar{P}_y(\tilde{A})$ of the fuzzy event $\tilde{A}$ is now determined by the intersection of the fyzzy sets $P_y^*(\tilde{A})$ and $\bar{P}_y^*(\tilde{A})$ modeled by the min-operator as in definition 2.1:

$$\bar{P}_y(\tilde{A})(w) = \begin{cases} 0 \ , \ w = 0 \\ 0.2 \ , \ w \in [0, 0.2] \\ 0.6 \ , \ w \in [0.2, 0.8] \\ 0.2 \ , \ w \in [0.8, 1] \end{cases}$$

For example consider a problem in which the objective function corresponds to three partial utility functions.

$$f_1(x) \;\; \Rightarrow \;\; \begin{cases} \tilde{u}_{11} = a_1\tilde{x}_1 + b_1\tilde{x}_2 + c_1\tilde{x}_3 & \Rightarrow \\ \tilde{u}_{12} = a_2\tilde{x}_1 + b_2\tilde{x}_2 + c_2\tilde{x}_3 & \Rightarrow \\ \tilde{u}_{13} = a_3\tilde{x}_1 + b_3\tilde{x}_2 + c_3\tilde{x}_3 & \Rightarrow \end{cases} \begin{cases} \tilde{M}_{u_{11}} = \{(x_1,\mu_1),(x_2,\mu_2),(x_3,\mu_3)\} \\ \tilde{M}_{u_{12}} = \{(x_1,\mu_1),(x_2,\mu_2),(x_3,\mu_3)\} \\ \tilde{M}_{u_{13}} = \{(x_1,\mu_1),(x_2,\mu_2),(x_3,\mu_3)\} \end{cases} \tag{2.2}$$

Combining the partial utility functions The next step for solving the problem is combining the partial utility functions and achieving the conditional utility function for each objective. To do this, a coefficient for each partial utility function is needed. To achieve the coefficients we can use the probabilities calculated earlier.

On the other hand, the probabilities of the partial utility functions are fuzzy sets and each member of these sets is a probability period with a degree of membership, while a crisp coefficient for each utility function is needed. So, to deal with this, some levels of probabilities can be used by cutting these fuzzy sets of probabilities based on the decision maker's opinion. Since the crisp probability for each variable were available at first for the whole planning period, so the probability of each partial utility function has been calculated for the whole planning period. But in the proposed method, it has been decided to divide the whole period to partial periods and then assign a partial utility function to each one of these partial periods. So it is rational to use some cuts on the fuzzy sets of probabilities.

For example consider a company with two shifts for its employees and the objective is maximizing the profit. After checking the conditions existing in the company, it's been decided that the objective should have two partial utility functions. On the other hand, the manager wants to achieve at least 60% of the company's profit in the first shift and if the profit of the first shift is less than 60% of the whole profit in the planning period, there will be no utility or satisfaction for the manager. So the partial utility function of the first shift should occur at least 60% of the whole planning period and in result, the partial utility function for the second shift should also occur at most 40% of the whole period. The manager's request should be applied on the probability sets and therefore those members of the probability set of the first partial utility function which are in the interval of 60% to 100% should be selected for this function and those members of the probability set of the second partial utility function which are in the interval 0% to 40% should be selected for the second utility function.

Now for calculating the coefficients, we can sum the degree of the memberships of the elements selected for each of the partial utility functions from the probability sets, call this summation "*s*". After that for each objective, we should sum all of these "*s*" and then it will be called cumulative value "*S*". By dividing each"*s*" by "*S*", a coefficient"*t*" for each partial utility function will be achieved.

Example 2.2. consider the probabilities for two partial utility functions $\tilde{u}_1$ and $\tilde{u}_2$

$$P_{\tilde{u}_1} = \{((0,0.6),0.3),((0.6,0.8),0.1),((0.8,1),0.3)\}$$
$$P_{\tilde{u}_2} = \{((0,0.2),0.3),((0.2,0.4),0.4),((0.4,1),0.2)\}$$

Consider that the manager requested that the function $\tilde{u}_1$ must occur at least 60% and the function $\tilde{u}_2$ must occur at most 40% of the whole period.

$s_1 = \mu_{(0.6,\,0.8)} + \mu_{(0.8,\,1)} = 0.1 + 0.3 = 0.4$ $\qquad s_2 = \mu_{(0,\,0.2)} + \mu_{(0.2,\,0.4)} = 0.3 + 0.4 = 0.7$

$S = s_1 + s_2 = 0.3 + 0.7 = 1.1$ $\qquad t_1 = \frac{s_1}{S} = 0.36 \quad , \quad t_2 = \frac{s_2}{S} = 0.64$.

The coefficients of the partial utility functions are as $f_1(x) \Rightarrow \left\{ \begin{array}{c} \tilde{u}_{11} \\ \tilde{u}_{12} \\ \tilde{u}_{13} \end{array} \right\} \begin{array}{c} \Rightarrow \\ \Rightarrow \\ \Rightarrow \end{array} \left\{ \begin{array}{c} t_{11} \\ t_{12} \\ t_{13} \end{array} \right\}$.

Since the partial utility functions are not utility independent from each other, for combining them and creating the conditional utility functions, we can simply add them together with their coefficients. $\tilde{U}_1$ is conditional utility function for $f_1(x)$.

$$\tilde{U}_1 = t_{11}\tilde{u}_{11} + t_{12}\tilde{u}_{12} + t_{13}\tilde{u}_{13} \tag{2.3}$$

Combining the conditional utility functions. Based on the Keeney and Raiffa [7], there are different forms to combine utility functions. In this paper, formation of the multiplicative utility function is used. Let us first define the mutual utility independence and then state Theorem 2.1 [7].

Definition 2.2. Attributes $X_1 , X_2 , \dots , X_n$ are mutually utility independent if every subset of $\{X_1 , X_2 , \dots , X_n\}$ is utility independent of its complement.

Theorem 2.1. *If attributes $X_1 , X_2 , \dots , X_n$ are mutually utility independent, then*

$$u\,(x) = \sum_{i=1}^{n} k_i u_i(x) + K \sum_{\substack{i=1 \\ j>i}}^{n} k_i . k_j . u_i(x_i) . u_j(x_j) + K^2 \sum_{\substack{i=1 \\ j>i \\ \ell>j}}^{n} k_i . k_j . k_\ell . u_i(x_i). \\ . u_j(x_j) . u_\ell(x_\ell) + \dots + K^{n-1} k_1 k_2 \ . \ . \ . \ k_n \, u_1(x_1)\, u_2(x_2) \ . \ . \ . \ u_n(x_n)\,, \tag{2.4}$$

where

i) *u is normalized by $u\,(x_1^0, x_2^0, \dots , x_n^0) = 0$ and $u\,(x_1^*, x_2^*, \dots , x_n^*) = 1$;*

ii) *$u_i(x_i)$ is a conditional utility function on X_i normalized by $u_i(x_i^0) = 0$ and $u_i(x_i^*) = 1$, $i = 1 , 2 , \dots , n$;*

iii) $k_i = u\,(x_i^*, \bar{x}_i^0)$;

iv) *K is a scaling constant that is a solution to:* $\qquad 1 + K = \prod_{i=1}^{n} (1 + K . k_i)$

Defuzzification of the conditional utility functions As known, the conditional utility functions of the problem we considered are fuzzy functions. So after combining these functions by using the multiplicative formation, the Total utility function will be a nonlinear fuzzy function that can't be solved with the existing methods. Therefor each conditional utility function should be defuzzified and be changed to a crisp function. There are many methods for defuzzification. One of these methods is using the α-cuts.

This method is based on the fuzzy set of each conditional utility function. In this paper, a fuzzy set to each partial utility function is considered. Then the coefficients for these functions were calculated to combine them and achieve a conditional utility function for each objective. For combining the partial utility functions, it has been decided to multiply every coefficient to its related function and then add these values to each other. Here the same can be done with the fuzzy sets of partial utility functions. In other words, the algebraic sum can be used to sum the fuzzy sets of the partial utility functions and achieve the fuzzy set of the conditional utility function.

$$\tilde{C} = \tilde{A} + \tilde{B} = (\tilde{B} + \tilde{A}) \quad \Rightarrow \qquad \mu_{\tilde{C}}(x) = \mu_{\tilde{A}+\tilde{B}}(x) = \mu_{\tilde{A}}(x) + \mu_{\tilde{B}}(x) - \mu_{\tilde{A}}(x).\mu_{\tilde{B}}(x) \tag{2.5}$$

Defuzzification by using α-cuts In this method, after achieving the fuzzy set of the conditional utility functions, the least degree of membership among the elements in this fuzzy set, must be selected and then based on this degree an α-cut will be applied. This action changes the fuzzy set to a crisp set. So, after that, the fuzzy conditional utility function pertaining to that crisp set will be a crisp conditional utility function. Consider the following fuzzy set:

$$\tilde{M}_{\tilde{U}_1} = \{(x_1,\ \mu_1)\,,\ (x_2,\ \mu_2)\,,\ (x_3,\ \mu_3)\} \qquad \Rightarrow \qquad M_{\alpha=\min \mu_i} = \{x_1, x_2, x_3\}\,.$$

Calculating the scaling constants. After the defuzzification and achieving the crisp conditional utility functions, calculating the scaling constants to combine the conditional utility functions is needed. Based on the Eq. (4), each conditional utility function has a scaling constant. To achieve these constants the indifference level of the decision maker should be calculated. By asking the decision maker, his indifference level is concluded as $(f_1, f_2) \approx (\beta,\ 0) \ \sim\ (0,\ \lambda)\,.$

Recall that $u\,(x_i)$ actually means $u\,(x_1^0,\ \dots\ ,\ x_{i-1}^0,\ x_i,\ x_{i+1}^0,\ \dots\ ,\ x_n^0)$. Hence

$$u\,(x_i) = k_i\, u_i(x_i) \tag{2.6}$$

$$U\,(f_1,\ 0) = k_1\, U_{f_1}\,(f_1,\ 0)\ ,$$
$$U\,(0,\ f_2) = k_2\, U_{f_2}\,(0,\ f_2)\ .$$

So by considering the indifference level of the decision maker
$U\,(\beta,\ 0) = U\,(0,\ \lambda) \qquad \Rightarrow \qquad k_1 U_{f_1}\,(\beta,\ 0) = k_2\, U_{f_2}\,(0,\ \lambda)$
In the above equation, the conditional utility functions can be substituted with their values. So the unknown parameters will be only k_1 and k_2. In this way a relation between k_1 and k_2 can be achieved. Then, the fourth property of the Theorem 2.1 to calculate all of the scaling constants can be used.

Total utility function. Based on Eq. (2.4), the total utility function "U(F)" is a non-fuzzy and non-linear

$$U(F) = k_1 U_{(f_1)} + k_2 U_{(f_2)} + K\,.\,k_1.\,k_2\,.\,U_{(f_1)}.U_{(f_2)}, \tag{2.7}$$

and the non-linear programming methods can be used to solve it

3. Numerical example

A numerical application of the proposed method with two objectives is presented in order to illustrate its capabilities in dealing with multi-objective optimization problems.

$$\begin{array}{l} \max\ \ f_1(x) = 2x_1 + 2.4x_2 + 3x_3 \\ \max\ \ f_2(x) = 1.8x_1 + 1.5x_2 \\ \left.\begin{array}{rl} s.t.\ \ 3x_1 + x_2 + 1.5x_3 & \leq 400 \\ x_1 + x_2 + \ \ 2x_3 & \leq 250 \\ 1.2x_2 + 0.4x_3 & \leq 150 \\ x_1, x_2, x_3 & \geq 0 \end{array}\right\} \ \rightarrow\ Z \end{array}$$

In order to use the utility function method, at first each one of these objectives should be solved as a single-objective problem to find the optimum value of them. These optimum values will be considered as the values which their utility functions equals to 1. In other words, if these objectives reach their optimum values, the utility level of the decision maker will be maximized which equals to 1.

$$\begin{cases} \max \ \ f_1(x) \\ s.t. \ \ x \in Z \end{cases} \Rightarrow \begin{cases} f_1^* = 520.435 \\ x_1^* = 81.52 \\ x_2^* = 116.3 \\ x_3^* = 26.09 \end{cases} \qquad \begin{cases} \max \ \ f_2(x) \\ s.t. \ \ x \in Z \end{cases} \Rightarrow \begin{cases} f_2^* = 352.5 \\ x_1^* = 91.67 \\ x_2^* = 125 \\ x_3^* = 0 \end{cases}$$

$$\Rightarrow \begin{cases} 0 \le f_1^* \le 520.435 \\ P_{x_1} = 0.36 \\ P_{x_2} = 0.52 \\ P_{x_3} = 0.12 \end{cases} \qquad \Rightarrow \begin{cases} 0 \le f_2^* \le 352.5 \\ P_{x_1} = 0.42 \\ P_{x_2} = 0.58 \\ P_{x_3} = 0 \end{cases}$$

Based on the questions asked from the decision maker pertaining to the objectives of the problem, it has been decided to define three utility functions for each objective.

$$f_1(x) \ \Rightarrow \begin{cases} \tilde{u}'_{11} = 2\tilde{x}_1 + 3\tilde{x}_2 + 2\tilde{x}_3 & \Rightarrow \\ \tilde{u}'_{12} = \tilde{x}_1 + 2\tilde{x}_2 + \tilde{x}_3 & \Rightarrow \\ \tilde{u}'_{13} = \tilde{x}_1 + \tilde{x}_2 + 3\tilde{x}_3 & \Rightarrow \end{cases} \begin{cases} \tilde{M}_{11} = \{(x_1, 0.5), (x_2, 0.3), (x_2, 0.4)\} \\ \tilde{M}_{12} = \{(x_1, 0.3), (x_2, 0.4), (x_2, 0.2)\} \\ \tilde{M}_{13} = \{(x_1, 0.2), (x_2, 0.3), (x_2, 0.4)\} \end{cases}$$

$$f_2(x) \ \Rightarrow \begin{cases} \tilde{u}'_{21} = \tilde{x}_1 + 2\tilde{x}_2 & \Rightarrow \\ \tilde{u}'_{22} = \tilde{x}_1 + \tilde{x}_2 & \Rightarrow \\ \tilde{u}'_{23} = 2\tilde{x}_1 + \tilde{x}_2 & \Rightarrow \end{cases} \begin{cases} \tilde{M}_{21} = \{(x_1, 0.6), (x_2, 0.3)\} \\ \tilde{M}_{22} = \{(x_1, 0.2), (x_2, 0.4)\} \\ \tilde{M}_{23} = \{(x_1, 0.2), (x_2, 0.3)\} \end{cases}$$

Then we should normalize these utility functions:

$$\bar{\tilde{u}}_1 = \tilde{u}'_{11} + \tilde{u}'_{12} + \tilde{u}'_{13} = 4\tilde{x}_1 + 6\tilde{x}_2 + 6\tilde{x}_3 \ , \qquad \bar{\tilde{u}}_2 = \tilde{u}'_{21} + \tilde{u}'_{22} + \tilde{u}'_{23} = 4\tilde{x}_1 + 4\tilde{x}_2 \ .$$

$$\begin{cases} \tilde{u}_{11} = \frac{\tilde{u}'_{11}}{\bar{\tilde{u}}_1} = \frac{2\tilde{x}_1+3\tilde{x}_2+2\tilde{x}_3}{4\tilde{x}_1+6\tilde{x}_2+6\tilde{x}_3} \\ \tilde{u}_{12} = \frac{\tilde{u}'_{12}}{\bar{\tilde{u}}_1} = \frac{\tilde{x}_1+2\tilde{x}_2+\tilde{x}_3}{4\tilde{x}_1+6\tilde{x}_2+6\tilde{x}_3} \\ \tilde{u}_{13} = \frac{\tilde{u}'_{13}}{\bar{\tilde{u}}_1} = \frac{\tilde{x}_1+\tilde{x}_2+3\tilde{x}_3}{4\tilde{x}_1+6\tilde{x}_2+6\tilde{x}_3} \end{cases} \begin{cases} \tilde{u}_{21} = \frac{\tilde{u}'_{21}}{\bar{\tilde{u}}_2} = \frac{\tilde{x}_1+2\tilde{x}_2}{4\tilde{x}_1+4\tilde{x}_2} \\ \tilde{u}_{22} = \frac{\tilde{u}'_{22}}{\bar{\tilde{u}}_2} = \frac{\tilde{x}_1+\tilde{x}_2}{4\tilde{x}_1+4\tilde{x}_2} \\ \tilde{u}_{23} = \frac{\tilde{u}'_{23}}{\bar{\tilde{u}}_2} = \frac{2\tilde{x}_1+\tilde{x}_2}{4\tilde{x}_1+4\tilde{x}_2} \end{cases}$$

The results presented in Table 1. are obtained by performing the Yager's Method [15]:

Table 1. Probabilities of the partial utility functions

$\tilde{u}$	$\bar{P}_y(\tilde{M})$
$\tilde{u}_{11}$	$\bar{P}_y(\tilde{M}_{11}) = \{(0, 0), ([0, 1], 0.3)\}$
$\tilde{u}_{12}$	$\bar{P}_y(\tilde{M}_{12}) = \begin{Bmatrix} (0, 0), ([0, 0.12], 0.2), ([0.12, 0.48], 0.3), ([0.48, 0.52], 0.4), \\ ([0.52, 0.88], 0.3), ([0.88, 1], 0.2) \end{Bmatrix}$
$\tilde{u}_{13}$	$\bar{P}_y(\tilde{M}_{13}) = \{ (0, 0), ([0, 0.36], 0.2), ([0.36, 0.64], 0.3), ([0.64, 1], 0.2) \}$
$\tilde{u}_{21}$	$\bar{P}_y(\tilde{M}_{21}) = \{(0, 0), ([0, 1], 0.3)\}$
$\tilde{u}_{22}$	$\bar{P}_y(\tilde{M}_{22}) = \{(0, 0), ([0, 1], 0.2)\}$
$\tilde{u}_{23}$	$\bar{P}_y(\tilde{M}_{13}) = \{ (0, 0), ([0, 0.42], 0.2), ([0.42, 0.58], 0.3), ([0.58, 1], 0.2) \}$

After that the cuts over the probability sets should be applied based on the decision maker's opinion to calculate the coefficients as given in Table 2.

Table 2. Coefficients of the partial utility functions

$\tilde{u}$	P	s	S_1	t	$\tilde{u}$	P	s	S_1	t
$\tilde{u}_{11}$	[0.6 , 1]	s_{11} = 0.3	0.3	t_{11}= 0.2	$\tilde{u}_{21}$	[0.7 , 1]	s_{21} = 0.3	0.3	t_{21}= 0.3
$\tilde{u}_{12}$	[0.3 , 0.6]	s_{12} = 1	1.3	$t_{12=}$ 0.67	$\tilde{u}_{22}$	[0.5 , 0.7]	s_{22} = 0.2	0.5	t_{22}= 0.2
$\tilde{u}_{13}$	[0 , 0.3]	s_{13} = 0.2	1.5	t_{13}= 0.13	$\tilde{u}_{23}$	[0 , 0.5]	s_{23} = 0.5	1	t_{23}= 0.5

$$\tilde{U}_{(f_1)} = t_{11}\tilde{u}_{11} + t_{12}\tilde{u}_{12} + t_{13}\tilde{u}_{13} \qquad \tilde{U}_{(f_2)} = t_{21}\tilde{u}_{21} + t_{22}\tilde{u}_{22} + t_{23}\tilde{u}_{23}$$
$$\tilde{U}_{(f_1)} = \frac{1.2\tilde{x}_1+2.07\tilde{x}_2+1.46\tilde{x}_3}{4\tilde{x}_1+6\tilde{x}_2+6\tilde{x}_3} \qquad \tilde{U}_{(f_2)} = \frac{1.5\tilde{x}_1+1.3\tilde{x}_2}{4\tilde{x}_1+4\tilde{x}_2}$$

The next step is the defuzzification of the utility functions.

$$\tilde{M}_{\tilde{U}_{(f_1)}} = \tilde{M}_{\tilde{u}_{11}} + \tilde{M}_{\tilde{u}_{12}} + \tilde{M}_{\tilde{u}_{13}} \qquad \tilde{M}_{\tilde{U}_{(f_2)}} = \tilde{M}_{\tilde{u}_{21}} + \tilde{M}_{\tilde{u}_{22}} + \tilde{M}_{\tilde{u}_{23}}$$
$$\tilde{M}_{\tilde{U}_{(f_1)}} = \{(x_1,\ 0.72),(x_2,\ 0.71),(x_3,\ 0.71)\} \Rightarrow M_{\alpha=\min \mu_i=0.71} = \{x_1,x_2,x_3\}$$
$$\tilde{M}_{\tilde{U}_{(f_2)}} = \{(x_1,\ 0.74),(x_2,\ 0.71)\} \Rightarrow M_{\alpha=\min \mu_i=0.71} = \{x_1,x_2\}$$
$$\tilde{U}_{(f_1)} = \frac{1.2\tilde{x}_1+2.07\tilde{x}_2+1.46\tilde{x}_3}{4\tilde{x}_1+6\tilde{x}_2+6\tilde{x}_3} \quad \Rightarrow U_{(f_1)} = \frac{1.2x_1+2.07x_2+1.46x_3}{4x_1+6x_2+6x_3}$$
$$\tilde{U}_{(f_2)} = \frac{1.5\tilde{x}_1+1.3\tilde{x}_2}{4\tilde{x}_1+4\tilde{x}_2} \Rightarrow U_{(f_2)} = \frac{1.5x_1+1.3x_2}{4x_1+4x_2}$$

After defuzzification of the utility functions the scaling constants should be calculated:

$$\begin{cases} U_{(F)}\ (0,\,0) = 0 \\ U_{(F)}\ (520.435\ ,\ \ 352.5) = 1 \end{cases} \Rightarrow \begin{cases} U_{(F)}\ (520.435\,,\ 0) = k_1 \\ U_{(F)}\ (0\,,\ 352.5) = k_2 \end{cases}$$

and $\begin{cases} U_{(F)}\ (f_1,\,0) = k_1 U_{(f_1)}\,(f_1\,,\,0) \\ U_{(F)}\ (0,\ f_2) = k_2 U_{(f_2)}\,(0\,,\ f_2)\ . \end{cases}$

The indifference level of the decision maker has been described as

$$(f_1,\,f_2)\ \approx\ (250\,,\ 0) \sim (0\,,\ 352.5)$$
$$U_{(F)}\,(250\,,\ 0) = U_{(F)}\,(0\,,\ 352.5)$$
$$k_1.\,U_{(f_1)}\,(250\,,\ 0) = k_2\,.\,U_{(f_2)}\,(0\,,\ 352.5)$$
$$k_1.\,U_{(f_1)}\,(250\,,\ 0) = k_2\ .$$

On the other hand, we know that

$$\begin{cases} U_{(f_1)}\ (0,\,0) = 0 \\ U_{(f_1)}\ (520.435\,,\ 0) = 1 \end{cases} \quad \begin{cases} U_{(f_2)}\ (0,\,0) = 0 \\ U_{(f_2)}\ (0\,,\ 352.5) = 1 \end{cases}$$

and $\frac{1}{520.435} = \frac{U_{(f_1)}(250\,,0)}{250} \quad \to \quad U_{(f_1)}\,(250,0) = 0.48 \quad \to \quad K = \frac{1-1.48k_1}{0.48k_1^2}$

In order to calculate k_1, the following procedure can be used. By asking from the decision maker we conclude that

$$\{(520.435\ ,\ \ 352.5) : (0\ ,\ 0)\}\ \sim\ (320\ ,\ 160)\ ;$$
$$(320\ ,\ 160) \begin{array}{l} \Rightarrow \frac{1}{2} \Rightarrow \ (520.435\ ,\ \ 352.5) \\ \Rightarrow \frac{1}{2} \Rightarrow \qquad\qquad (0\ ,\ 0) \end{array}$$
$$U_{(F)}(320,160) = \tfrac{1}{2}U_{(F)}(520.435,352.5) + \tfrac{1}{2}U_{(F)}(0,0) = 0.5$$

So by considering this value in the Eq. (2.7), we have

$$U(F) = 0.5 = k_1 U_{(f_1=320)} + k_2 U_{(f_2=160)} + K\,.\,k_1.\,k_2\,.\,U_{(f_1=320)}.\,U_{(f_2=160)} \tag{3.8}$$

On the other hand it is known that $U_{(f_1)}\,(320\,,\ 0) = 0.61 \quad , \quad U_{(f_2)}\,(0\,,\ 160) = 0.45$

So by solving Eq. (3.8) $\quad k_1 = 0.54\ ,\quad k_2 = 0.26\ ,\quad K = 1.43\ .$
Finally, the total utility function will be expressed as

$$U(f_1, f_2) = \left(\frac{0.65x_1+1.12x_2+0.79x_3}{4x_1+6x_2+6x_3}\right) + \left(\frac{0.39x_1+0.34x_2}{4x_1+4x_2}\right)$$
$$+\left(\frac{0.36x_1^2+0.93x_1x_2+0.54x_2^2+0.38x_2x_3+0.44x_1x_3}{16x_1^2+40x_1x_2+24x_2^2+24x_1x_3+24x_2x_3}\right)$$

and the problem will be changed to the following form

$$\begin{array}{ll} \max & U(f_1, f_2) \\ s.t. & 3x_1 + x_2 + 1.5x_3 \le 400 \\ & x_1 + x_2 + \quad 2x_3 \le 250 \\ & 1.2x_2 + 0.4x_3 \le 150 \\ & x_1, x_2, x_3 \ge 0 \quad . \end{array} \qquad \begin{array}{l} U^*(f_1, f_2) = 0.294 \\ x^* = (0\ ,\ 125\ ,\ 0) \\ F^* = (300\ ,\ 187.5) \end{array}$$

Note that the numerical example is based on a production system and the related data are taken from the normal production shift. The utility functions are formed regarding to the management policies. The example can be designed with respect to other utilities or other objectives whether minimize or maximize, and the procedure can be repeated to obtain the solution. For example one can substitute this problem with a more complicated constrained multi objective problem which has so many factors to be considered in the process of the algorithm., i.e., the complexity does not influence the solution capability.

4. MAIN RESULTS

This paper proposed a new method to solve multi objective problem using multiple utility functions for each objective. In the classic utility method there is only one utility function for each objective. Based on different situations in the programming environment and the uncertainty in decision making, fuzzy consideration was included. To solve the problem, Yager's method in the field of fuzzy probabilistic and the combination formation of conditional utility functions suggested by Keeney and Raiffa [15, 16] called multiplicative, were adapted. Since the conditional utility functions were in the fuzzy environment, the α-cut method was used to defuzzify the functions. Finally the total utility function of the problem was optimized to achieve the best solutions for the objectives and maximize the utility of the decision maker.

To check the capability of the proposed method, a numerical illustration extracted from an industrial unit has been applied in the form of a constrained multi objective problem. Based on the experimental results it can be concluded that the proposed methodology enables the decision making process for optimizing a constrained multi-objective problem using the utility function method under uncertainty.

As future researches the following topics could be of interest: Here we considered linear utility functions while exponential utility functions could also be worked on, using linguistic variables as utility based on preferences of decision makers, surveying the fuzzy transition matrix among multiple utility functions.

REFERENCES

[1] Arroyo, J. E. C. and Armentano, V. A., *Genetic local search for multi-objective flowshop scheduling problems*, European J. Oper. Res., **167** (2005), 717–738

[2] Bernardo, J. J. and Lin, K.-S., *An interactive procedure forbi-criteria production scheduling*, Computers and Oper. Res., **21** (1994), 677–688

[3] Gupta, J. N. D., Palanimuthu, N. and Chen, C.-L., *Designing a tabu search algorithm for the two-stage flow shop problem with secondary criterion*, Production Planning and Control, **10** (1999), 251–265

[4] Gupta, J. N. D., Neppalli, V. R. and Werner, F., *Minimizing total flow time in a two-machine flowshop problem with minimum makespan*, Int. J. Production Economics, **69** (2006), 323–338

[5] Gupta, J. N. D., Hennig, K. and Werner, F., *Local search heuristics for two-stage flow shop problems with secondary criterion*, Computers and Oper. Res., **2** (2002), 123–149

[6] Jozefowiez, N., Semet, F. and Talbi, E., *Multi-objective vehicle routing problems*, European J. Oper. Res., **189** (2008), 293–309

[7] Keeney and Raiffa, *Decisions with Multiple Objectives*, Cambridge Uni. Press, (1993), 289–305

[8] Mirzapour Al-e-hashem, S. M. J., Malekly, H. and Aryanezhad, M. B., *A multi-objective robust optimization model for multi-product multi-site aggregate production planning in a supply chain under uncertainty*, Int. J. Production Economics, **134** (2011), 28–42

[9] Nagar, A., Heragu, S. and Haddock, J., *A branch and bound approach for a 2-machine flowshop scheduling problem*, J. Oper. Res. Society, **46** (1995b), 721–734

[10] Rajendran, C., *Heuristics for scheduling in flowshop with multiple objectives*, European J. Oper. Res., **82** (1995), 540–555

[11] Rajendran, C., *Two-stage flow shop scheduling problem with bi criteria*, J. Oper. Res. Society, **43** (1992), 871–884

[12] Serifoglu, F. S. and Ulusoy, G., *A bicriteria two-machine permutation flowshop problem*, European J. Oper. Res., **107** (1998), 414–430

[13] T'kindt, V. and Billaut, J.-C., *Multicriteria scheduling problems:A survey*, RAIRO Oper. Res., **35** (2001), 143–163

[14] Yang, L., Jones, B. F. and Yang, S., *A fuzzy multi-objective programming for optimization of fire station locations through genetic algorithms*, European J. Oper. Res., **181** (2007), 903–915

[15] Yager, R. R., *A representation of the probability of fuzzy subsets*, FSS, **13** (1984), 273–283

[16] Zimmermann, H. J., *Fuzzy Set Theory and Its Application*, Kluwer Academic Publishers Group, (1996), 109–121

Mazandaran University of Science and Technology
Department of Industrial Engineering
Babol, Iran
E-mail address: hfazl@iust.ac.ir