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The $\Re_a$ operator in ideal topological spaces

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ABSTRACT. Given a topological space (X,τ) an ideal $\mathcal{I}$ on X and $A \subseteq X$, the concept of a-local function is defined as follows $A^{a^*}(\mathcal{I},\tau) = \{x \in X : U \cap A \notin \mathcal{I}$, for every $U \in \tau^a(x)\}$. In this paper a new type of space has been introduced with the help of a-open sets and the ideal topological space called a-ideal space. We introduce an operator $\Re_a : \wp(X) \to \tau$, for every $A \in \wp(X)$, and we use it to define some interesting generalized a-open sets and study their properties.

1. INTRODUCTION

Ideals in topological spaces have been considered since 1930. This topic has won its importance by Vaidaynathaswamy [20, 11]. In [10], Jankovic and Hamlett investigated further properties of ideal topological spaces. Applications to various fields were further investigated by Hamlett and Jankovic [9]; Dontchev et al. [5]; Mukherjee et al. [14]; Arenas et al. [4]; Navaneethakrishnan et al. [16]; Nasef and Mahmoud [17] etc.

In an ideal topological space Kuratowski [11] has defined the concept of local function. T. Natkaniec [15] in 1986 introduced another operator ψ in ideal topological spaces. Hamlett and Jankovic in [9] and Modak and Bandyopadhyay in [13] have considered the operator ψ and discussed its properties in detail. They also have shown that the operator ψ gives an interior operator which is the interior operator of the topology defined by Jankovic and Hamlett in [10].

In this paper we consider a new type of space, called **a**-ideal space, by replacing τ of the ideal topological space $(X,\tau,\mathcal{I})$ by the set of all a-open sets of (X,τ). With the help of a-local functions [1] we shall also define and study an operator $\Re_a : \wp(X) \to \tau$ which is defined as $\Re_a(A) = \{x \in X :$ there exists $U_x \in \tau^a$ containing x such that $U_x - A \in \mathcal{I}\}$, for every $A \in \wp(X)$. Its equivalent definition is $\Re_a(A) = X - (X - A)^{a^*}$. Further, we also discuss the properties of this operator.

2. PRELIMINARIES

Throughout this paper, (X,τ) and (Y,σ) always mean topological spaces on which no separation axiom is assumed. A subset A of a space (X,τ) is said to be regular open (resp. regular closed) [19] if $A = Int(Cl(A))$ $(A = Cl(Int(A))$, respectively).

A is called δ-open [21] if, for each $x \in A$, there exists a regular open set G such that $x \in G \subset A$. The complement of δ-open set is called δ-closed. A point $x \in X$ is called a δ-cluster point of A if $Int(Cl(U)) \cap A \neq \emptyset$, for each open set V containing x. The set of all δ-cluster points of A is called the δ-closure of A and is denoted by $Cl_\delta(A)$ [21]. The set δ-interior of A is the union of all regular open sets of X contained in A and its denoted by $Int_\delta(A)$ [21]. A is δ-open if $Int_\delta(A) = A$. δ-open sets form a topology τ^δ. The collection of all δ-open sets in X is denoted by $\delta O(X)$. For more details see [2, 3].

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A subset A of a space (X, τ) is said to be a-open (resp. a-closed) [6] if $A \subset Int(Cl(\delta - Int(A)))$ $(Cl(Int(\delta - Cl(A))) \subset A$, respectively), or $A \subset Int(Cl(\delta Int(A)))$ $(Cl(Int(\delta Cl(A)))$ $\subset A$, respectively). The family of a-open sets of X forms a topology on X, denoted by τ^a [7]. The family of all a-open (respectively a-closed) sets containing x is denoted by $\tau^a(x)$ $(aC(X, x)$, respectively).

If A is a subset of a space X, then the intersection of all a-closed sets containing A is called the a-closure of A and is denoted by $aCl(H)$. The a-interior of A, denoted by $aInt(A)$, is defined by the union of all a-open sets contained in A [6].

An ideal $\mathcal{I}$ on a topological space $(X, \mathcal{I})$ is a nonempty collection of subsets of X which satisfies the following conditions:

$A \in \mathcal{I}$ *and* $B \subset A$ *implies* $B \in \mathcal{I}$; $A \in \mathcal{I}$ *and* $B \subset A$ *implies* $A \cup B \in \mathcal{I}$.

Applications of these concepts to various fields were further investigated by Jankovic and Hamlett [10] Dontchev et al. [5]; Mukherjee et al. [14]; Arenas et al. [4]; Navaneethakrishnan et al. [16]; Nasef and Mahmoud [17] etc.

Given a topological space $(X, \mathcal{I})$ with an ideal $\mathcal{I}$ on X and if $\wp(X)$ denotes the set of all subsets of X, a set operator $(.)^* : \wp(X) \to \wp(X)$, called a local function [22, 10] of A with respect to τ and $\mathcal{I}$, is defined as follows: for $A \subseteq X$,

$$A^*(\mathcal{I}, \tau) = \{x \in \tau \mid U \cap A \notin \text{for every U} \in \tau(x)\},$$

where $\tau(x) = \{U \in \tau \mid x \in U\}$.

A Kuratowski closure operator is $Cl^*(x) = A \cup A^*(\mathcal{I}, \tau)$. When there is no chance for confusion, we will simply write A^* for $A^*(\mathcal{I}, \tau)$. X^* is often a proper subset of X. The hypothesis $X = X^*$ is equivalent to the hypothesis $\tau \cap \mathcal{I} = \emptyset$ and τ^* for $\tau^*(\mathcal{I}, \tau)$. If $\mathcal{I}$ is an ideal on X, then $(X, \tau, \mathcal{I})$ is called an ideal space. N is the ideal of all nowhere dense subsets in (X, τ). A subset A of an ideal space $(X, \tau, \mathcal{I})$ is $\star$-dense in itself [8], if $A \subseteq A^*$.

For every ideal topological space, there exists a topology $\tau^*(\mathcal{I})$ finer than τ generated by $\beta(\mathcal{I}, \tau) = \{U - A \mid U \in \tau \text{ and } A \in \mathcal{I}\}$, but in general $\beta(\mathcal{I}, \tau)$ is not always a topology [10].

Let $(X, \tau, \mathcal{I})$ be an ideal topological space and A be a subset of of X. Then $A_*(\mathcal{I}, \tau) = \{x \in X : U \cap A \notin \mathcal{I} \text{ for every } U \in SO(X, x)\}$ is called a semi local function of A with respect to $\mathcal{I}$ and τ [12]. Let $(X, \mathcal{I}, \tau)$ be an ideal topological space. We say that the topology τ is *compatible* with the $\mathcal{I}$, denoted $\tau \sim \mathcal{I}$, if the following hold for every $A \subset X$: if for every $x \in A$ there exists a $U \in \tau$ such that $U \cap A \in \mathcal{I}$, then $A \in \mathcal{I}$.

3. a-LOCAL FUNCTION

Quite recently, W. Al-Omeri, Mohd. Salmi Md. Noorani and A. Al-Omari [1] defined the a-local function and obtained some interesting results. Let $(X, \tau, \mathcal{I})$ be an ideal topological space and A be a subset of X. Then $A^{a^*}(\mathcal{I}, \tau) = \{x \in X : U \cap A \notin \mathcal{I}, \text{ for every } U \in \tau^a(x)\}$ is called a-local function [1] of A with respect to $\mathcal{I}$ and τ, where $\tau^a(x) = \{U \in \tau^a : x \in U\}$. We denote simply A^{a^*} for $A^{a^*}(\mathcal{I}, \tau)$.

Remark 3.1. [1]

i) The minimal ideals are $\{\emptyset\}$ in any topological space (X, τ) and the maximal ideal is $\wp(X)$. It can be deduced that $A^{a^*}(\{\emptyset\}) = Cl_a(A) \neq Cl(A)$ and $A^{a^*}(\wp(X)) = \emptyset$, for every $A \subset X$.

ii) If $A \in \mathcal{I}$, then $A^{a^*} = \emptyset$.

iii) $A \not\subseteq A^{a^*}$ and $A^{a^*} \not\subseteq A$, in general.

Theorem 3.1. [1] *Let $(X, \tau, \mathcal{I})$ be an ideal in topological space and A, B be subsets of X. Then for a-local functions the following properties hold:*

i) $\tau^a \cap \mathcal{I} = \emptyset$;
ii) *If* $I \in \mathcal{I}$ *then* $aInt(I) = \emptyset$;
iii) *For every* $G \in \tau^a$ *then* $G \subseteq G^{a^*}$;
iv) $X = X^{a^*}$.

Theorem 3.2. [1] *Let* $(X, \tau, \mathcal{I})$ *be an ideal in topological space and* A, B *subsets of* X. *Then for a-local functions the following properties hold:*

i) *If* $A \subset B$, *then* $A^{a^*} \subset B^{a^*}$;
ii) *For another ideal* $J \supset \mathcal{I}$ *on X,* $A^{a^*}(J) \subset A^{a^*}(\mathcal{I})$;
iii) $A^{a^*} \subset aCl(A)$;
iv) $A^{a^*}(\mathcal{I}) = aCl(A^{a^*}) \subset aCl(A)$ $(i.e$ A^{a^*} *is a a-closed subset of* $aCl(A))$;
v) $(A^{a^*})^{a^*} \subset A^{a^*}$;
vi) $(A \cup B)^{a^*} = A^{a^*} \cup B^{a^*}$;
vii) $A^{a^*} - B^{a^*} = (A - B)^{a^*} - B^{a^*} \subset (A - B)^{a^*}$;
viii) *If* $U \in \tau^a$, *then* $U \cap A^{a^*} = U \cap (U \cap A)^{a^*} \subset (U \cap A)^{a^*}$;
ix) *If* $U \in \tau^a$, *then* $(A - U)^{a^*} = A^{a^*} = (A \cup U)^{a^*}$;
x) *If* $A \subseteq A^{a^*}$, *then* $A^{a^*}(\mathcal{I}) = aCl(A^{a^*}) = aCl(A)$.

Theorem 3.3. [1] *Let* $(X, \tau, \mathcal{I})$ *be an ideal topological space and* A *a subset of* X. *Then the following are equivalent:*

i) $\mathcal{I} \sim^a \tau$;
ii) *If a subset* A *of* X *has a cover a-open sets of whose intersection with* A *is in* $\mathcal{I}$, *then* A *is in* $\mathcal{I}$, *in other words* $A^{a^*} = \emptyset$, *then* $A \in \mathcal{I}$;
iii) *For every* $A \subset X$, *if* $A \cap A^{a^*} = \emptyset$, $A \in \mathcal{I}$;
iv) *For every* $A \subset X$, $A - A^{a^*} \in \mathcal{I}$;
v) *For every* $A \subset X$, *if* A *contains no nonempty subset* B *with* $B \subset B^{a^*}$, *then* $A \in \mathcal{I}$.

Theorem 3.4. [1] *Let* $(X, \mathcal{I}, \tau)$ *be an ideal topological space. Then* $\beta(\mathcal{I}, \tau)$ *is a basis for* τ^{a^*}. $\beta(\mathcal{I}, \tau) = \{V - I_i : V \in \tau^a(x),\ I_i \in \mathcal{I}\}$ *and* β *is not, in general, a topology.*

Theorem 3.5. [1] *Let* $(X, \tau, \mathcal{I})$ *be an ideal topological space and* A *a subset of* X. *If* τ *is a-compatible with* $\mathcal{I}$, *then the following are equivalent:*

i) *For every* $A \subset X$, *if* $A \cap A^{a^*} = \emptyset$ *implies* $A^{a^*} = \emptyset$;
ii) *For every* $A \subset X$, $(A - A^{a^*})^{a^*} = \emptyset$;
iii) *For every* $A \subset X$,$(A \cap A^{a^*})^{a^*} = A^{a^*}$.

Remark 3.2. [1] The notion of local function and semi local function are independent of that of a-local function, as illustrated by the following example.

Example 3.1. [1] Let $X = \{x, y, w, z\}$ with a topology $\tau = \{\emptyset, X, \{x, y\}\}$ and $\mathcal{I} = \{\emptyset, \{x\}, \{y\}, \{x, y\}\}$. Take $A = \{x, z\}$. Then $A^* = \{\emptyset\}$, $A_* = \{z\}$, $A^{a^*} = X$.

4. ON $\Re_a$-OPERATOR IN IDEAL TOPOLOGICAL SPACES

In this section we shall introduce the operator $\Re_a$ in $(X, \tau, \mathcal{I})$. In [11], Kuratowski has shown that $Cl(A) = X - Int(X - A)$. This relation is the motivation of defining the operator $\Re_a$. We shall also discuss the behaviour of this operator.

Definition 4.1. Let $(X, \tau, \mathcal{I})$ be an **a**–ideal space. An operator $\Re_a : \wp(X) \to \tau$ is defined as follows: for every $A \in \wp(X)$,

$$\Re_a(A) = \{x \in X : \text{ there exists} U_x \in \tau^a \text{ such that } U_x - A \in \mathcal{I}\}.$$

Let us observes that $\Re_a(A) = X - (X - A)^{a*}$.

Theorem 4.6. *Let $(X, \tau, \mathcal{I})$ be an **a**-ideal space. Then for $A \in \wp(X)$, $\Re_a(A) = X - (X - A)^{a*}$.*

Proof. Let $x \in \Re_a(A)$. Then there exists an a-open set U_x containing x such that $U_x - A \in \mathcal{I}$. Then $X \cap (U_x - A) \in \mathcal{I}$ implies that $U_x \cap (X - A) \in \mathcal{I}$. So $x \notin (X - A)^{a^*}$ and hence $x \in X - (X - A)^{a^*}$. Therefore

$$\Re_a(A) \subset X - (X - A)^{a*}. \tag{4.1}$$

For the reverse inclusion, consider $x \in X - (X - A)^{a*}$. Then $x \notin (X - A)^{a^*}$. Therefore there exists an a-open set U_x containing x such that $U_x \cap (X - A) \in \mathcal{I}$. This implies that $U_x - A \in \mathcal{I}$. Hence $x \in \Re_a(A)$. So

$$X - (X - A)^{a*} \subset \Re_a(A). \tag{4.2}$$

From (4.1) and (4.2), we get $\Re_a(A) = X - (X - A)^{a*}$.

Actually the relation $\Re_a(A) = X - (X - A)^{a*}$ is equivalent to the definition of the operator $\Re_a$. □

Theorem 4.7. *Let $(X, \tau, \mathcal{I})$ be an **a**–ideal space. Then the following properties hold:*

i) If $A \subset X$, then $\Re_a(A)$ is a-open.
ii) If $A \subset B$, then $\Re_a(A) \subseteq \Re_a(B)$.
iii) If $A, B \in \wp(X)$, then $\Re_a(A \cap B) = \Re_a(A) \cap \Re_a(B)$.
iv) If $U \in \tau^{a^*}$, then $U \subseteq \Re_a(U)$.
v) If $A \subset X$, then $\Re_a(A) \subseteq \Re_a(\Re_a(A))$.
vi) If $A \subset X$, then $\Re_a(A) = \Re_a(\Re_a(A))$ if and only if $(X - A)^{a^*} = ((X - A)^{a^*})^{a^*}$.
vii) If $A \in \mathcal{I}$, then $\Re_a(A) = X - X^{a^*}$.
viii) If $A \subset X$, then $A \cap \Re_a(A) = Int^{a^*}(A)$, where Int^{a^*} is the interior of τ^{a^*} (Theorem 3.4).
ix) If $A \subset X$, $I \in \mathcal{I}$, then $\Re_a(A - I) = \Re_a(A)$.
x) If $A \subset X$, $I \in \mathcal{I}$, then $\Re_a(A \cup I) = \Re_a(A)$.
xi) If $(A - B) \cup (B - A) \in \mathcal{I}$, then $\Re_a(A) = \Re_a(B)$.
xii) If $A, B \in \wp(X)$, then $\Re_a(A \cap B) \subset \Re_a(A) \cap \Re_a(B)$.

Proof. (1) This follows by Theorem 4 (2).

(2) Given $A \subseteq B$, then $(X - A) \supseteq (X - B)$. Then, from Theorem 4 (1), $(X - A)^{a^*} \supseteq (X - B)^{a^*}$ and hence $\Re_a(A) \subseteq \Re_a(B)$.

(3) $\Re_a(A \cap B) = X - (X - A \cap B)^{a^*} = X - ((X - A) \cup (X - B))^{a^*}$. This implies that $\Re_a(A \cap B) = X - (X - A)^{a^*} \cup (X - B)^{a^*}$, from Theorem 4 (7). Therefore $\Re_a(A \cap B) = (X - (X - A)^{a^*}) \cap (X - (X - B)^{a^*})$, and hence $\Re_a(A \cap B) = \Re_a(A) \cap \Re_a(B)$.

(4) If $U \in \tau^{a^*}$, then $X - U$ is τ^{a^*}-closed which implies $(X - U)^{a^*} \subseteq (X - U)$ and hence $U \subseteq X - (X - U)^{a^*} = \Re_a(U)$.

(5) From (1) $\Re_a(A)$ is a-open, and from (4) $\Re_a(A) \subseteq \Re_a(\Re_a(A))$.

(6) This follows from the facts:

i) $\Re_a(A) = X - (X - A)^{a^*}$.
ii) $\Re_a(\Re_a(A)) = X - [X - X - (X - A)^{a^*})]^{a^*} = X - ((X - A)^{a^*})^{a^*}$.

(7) By Theorem 4 (10), we obtain that $(X - A)^{a^*} = X^{a^*}$ if $A \in \mathcal{I}$. Then $\Re_a(A) = X - (X - A)^{a^*} = X - X^{a^*}$.

(8) If $x \in A \cap \Re_a(A)$, then $x \in A$ and there exist a $U_x \in \tau^a(x)$ such that $U_x - A \in \mathcal{I}$. Then by Theorem 3.4, $U_x - (U_x - A)$ is an τ^{a^*}-open neighbourhood of x and $x \in Int^{a^*}(A)$. On the other hand, if $x \in Int^{a^*}(A)$, there exists a basic τ^{a^*}-open neighbourhood $V_x - I$ of x, where $V_x \in \tau^a$ and $I \in \mathcal{I}$, such that $x \in V_x - I \subseteq A$ which implies $x \in V_x - A \subseteq I$ and

hence $V_x - A \in \mathcal{I}$. Hence $x \in A \cap \Re_a(A)$.
(9) We know that $X - [X - (A - I)]^{a^*} = X - [(X - A) \cup I]^{a^*} = X - (X - A)^{a^*}$ (from Theorem 4(10)). So $\Re_a(A - I) = \Re_a(A)$ by (9) and (10).
(10) This follows from Theorem (9) and $X - [X - (A \cup I)]^{a^*} = X - [(X - A) - I]^{a^*} = X - (X - A)^{a^*} = \Re_a(A)$.
(11) Given that $(A - B) \cup (B - A) \in I$, and let $A - B = I_1, B - A = I_2$. We observe that I_1 and $I_2 \in I$ by heredity. Also observe that $B = (A - I_1) \cup I_2$. Thus $\Re_a(A) = \Re_a(A - I_1) = \Re_a[(A - I_1) \cup I_2] = \Re_a(B)$.
(12) Proof is obvious from (2). □

Corollary 4.1. Let $(X, \tau, \mathcal{I})$ be an **a**–ideal space. Then $U \subseteq \Re_a(U)$ for every a-open set U.

Proof. We know that $\Re_a(A) = X - (X - A)^{a^*}$. Now by Theorem 4(4) $(X - U)^{a^*} \subseteq aCl(X - U) = X - U$, since $X - U$ is a-closed. Therefore, $U = X - (X - U) \subseteq X - (X - U)^{a^*} = \Re_a(U)$ □

Theorem 4.8. *Let $(X, \tau, \mathcal{I})$ be an **a**-ideal space and $A \subseteq X$. Then the following properties hold:*

i) $\Re_a(A) = \{x \in X :$ *there exists* $U_x \in \tau^a(x)$*:* $U_x - A \in \mathcal{I}\}$.
ii) $\Re_a(A) = \cup\{U \in \tau^a : U - A \in \mathcal{I}\}$.

Proof. (1) $x \in \Re_a(A) \Leftrightarrow x \notin (X - A)^{a^*} \Leftrightarrow$ there exists $U_x \in \tau^a(x)$ such that $U_x - A = U_x \cap (X - A) \in \mathcal{I} \Leftrightarrow \{x \in X :$ there exist $U_x \in \tau^a(x) : U_x - A \in \mathcal{I}\}$.
(2) Let $H = \cup\{U \in \tau^a : U - A \in \mathcal{I}\}$. Now $x \in H$ implies that there exist $U \in \tau^a$ with $x \in U$ such that $U - A \in \mathcal{I}$. Thus by (1), $x \in \Re_a(A)$. From the expression of $\Re_a(A)$ in (1) it is clear that $\Re_a(A) \subseteq H$ □

Remark 4.3. Let $\mathcal{I} = \emptyset$, then by Theorem 4.8(2) $\Re_a(A) = \cup\{U \in \tau^a : U - A = \emptyset\} = \cup\{U \in \tau^a : U \subseteq A\} = aInt(A)$, for any space (X, τ).

Theorem 4.9. *Let $(X, \tau, \mathcal{I})$ be an **a**–ideal space. If $\Psi = \{A \subseteq X : A \subseteq \Re_a(A)\}$, then Ψ is a topology for X and $\Psi = \tau^{a^*}$.*

Proof. Let $\Psi = \{A \subseteq X : A \subseteq \Re_a(A)\}$. First, we show that Ψ is a topology. Observe that $\emptyset \subseteq \Re_a(\emptyset)$ and $X \subseteq \Re_a(X) = X$, and thus $\emptyset$ and $X \in \Psi$. Now if $A, B \in \Psi$, then $A \cap B \subseteq \Re_a(A) \cap \Re_a(B) = \Re_a(A \cap B)$, which implies that $A \cap B \in \Psi$. If $\{A_\alpha : \alpha \in \Re_a\} \subseteq \Psi$, then $A_\alpha \subseteq \Re_a(A_\alpha) \subseteq \Re_a(\cup A_\alpha)$ for every α and hence $\cup A_\alpha \subseteq \Re_a(\cup A_\alpha)$. This shows that Ψ is a topology. Now if $U \in \tau^{a^*}$ and $x \in U$, then by Theorem 3.4 there exist $V \in \tau^a(x)$ and $I \in \mathcal{I}$ such that $x \in V - I \subseteq U$. Clearly $V - U \subseteq I$ so that $V - U \in \mathcal{I}$ by heredity and hence $x \in \Re_a(U)$. Thus $U \subseteq \Re_a(U)$ and we have shown $\tau^{a^*} \subseteq \Psi$. Now let $A \in \Psi$, then we have $A \subseteq \Re_a(A)$, that is, $A \subseteq X - (X - A)^{a^*}$ and $(X - A)^{a^*} \subseteq X - A$. This shows that $X - A$ is τ^{a^*}-closed and hence $A \in \tau^{a^*}$. Thus $\Psi \subseteq \tau^{a^*}$ and hence $\Psi = \tau^{a^*}$. □

Theorem 4.10. *Let $(X, \tau, \mathcal{I})$ be an **a**–ideal space and $A \subseteq X$. Then for any a-open set A of X, $\Re_a(A) = \cup\{U \in \tau^a : (U - A) \cup (A - U) \in \mathcal{I}\}$.*

Proof. Let $H = \cup\{U \in \tau^a : (U - A) \cup (A - U) \in \mathcal{I}\}$. Since $\mathcal{I}$ is heredity, it is obvious that $H = \cup\{U \in \tau^a : (U - A) \cup (A - U) \in \mathcal{I}\} \subseteq \cup\{U \in \tau^a : (U - A) \in \mathcal{I}\} = \Re_a(A)$ for every $A \subseteq X$. Now, let $x \in \Re_a(A)$, then there exist $U \in \tau^a(x)$ such that $U - A \in \mathcal{I}$. Let $V = U \cup A \in \tau^a$, then $(V - A) \cup (A - V) = U - A \in \mathcal{I}$ and $x \in V \in \tau^a$. Thus $x \in H$. □

Definition 4.2. [1] Let $(X, \tau, \mathcal{I})$ be an ideal topological space. Then τ is said to be a-compatible with respect to $\mathcal{I}$, denoted by $\tau \sim^a \mathcal{I}$, if and only if, for every $x \in A$ there exist $U \in \tau^a(x)$ such that $U \cap A \in \mathcal{I}$, then $A \in \mathcal{I}$.

Theorem 4.11. *Let $(X, \tau, \mathcal{I})$ be an* **a***-ideal space. Then $\tau \sim^a \mathcal{I}$ if and only if $\Re_a(A) - A \in \mathcal{I}$, for every $A \subseteq X$.*

Proof. (Necessity). Assume $\tau \sim^a \mathcal{I}$ and let $A \subseteq X$. Observe that $x \in \Re_a(A) - A \in \mathcal{I}$ if and only if $x \notin A$ and $x \notin (X - A)^{a^*}$ if and only if $x \notin A$ and there exists $U_x \in \tau^a(x)$ such that $U_x - A = U_x \cap (X - A) \in \mathcal{I}$ if and only if there exists $U_x \in \tau^a(x)$ such that $x \in U_x - A$. Now, for each $x \in \Re_a(A) - A$ and $U_x \in \tau^a(x)$, $U_x \cap (\Re_a(A) - A) \in \mathcal{I}$ by heredity so that $\Re_a(A) - A \in \mathcal{I}$ by assumption that $\tau \sim^a \mathcal{I}$.
(sufficiency). Let $A \subseteq X$ and assume for each $x \in A$ there exists $U_x \in \tau^a(x)$ such that $U_x \cap A \in \mathcal{I}$. Observe that $\Re_a(X - A) - (X - A) = \{x \in X : \text{there exist}\, U_x \in \tau^a(x) : U_x - A \in \mathcal{I}\}$. Thus we have $A \subseteq \Re_a(X - A) - (X - A) \in \mathcal{I}$ and hence $A \in \mathcal{I}$ by heredity of $\mathcal{I}$. □

Proposition 4.1. *Let $(X, \tau, \mathcal{I})$ be an* **a***–ideal space. Then $\tau \sim^a \mathcal{I}$, $A \subseteq X$. If N is a nonempty a-open subset of $A^* \cap \Re_a(A)$, then $N - A \in \mathcal{I}$ and $N \cap A \notin \mathcal{I}$.*

Proof. If $N \subseteq A^{a^*} \cap \Re_a(A)$, then $N - A \subseteq \Re_a(A) - A \in \mathcal{I}$ by Theorem 4.11 and hence $N - A \in \mathcal{I}$ by heredity. Since $N \in \tau^a - \{\emptyset\}$ and $N \subseteq A^{a^*}$, we have $N \cap A \notin \mathcal{I}$ by the definition of A^{a^*}. □

As a consequence of Theorem 4.11, we have the following.

Corollary 4.2. *Let $(X, \tau, \mathcal{I})$ be an* **a***-ideal space with $\tau \sim^a \mathcal{I}$. Then $\Re_a(\Re_a(A)) = \Re_a(A)$, for every $A \subseteq X$.*

Proof. $\Re_a(A) \subseteq \Re_a(\Re_a(A))$ follows from Theorem 4.7(5). Since $\tau \sim^a \mathcal{I}$, $\Re_a(A) = A \cup I$ for some $I \in \mathcal{I}$ (Theorem 4.11) and hence $\Re_a(\Re_a(A)) = \Re_a(A)$ by Theorem 4.7(10). □

To see that the converse of Corollary 4 does not hold, let X be an infinite discrete space with $\mathcal{I}$ the ideal of finite subsets. For each $A \subseteq X$, $(\Re_a(A)) = X$, and hence $\Re_a(\Re_a(A)) = \Re_a(A) = X$, but $\mathcal{I}$ is not a-compatible with τ.

Theorem 4.12. *Let $(X, \tau, \mathcal{I})$ be an* **a***-ideal space with $\tau \sim^a \mathcal{I}$. Then $\Re_a(A) = \cup\{\Re_a(U) : U \in \tau^a, \Re_a(U) - A \in \mathcal{I}\}$.*

Proof. Let $H(A) = \cup\{\Re_a(U) : U \in \tau^a, \Re_a(U) - A \in \mathcal{I}\}$. clearly, $H(A) \subseteq \Re_a(A)$. Now let $x \in \Re_a(A)$. Then there exists $U \in \tau^a(x)$ such that $U - A \in \mathcal{I}$. By corollary 4.1, $U \subseteq \Re_a(U$ and $\Re_a(U) - A \subseteq [\Re_a(U) - U] \cup [U - A]$. By Theorem 4.11, $\Re_a(U) - U \in \mathcal{I}$ and hence $\Re_a(U) - A \in \mathcal{I}$ hence $x \in H(A)$ and $H(A) \supseteq \Re_a(A)$. Consequently, we obtain $H(A) = \Re_a(A)$. □

Newcomb defined In [18], $A = B[mod \mathcal{I}]$ if $(A - B) \cup (B - A) \in \mathcal{I}$ and observed that $= [mod \mathcal{I}]$ is an equivalence relation. By Theorem 4.7(11), we have that if $A = B[mod \mathcal{I}]$, then $\Re_a(A) = \Re_a(B)$.

Definition 4.3. Let $(X, \tau, \mathcal{I})$ be an **a**–ideal space and with set $A \subseteq X$. A is said to be a Baire set with respect to τ^a and $\mathcal{I}$, denoted $A \in \mathcal{B}_r(X, \tau, \mathcal{I})$, if there exist a a-open set $U \in \tau^a$ such that $A = U[mod\ \mathcal{I}]$.

Lemma 4.1. *Let $(X, \tau, \mathcal{I})$ be an* **a***-ideal space with $\tau \sim^a \mathcal{I}$. If $U, V \in \tau^a$ and $\Re_a(U) = \Re_a(V)$, then $U = V[mod\ \mathcal{I}]$.*

Proof. Since $U \in \tau^a$, by Theorem 4.7(4) $U \subseteq \Re_a(U)$ and hence $U - V \subseteq \Re_a(U) - V = \Re_a(V) - V \in \mathcal{I}$ by Theorem 4.11. Similarly $V - U \in \mathcal{I}$. Now $(U - V) \cup (V - U) \in \mathcal{I}$ by additivity. Hence $U = V[mod \mathcal{I}]$. □

Theorem 4.13. *Let $(X, \tau, \mathcal{I})$ be an **a**-ideal space with $\tau \sim^a \mathcal{I}$. If $A, B \in \mathcal{B}_r(X, \tau, \mathcal{I})$, and $\Re_a(A) = \Re_a(B)$, then $A = B[mod\ \mathcal{I}]$.*

Proof. $U, V \in \tau^a$ such that $A = U[mod\ \mathcal{I}]$ and $B = V[mod\ \mathcal{I}]$. Now $\Re_a(A) = \Re_a(U)$ and $\Re_a(B) = \Re_a(V)$ by Theorem 4.7(11). $\Re_a(U) = \Re_a(V)$ and hence $U = V[mod\ \mathcal{I}]$ by Lemma 4.1 then $A = B[mod\ \mathcal{I}]$ by transitivity. □

Proposition 4.2. *Let $(X, \tau, \mathcal{I})$ be an **a**-ideal space with $\tau^a \cap \mathcal{I} = \emptyset$. If $B \in \mathcal{B}_r(X, \tau, \mathcal{I}) - \mathcal{I}$, then $\Re_a(B) \cap aInt(B^{a^*}) \neq \emptyset$.*

Proof. Assume $B \in \mathcal{B}_r(X, \tau, \mathcal{I}) - \mathcal{I}$, then by Proposition 4.3(1) there exists $A \in \tau^a - \{\emptyset\}$ such that $B = A[mod\ \mathcal{I}]$. This implies $\emptyset \neq A \subset A^{a^*} = ((B - J) \cup I)^{a^*} = B^{a^*}$, where $J = B - A$, $I = A - B \in \mathcal{I}$ by Theorem 3.1 and Theorem 4(9). Also $\emptyset \neq A \subset \Re_a(A) = \Re_a(B)$ by Theorem 4.7(11), so that $A \subseteq \Re_a(B) \cap aInt(B^{a^*})$. □

Proposition 4.3. *Let $(X, \tau, \mathcal{I})$ be an **a**-ideal space*

i) If $B \in \mathcal{B}_r(X, \tau, \mathcal{I}) - \mathcal{I}$, then there exists $A \in \tau^a - \{\emptyset\}$ such that $B = A[mod\ \mathcal{I}]$
ii) If $\tau^a \cap \mathcal{I} = \emptyset$, then $B \in \mathcal{B}_r(X, \tau, \mathcal{I}) - \mathcal{I}$ if and only if there exists $A \in \tau^a - \{\emptyset\}$ such that $B = A[mod\ \mathcal{I}]$.

Proof. (1) Assume $B \in \mathcal{B}_r(X, \tau, \mathcal{I}) - \mathcal{I}$, then $B \in \mathcal{B}_r(X, \tau, \mathcal{I})$. Now, if there does not exist $A \in \tau^a - \{\emptyset\}$ such that $B = A[mod\ \mathcal{I}]$, this implies that $B \in \mathcal{I}$, which is a contradiction.

(2) Assume that there exists $A \in \tau^a - \{\emptyset\}$ such that $B = A[mod\ \mathcal{I}]$. Then $A = (B - J) \cup I$, where $J = B - A$, $I = A - B \in \mathcal{I}$. If $B \in \mathcal{I}$ then $A \in \mathcal{I}$ by heredity and additivity, which is contrast to $\tau^a \cap \mathcal{I} = \emptyset$. □

Theorem 4.14. *Let $(X, \tau, \mathcal{I})$ be an **a**–ideal space with $\tau^a \cap \mathcal{I} = \emptyset$. Then, for $A \subseteq X$, $\Re_a(A) \subseteq A^{a^*}$.*

Proof. Suppose $x \in \Re_a(A)$ and $x \notin A^{a^*}$. Then there exists U containing x such that $U_x \in aO(x)$ and $U_x \cap A \in \mathcal{I}$. Since $x \in \tau^a$, then by Theorem 4.8(2) $\Re_a(A) = \cup\{U \in \tau^a : U - A \in \mathcal{I}\}$ and there exist $V \in \tau^a$ such that $x \in V$ and $V - A \in \mathcal{I}$. Then we have $U_x \cap V \in aO(x)$, $(U_x \cap V) - A \in \mathcal{I}$ and $U_x \cap V \cap A \in \mathcal{I}$ by heredity. Hence by finite additivity we have $(U_x \cap V \cap A) \cup ((U_x \cap V) - A) = U_x \cap V) \in \mathcal{I}$. Since $(U_x \cap V) \in aO(x)$, this is contrary with $\tau^a \cap \mathcal{I} = \emptyset$. So $x \in A^{a^*}$. This implies $\Re_a(A) \subseteq A^{a^*}$. □

Given an ideal topology space $(X,\tau,\mathcal{I})$, let $\mathcal{U}(X,\tau,\mathcal{I})$ denoted by $\{A \subset X$: there exists $B \in \mathcal{B}_r(X,\tau,\mathcal{I}) - \mathcal{I}$ such that $B \subseteq A\}$.

Proposition 4.4. *Let $(X, \tau, \mathcal{I})$ be an **a**-ideal space with $\tau^a \cap \mathcal{I} = \emptyset$. Then the following are equivalent:*

i) $A \in \mathcal{U}(X, \tau, \mathcal{I})$;
ii) $\Re_a(A) \cap aInt(A^{a^*}) \neq \emptyset$;
iii) $\Re_a(A) \cap A^{a^*} \neq \emptyset$;
iv) $\Re_a(A) \neq \emptyset$;
v) $Int^{a^*}(A) \neq \emptyset$;
vi) There exists $N \in \tau^a - \{\emptyset\}$ such that $N - A \in \mathcal{I}$ and $N \cap A \notin \mathcal{I}$.

Proof. (1)⇒(2): Let $B \in \mathcal{B}_r(X, \tau, \mathcal{I}) - \mathcal{I}$ such that $B \subset A$. Then $aInt(B^{a^*}) \subset aInt(A^{a^*})$ and $\Re_a(B) \subset \Re_a(A)$ and hence $\Re_a(B) \cap aInt(B^{a^*}) \subset \Re_a(A) \cap aInt(A^{a^*})$. By the proposition 4.3, we have $\Re_a(A) \cap aInt(A^{a^*}) \neq \emptyset$.

(2)⇒(3): The proof is obvious.

(3)⇒(4): The proof is obvious.

(4)⇒(5): If $\Re_a(A) \neq \emptyset$, then there exists $U \in \tau^a - \emptyset$ such that $U - A \in \mathcal{I}$. Since $U \notin \mathcal{I}$ and $U = (U-A) \cup (U \cap A)$, we have $U \cap A \notin \mathcal{I}$. By Theorem 4.7 $\emptyset \neq (U \cap A) \subset \Re_a(U) \cap A = \Re_a((U - A) \cup (U \cap A)) \cap A = \Re_a((U \cap A) \cap A = Int^{a^*}(A)$. Hence $Int^{a^*}(A) \neq \emptyset$.

(5)⇒(6): If $Int^{a^*}(A) \neq \emptyset$, then there exists $N \in \tau^a - \emptyset$ and $I \in \mathcal{I}$ such that $\emptyset \neq N - A \subset A$. We have $N - A \in \mathcal{I}$, $N = (N - A) \cup (N \cap A)$ *and* $N \notin \mathcal{I}$. This implies that $N \cap A \notin \mathcal{I}$.

(6)⇒(1): Let $B = N \cap A \notin \mathcal{I}$ with $N \in \tau^a - \emptyset$ and $(N - A) \in \mathcal{I}$. Then $B \in \mathcal{B}_r(X, \tau, \mathcal{I}) - \mathcal{I}$ since $B \notin \mathcal{I}$ and $(B - N) \cup (N - B) = N - A \in \mathcal{I}$. □

Corollary 4.3. *Let $(X, \tau, \mathcal{I})$ be an **a**–ideal space with $\tau^a \cap \mathcal{I} = \emptyset$. Then for $A \subseteq X$, $\Re_a(A) \subseteq aCl(A^{a^*})$.*

Theorem 4.15. *Let $(X, \tau, \mathcal{I})$ be an **a**–ideal space. Then the following properties are equivalent:*

i) $\tau^a \cap \mathcal{I} = \emptyset$;
ii) $\Re_a(\emptyset) = \emptyset$;
iii) If $A \subseteq X$ is a-closed, then , $\Re_a(A) - A = \emptyset$;
iv) If $I \in \mathcal{I}$. then $\Re_a(I) = \emptyset$.

Proof. (1)⇒(2): Since $\tau^a \cap \mathcal{I} = \emptyset$, then by Theorem 4.8(2) $\Re_a(A) = \cup\{U \in \tau^a : U - A \in \mathcal{I}\} = \emptyset$.

(2)⇒(3): Suppose $\Re_a(A) - A$, then there exist a $U_x \in \tau^a(x)$ such that $x \in U_x - A \in \mathcal{I}$ and $U_x - A \in \tau^a$. But then $U_x - A \in \{U_x \in \tau^a : U \in \mathcal{I}\}$ which implies that $\Re_a(\emptyset) \neq \emptyset$. Hence $\Re_a(A) - A = \emptyset$.

(3)⇒(4): Let $I \in \mathcal{I}$ and since $\emptyset$ is a-closed, then $\Re_a(I) = \Re_a(I \cup \emptyset) = \Re_a(I) = \emptyset$.

(4)⇒(1): Suppose $A \in \tau^a \cap \mathcal{I}$, then $A \in \mathcal{I}$ and by (4) $\Re_a(A) = \emptyset$. Since $A \in \tau^a$, by Corollary 4.1 we have $A \subseteq \Re_a(A) = \emptyset$. Hence $\tau^a \cap \mathcal{I} = \emptyset$. □

Definition 4.4. A subset in an **a**-ideal space $(X, \tau, \mathcal{I})$ is said to be $\mathcal{I}_a$-dense if $A^{a^*} = X$.

The set of all $\mathcal{I}_a$-dense in $(X, \tau, \mathcal{I})$ is denoted by $\mathcal{I}_a\mathcal{D}(X, \tau)$. The collection of all dense sets in (X, τ) is denoted by $\mathcal{D}(X, \tau)$. Now we show that the collection of dense sets in a topological space (X, τ^{a^*}) which is denoted by $\mathcal{D}(X, \tau^{a^*})$ and the collection of $\mathcal{I}_a$-dense ideal in a topological space $(X, \tau, \mathcal{I})$ are equal if $\tau^a \cap \mathcal{I} = \emptyset$.

Theorem 4.16. *Let $(X, \tau, \mathcal{I})$ be an **a**–ideal space. Then for $x \in X$, $X - \{x\}$ is $\mathcal{I}$-dense if and only if $\Re_a(\{x\}) = \emptyset$.*

Proof. The proof follows from the definition 4.4, since $\Re_a(\{x\}) = (X - \{x\})^{a^*} = \emptyset$. if and only if $X = (X - \{x\})^{a^*}$. □

Theorem 4.17. *Let $(X, \tau, \mathcal{I})$ be an **a**-ideal space. If $\tau^a \cap \mathcal{I} = \emptyset$, then $\mathcal{I}_a\mathcal{D}(X, \tau) = \mathcal{D}(X, \tau^{a^*})$.*

Proof. Let $D \in \mathcal{I}_a\mathcal{D}(X, \tau)$. Then $aCl^*(D) = D \cup D^{a^*} = X$, i.e. $D \in \mathcal{D}(X, \tau^{a^*})$. Therefore $\mathcal{I}_a\mathcal{D}(X, \tau) \subseteq \mathcal{D}(X, \tau^{a^*})$.

Conversely, let $D \in \mathcal{D}(X, \tau)^{a^*}$. Then $aCl^*(D) = D \cup D^{a^*} = X$. We prove that $D^{a^*} = X$. Let $x \in X$ such that $x \notin D^{a^*}$. Therefore there exists $\emptyset \neq U \in \tau^a$ such that $U \cap D \in \mathcal{I}$. Since $U \notin \mathcal{I}$, $U \cap (X - D) \notin \mathcal{I}$ and hence $U \cap (X - D) \neq \emptyset$. Let $x_0 \in U \cap (X - D)$. Then $x_0 \notin D$ and also $x_0 \in D^{a^*}$. Because $x_0 \in D^{a^*}$ implies that $U \cap D \notin \mathcal{I}$ which is contrary to $U \cap D \in \mathcal{I}$. Thus $x_0 \in D \cup D^{a^*} = aCl^*(D) = X$. This is a contradiction. Therefore, we obtain $D \in \mathcal{I}_a\mathcal{D}(X, \tau)$. Therefore, $D \in \mathcal{D}(X, \tau^{a^*}) \subseteq D \in \mathcal{I}_a\mathcal{D}(X, \tau)$. Hence $\mathcal{I}_a\mathcal{D}(X, \tau) = \mathcal{D}(X, \tau^{a^*})$. □

Proposition 4.5. *Let $(X, \tau, \mathcal{I})$ be an **a**–ideal space. If $\tau^a \cap \mathcal{I} = \emptyset$, then $\Re_a(A) \neq \emptyset$ if and only if A contains a nonempty τ^{a^*}-interior.*

Proof. Let $\Re_a(A) \neq \emptyset$ By Theorem 4.8(2), $\Re_a(A) = \cup\{U \in \tau^a : U - A \in \mathcal{I}\}$ and there exist a nonempty set $U \in \tau^a$ such that $U - A \in \mathcal{I}$. Let $U - A = P$, where $P \in \mathcal{I}$. Now $U - P \subseteq A$. By Theorem 3.4 $U - P \in \tau^{a^*}$ and A contains a nonempty τ^{a^*}-interior.

Conversely, suppose that A contains a nonempty τ^{a^*}-interior. Hence there exist a nonempty set $U \in \tau^a$ and $P \in \mathcal{I}$ such that $U - P \subseteq A$. So $U - A \subseteq P$. Let $M = U - A \subseteq P$, then $M \in \mathcal{I}$. Hence $\cup\{U \in \tau^a : U - A \in \mathcal{I}\} = \Re_a(A) \neq \emptyset$. □

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