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On $H_3(1)$ Hankel determinant for concave univalent functions

ŞAHSENE ALTINKAYA and SIBEL YALÇIN

ABSTRACT. In this paper we study the class $C_0(\alpha)$ ($\alpha \in (1, 2]$) - the class of the so-called concave univalent functions. The main aim is to obtain an upper bound to the third Hankel determinant for concave univalent functions.

1. INTRODUCTION

Let A denote the class of analytic functions $f(z)$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

in the open unit disk

$$U = \{z : |z| < 1\}.$$

We shall denote by S the class of all functions $f(z) \in A$ which are univalent in U.

We also denote by S^* the subclass of starlike functions of S, i.e., the class of functions $f \in S$ satisfying

$$Re \frac{zf'(z)}{f(z)} > 0,$$

for $z \in U$. In 1976, Noonan and Thomas [9] defined the q^{th} Hankel determinant of f, for $n \geq 0$ and $q \geq 1$, by

$$H_q(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2q-2} \end{vmatrix} \qquad (a_1 = 1).$$

This determinant has also been considered by several authors. For example, Noor [10] determined the rate of growth of $H_q(n)$ as $n \to \infty$ for functions f given by (1.1) with bounded boundary. In particular, sharp upper bounds on $H_2(2)$ were obtained by the authors of articles ([10], [12]) for different classes of functions.

Note that

$$H_2(1) = \begin{vmatrix} a_1 & a_2 \\ a_2 & a_3 \end{vmatrix} = a_3 - a_2^2$$

and

$$H_2(2) = \begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix} = a_2 a_4 - a_3^2.$$

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Corresponding author: Şahsene Altınkaya; sahsene@uludag.edu.tr

The Hankel determinant $H_2(1) = a_3 - a_2^2$ is well-known as Fekete-Szegö functional. The Fekete-Szegö functional $\left|a_3 - \mu a_2^2\right|$ for normalized univalent functions

$$f(z) = z + a_2 z^2 + \cdots$$

is well known for its rich history in the theory of geometric functions. Its origin was in the disproof by Fekete and Szegö of the 1933 conjecture of Littlewood and Paley that the coefficients of odd univalent functions are bounded by unity (see [8]). The functional has since received great attention, particularly in many subclasses of the family of univalent functions. Nowadays, it seems that this topic had become of interest among the researchers (see, for example, [2], [3], [6], [11], [13]).

For our discussion in this paper, we consider the Hankel determinant in the case $q = 3$ and $n = 1$, denoted by $H_3(1)$, and given by

$$H_3(1) = \begin{vmatrix} a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \\ a_3 & a_4 & a_5 \end{vmatrix}.$$

For $f \in A$, $a_1 = 1$, we have

$$H_3(1) = a_3(a_2 a_4 - a_3^2) - a_4(a_4 - a_2 a_3) + a_5(a_3 - a_2^2)$$

and. by applying triangle inequality, we get

$$|H_3(1)| \leq |a_3| \left|a_2 a_4 - a_3^2\right| + |a_4| \left|a_4 - a_2 a_3\right| + |a_5| \left|a_3 - a_2^2\right|.$$

2. Preliminaries

A function $f : U \to \mathbb{C}$ is said to belong to the family $C_0(\alpha)$ if f satisfies the following conditions:

- f is analytic in U with the standard normalization $f(0) = f'(0) - 1 = 0$. In addition, it satisfies $f(1) = \infty$.
- f maps U conformally onto a set whose complement with respect to $\mathbb{C}$ is convex.
- The opening angle of $f(U)$ at ∞ is less than or equal to $\pi\alpha,\ \alpha \in (1, 2]$.

The class $C_0(\alpha)$ is referred to as the class of concave univalent functions and for a detailed discussion about concave functions, we refer to Avkhadiev et al. [1], Cruz and Pommerenke [7] and references therein.

Bhowmik et al. [4] showed that an analytic function f maps U onto a concave domain of angle $\pi\alpha$, if and only if

$$Re\left\{\frac{2}{\alpha - 1}\left[\frac{\alpha + 1}{2}\frac{1 + z}{1 - z} - 1 - z\frac{f''(z)}{f'(z)}\right]\right\} > 0$$

Before we establish our main results, we recall the following theorem and lemma, which will be helpful in the proofs.

Theorem 2.1. (see [4]) *Let $\alpha \in (1, 2]$. A function $f \in Co(\alpha)$ if and only if there exists a function $\phi \in S^*$ such that $f(z) = \Pi_\phi(z)$, where*

$$\Pi_\phi(z) = \int_0^z \frac{1}{(1 - t)^{\alpha + 1}} \left(\frac{t}{\phi(t)}\right)^{\frac{\alpha - 1}{2}} dt.$$

Lemma 2.1. (Bieberbach's Conjecture, see [5]) *If $f \in S,\ f(z) = z + \sum\limits_{n=2}^{\infty} a_n z^n,\ \ z \in U$, then*

$$|a_n| \leq n, \quad for \quad n = 2, 3, \ldots.$$

3. Main results

Theorem 3.2. *Let f given by (1.1) be in the class $C_0(\alpha)$. Then*

$$|a_2| \leq \alpha,$$

$$|a_3| \leq \frac{2\alpha^2+3\alpha-2}{3},$$

and

$$|a_4| \leq \frac{2\alpha^3+9\alpha^2+4\alpha-9}{6},$$

$$|a_5| \leq \frac{4\alpha^4+36\alpha^3+83\alpha^2-6\alpha-87}{30}.$$

Proof. We recall from Theorem 2.1 that $f \in C_0(\alpha)$ if and only if there exists a function $\phi \in S^*$ of the form

$$\phi(z) = z + \sum_{n=2}^{\infty} \phi_n z^n$$

such that

$$f'(z) = \frac{1}{(1-z)^{\alpha+1}} \left(\frac{z}{\phi(z)}\right)^{\frac{\alpha-1}{2}} \tag{3.2}$$

where f has the form given by (1.1). Equating coefficients in (3.2) yields

$$a_2 = \frac{\alpha+1}{2} - \frac{\alpha-1}{4}\phi_2, \tag{3.3}$$

$$a_3 = \frac{(\alpha+1)(\alpha+2)}{6} - \frac{\alpha^2-1}{6}\phi_2 - \frac{\alpha-1}{6}\phi_3 + \frac{\alpha^2-1}{24}\phi_2^2, \tag{3.4}$$

and

$$\begin{aligned} a_4 &= \tfrac{(\alpha+1)(\alpha+2)(\alpha+3)}{24} - \tfrac{\alpha-1}{8}\phi_4 - \tfrac{\alpha^2-1}{8}\phi_3 - \tfrac{\left(\alpha^2-1\right)(\alpha+2)}{16}\phi_2 \\ &\quad + \tfrac{(\alpha^2-1)(\alpha+1)}{32}\phi_2^2 - \tfrac{(\alpha^2-1)(\alpha+3)}{192}\phi_2^3 + \tfrac{\alpha^2-1}{16}\phi_2\phi_3. \end{aligned} \tag{3.5}$$

$$\begin{aligned} a_5 &= \tfrac{(\alpha+1)(\alpha+2)(\alpha+3)(\alpha+4)}{120} - \tfrac{\alpha-1}{10}\phi_5 - \tfrac{\alpha^2-1}{10}\phi_4 - \tfrac{(\alpha^2-1)(\alpha+2)}{20}\phi_3 - \tfrac{\left(\alpha^2-1\right)(\alpha+2)(\alpha+3)}{60}\phi_2 \\ &\quad + \tfrac{(\alpha^2-1)(\alpha+1)(\alpha+2)}{80}\phi_2^2 - \tfrac{(\alpha^2-1)(\alpha+1)(\alpha+3)}{240}\phi_2^3 + \tfrac{(\alpha^2-1)(\alpha+3)(\alpha+5)}{1920}\phi_2^4 \\ &\quad + \tfrac{(\alpha^2-1)(\alpha+1)}{20}\phi_2\phi_3 + \tfrac{\alpha^2-1}{20}\phi_2\phi_4 + \tfrac{\alpha^2-1}{40}\phi_3^2 - \tfrac{(\alpha^2-1)(\alpha+3)}{80}\phi_2^2\phi_3. \end{aligned} \tag{3.6}$$

and the results follow by triangle inequality and using Lemma 2.1. □

Theorem 3.3. *Let f given by (1.1) be in the class $C_0(\alpha)$. Then*

$$|a_2a_3 - a_4| \leq \begin{cases} \frac{(\alpha-1)(5\alpha+11)}{6} & 1 < \alpha < \frac{5}{3} \\ \frac{(\alpha-1)(3\alpha^2+8\alpha+17)}{12} & \frac{5}{3} \leq \alpha \leq 2 \end{cases}. \tag{3.7}$$

Proof. From (3.3) to (3.5) we find that

$$a_2a_3 - a_4 = \begin{aligned} &\left\{ \tfrac{\left(\alpha^2-1\right)(\alpha+2)}{24} - \tfrac{(\alpha^2-1)(3\alpha+2)}{48}\phi_2 + \tfrac{\alpha^2-1}{24}\phi_3 + \tfrac{(\alpha^2-1)(3\alpha-5)}{96}\phi_2^2 \right. \\ &\left. + \tfrac{\alpha-1}{8}\phi_4 - \tfrac{(\alpha-1)(\alpha+5)}{48}\phi_2\phi_3 - \tfrac{(\alpha^2-1)(\alpha-5)}{192}\phi_2^3 \right\} \end{aligned}$$

from which we get the asserted upper bounds in (3.7), by triangle inequality and using Lemma 2.1 in (3.6). □

Theorem 3.4. *Let f given by (1.1) be in the class $C_0(\alpha)$. Then we obtain*

$$\left|a_3 - a_2^2\right| \leq \frac{\alpha^2 + 6\alpha - 7}{6}.$$

Proof. Since $f \in C_0(\alpha)$, From (3.3) and (3.4) we find that

$$a_3 - a_2^2 = -\frac{\alpha^2 - 1}{12} + \frac{\alpha^2 - 1}{12}\phi_2 - \frac{\alpha - 1}{6}\phi_3 - \frac{(\alpha-1)(\alpha-5)}{48}\phi_2^2.$$

The rest of the proof follows as in Theorem 3.2. □

Theorem 3.5. *Let f given by (1.1) be in the class $C_0(\alpha)$. Then we obtain*

$$\left|a_2 a_4 - a_3^2\right| \leq \frac{\alpha^4 + 6\alpha^3 + 46\alpha^2 - 42\alpha - 11}{36}.$$

Theorem 3.6. *Let $f \in C_0(\alpha)$. Then*

$$|H_3(1)| \leq \frac{1}{45}\alpha^7 + \frac{343}{4320}\alpha^6 + \frac{43}{45}\alpha^5 + \frac{677}{360}\alpha^4 + \frac{1603}{480}\alpha^3 + \frac{2761}{720}\alpha^2 + \frac{407}{96}\alpha + \frac{11039}{4320}.$$

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Department of Mathematics
Uludag University
16059 Bursa, Turkey
Email address: sahsene@uludag.edu.tr
Email address: syalcin@uludag.edu.tr