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A note on Baskakov operators based on a function ϑ

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ABSTRACT. In this paper, we consider a modification of the classical Baskakov operators based on a function ϑ. Basic qualitative and quantitative Korovkin results are stated in weighted spaces. We prove a quantitative Voronovskaya-type theorem and present some results on the monotonic convergence of the sequence. Finally, we show a shape preserving property and further direct convergence theorems. Weighted modulus of continuity of first order and the notion of ϑ-convexity are used throughout the paper.

1. INTRODUCTION

In the last few years, within the so-called Korovkin-type approximation theory, there has been an increasing interest in studying modifications of some classical sequences of linear operators, in such a way that the resulting sequences represent new approximation processes where the functions of the classical Korovkin set $\{e_0, e_1, e_2\}$ $(e_i(t) = t^i)$ are replaced by the powers of certain function ϑ, namely ϑ^i, $i = 0, 1, 2$. These modified operators constructed with the help the function ϑ possess nice shape preserving properties and present a good behavior on approximating functions from certain weighted spaces.

A first study in this direction was made in [6], where the authors defined the following sequence of Bernstein type operators:

$$B_n^\tau f(x) := B_n\left(f \circ \tau^{-1}\right)(\tau(x)) = \sum_{k=0}^{n} \binom{n}{k} \tau^k(x)\left(1 - \tau(x)\right)^{n-k} \left(f \circ \tau^{-1}\right)\left(\frac{k}{n}\right), \tag{1.1}$$

for $f \in C[0,1]$ and $x \in [0,1]$, where B_n is the classical Bernstein operator and τ is a continuously ∞ times differentiable function on $[0,1]$ such that $\tau(0) = 0$, $\tau(1) = 1$ and $\tau'(x) > 0$ for $x \in [0,1]$.

In the recent note [1], the authors proved an asymptotic formula and a quantitative type asymptotic formula for Durrmeyer type generalization of (1.1). A similar construction of (1.1) for the Szász-Mirakyan operators has been recently introduced in [4].

Inspired by these studies, in this paper we introduce a generalization of the classical Baskakov operators [5], introduced in 1957 and defined, for $x \in \mathbb{R}^+ = [0,\infty)$ and a suitable function $f : \mathbb{R}^+ \to \mathbb{R}$, by

$$A_n(f; x) := \sum_{k=0}^{\infty} \binom{n+k-1}{k} f\left(\frac{k}{n}\right) x^k (1+x)^{-n-k}. \tag{1.2}$$

The proposed modification is defined by

$$A_n^\vartheta(f; x) := A_n\left(f \circ \vartheta^{-1}; \vartheta(x)\right),$$

where ϑ is a function satisfying the following properties:

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(Θ_1) ϑ is continuously differentiable on $\mathbb{R}^+$,

(Θ_2) $\vartheta(0)=0$, $\inf\limits_{x\in\mathbb{R}^+}\vartheta'(x)\geq 1$.

We note that these are the same conditions given for generalized Szász-Mirakyan operators (see [4]). We shall use the following definition of modified Baskakov operators:

$$\begin{aligned} A_n^{\vartheta}(f;x) &= \sum_{k=0}^{\infty}\binom{n+k-1}{k}\left(f\circ\vartheta^{-1}\right)\left(\frac{k}{n}\right)\vartheta^k(x)\left(1+\vartheta(x)\right)^{-n-k} \\ &= \sum_{k=0}^{\infty} f\left(\vartheta^{-1}\left(\frac{k}{n}\right)\right)\mathcal{P}_{\vartheta,n,k}(x), \end{aligned} \tag{1.3}$$

where

$$\mathcal{P}_{\vartheta,n,k}(x)=\binom{n+k-1}{k}\vartheta^k(x)\left(1+\vartheta(x)\right)^{-n-k}.$$

The paper is organized as follows. In Section 2, we give some lemmas which will be necessary to prove our main results. Section 3 contains the proof of weighted uniform convergence of the operators and also a statement concerning the degree of this weighted uniform convergence. A quantitative Voronovskaya-type theorem and a result about the convergence of the first derivative of A_n^{ϑ} are given in Section 4. In the last Section 5, we study some shape preserving properties, the property of monotonicity and local direct estimates using Lipschitz type function related to ϑ and generalized Lipschitz-type maximal function of order α, $\alpha\in(0,1]$.

2. Preliminary Results

For the main results we shall need the following auxiliary results:

Lemma 2.1. *We have*

$$A_n^{\vartheta}\left(\vartheta^0;x\right) = A_n^{\vartheta}(e_0;x)=1, \tag{2.4}$$

$$A_n^{\vartheta}\left(\vartheta^1;x\right) = \vartheta(x), \tag{2.5}$$

$$A_n^{\vartheta}\left(\vartheta^2;x\right) = \frac{\vartheta(x)\left(1+\vartheta(x)\right)}{n}+\vartheta^2(x), \tag{2.6}$$

and

$$A_n^{\vartheta}\left(\vartheta^3;x\right) = \frac{\vartheta(x)}{n^2}\left[\begin{array}{c} 1+3\vartheta(x)+3n\vartheta(x)+2\vartheta^2(x)+3n\vartheta^2(x) \\ +n^2\vartheta^2(x)\end{array}\right], \tag{2.7}$$

$$\begin{aligned} A_n^{\vartheta}\left(\vartheta^4;x\right) &= \vartheta^4(x)+\frac{6}{n}\left(\vartheta^4(x)+\vartheta^3(x)\right)+\frac{11\vartheta^4(x)+18\vartheta^3(x)+7\vartheta^2(x)}{n^2} \\ &\quad +\frac{6\vartheta^4(x)+12\vartheta^3(x)+7\vartheta^2(x)+\vartheta(x)}{n^3}. \end{aligned}$$

Proof. Using (1.2) and (1.3), we can write

$$A_n^{\vartheta}\left(\vartheta^0;x\right)=A_n^{\vartheta}(e_0;x)=A_n\left(e_0;\vartheta(x)\right)=1,$$

$$\begin{aligned}
A_n^{\vartheta}(\vartheta;\, x) &= \sum_{k=1}^{\infty} \frac{k}{n} \mathcal{P}_{\vartheta,n,k}(x) \\
&= \sum_{k=1}^{\infty} \binom{n+k-1}{k} \frac{k}{n} \vartheta^k(x)\,(1+\vartheta(x))^{-n-k} \\
&= \vartheta(x) \sum_{k=0}^{\infty} \binom{n+k}{k} \vartheta^k(x)\,(1+\vartheta(x))^{-(n+1)-k} \\
&= \vartheta(x) \sum_{k=0}^{\infty} \mathcal{P}_{\vartheta,n+1,k}(x) = \vartheta(x)\, A_{n+1}(e_0;\, \vartheta(x)) \\
&= \vartheta(x)
\end{aligned}$$

and

$$\begin{aligned}
A_n^{\vartheta}\left(\vartheta^2;\, x\right) &= \sum_{k=0}^{\infty} \frac{k^2}{n^2} \mathcal{P}_{\vartheta,n,k}(x) \\
&= \sum_{k=2}^{\infty} \frac{k(k-1)}{n^2} \mathcal{P}_{\vartheta,n,k}(x) + \sum_{k=1}^{\infty} \frac{k}{n^2} \mathcal{P}_{\vartheta,n,k}(x) \\
&= \left(\frac{n+1}{n}\right) \vartheta^2(x) \sum_{k=0}^{\infty} \mathcal{P}_{\vartheta,n+2,k}(x) + \frac{\vartheta(x)}{n} \sum_{k=0}^{\infty} \mathcal{P}_{\vartheta,n+1,k}(x) \\
&= \left(\frac{n+1}{n}\right) \vartheta^2(x)\, A_{n+2}(e_0;\, \vartheta(x)) + \frac{\vartheta(x)}{n} A_{n+1}(e_0;\, \vartheta(x)) \\
&= \frac{\vartheta(x)(1+\vartheta(x))}{n} + \vartheta^2(x)\,.
\end{aligned}$$

Since other calculations are similar, we omit the details. □

We deal with the ϑ-central moments of order $k \in \mathbb{N}$ of the operators A_n^{ϑ}, defined by

$$M_{n,k}^{\vartheta}(x) := A_n^{\vartheta}\left(\vartheta_x^k;\, x\right),$$

where $\vartheta_x^k(t) := (\vartheta(t) - \vartheta(x))^k$.

Using Lemma 2.1 , we can state the following

Lemma 2.2. *We have*

i) $M_{n,2}^{\vartheta}(x) = \frac{\vartheta^2(x)+\vartheta(x)}{n}$,

ii) $M_{n,4}^{\vartheta}(x) = \frac{3\vartheta^4(x)+6\vartheta^3(x)+3\vartheta^2(x)}{n^2} + \frac{6\vartheta^4(x)+12\vartheta^3(x)+7\vartheta^2(x)+\vartheta(x)}{n^3}$,

iii) $M_{n,6}^{\vartheta}(x) = \frac{\vartheta(x)(1+\vartheta(x))}{n^5} \left[\begin{array}{c} 1+5(6+5n)\vartheta(x)+5(30+31n+3n^2)\vartheta^2(x) \\ +10(24+26n+3n^2)\vartheta^3(x)+5(24+26n+3n^2)\vartheta^4(x) \end{array} \right]$.

Remark 2.1. The moments and the central moments of new operators can be checked just taking $\vartheta = e_1$. They were already presented for the classical Baskakov operators in some papers (for instance [2]).

3. On the convergence of $A_n^{\vartheta}(f;x)$ in weighted spaces

Korovkin-type theorems give simple and useful tools for investigating whether a given sequence of positive linear operators is an approximation process, or equivalently, converges strongly to the identity operator. These theorems exhibit a variety of subsets of test functions which guarantee that the approximation (or the convergence) property holds on the whole space provided it holds on them (see [3]).

The first weighted Korovkin-type theorems were proved by Gadzhiev in [7] and [8]. Recently, in the aforesaid papers [9] and [12], some quantitative versions were stated by introducing different weighted modulus of continuity.

Now, we present the weighted spaces of functions that appear in the paper. With this purpose we firstly introduce the function

$$\varphi\left(x\right):=1+\vartheta^{2}\left(x\right),$$

and define the weighted space

$$B_{\varphi}\left(\mathbb{R}^{+}\right):=\{f:\mathbb{R}^{+}\rightarrow\mathbb{R}\ :\ \left|f\left(x\right)\right|\leq M_{f}\ \varphi\left(x\right),x\geq0\},$$

where M_f is a constant depending on f. $B_{\varphi}\left(\mathbb{R}^{+}\right)$ is a normed space with the norm

$$\left\|f\right\|_{\varphi}=\sup_{x\in\mathbb{R}^{+}}\frac{\left|f\left(x\right)\right|}{\varphi\left(x\right)}.$$

$C_{\varphi}\left(\mathbb{R}^{+}\right)$ denotes the subspace of all continuous function in $B_{\varphi}\left(\mathbb{R}^{+}\right)$, and $C_{\varphi}^{k}\left(\mathbb{R}^{+}\right)$ denotes the subspace of all functions $f\in C_{\varphi}\left(\mathbb{R}^{+}\right)$ with $\lim\limits_{x\to\infty}\frac{f(x)}{\varphi(x)}=k_f$, where k_f is a constant depending on f. Also let $U_{\varphi}\left(\mathbb{R}^{+}\right)$ be the space of functions $f\in C_{\varphi}\left(\mathbb{R}^{+}\right)$ such that $\frac{f(x)}{\varphi(x)}$ is uniformly continuous. It is obvious that

$$C_{\varphi}^{k}\left(\mathbb{R}^{+}\right)\subset U_{\varphi}\left(\mathbb{R}^{+}\right)\subset C_{\varphi}\left(\mathbb{R}^{+}\right)\subset B_{\varphi}\left(\mathbb{R}^{+}\right).$$

Finally, we recover from [12] the weighted modulus of continuity that we shall use to estimate the rate of convergence for functions from $C_{\varphi}(R^{+})$. It is worthy mentioning that another modulus of this type was introduced in [9]. Obviously, it is useless to consider the first classical modulus of continuity because we are dealing with non bounded functions.

For $f\in C_{\varphi}(R^{+})$ and for every $\delta\geq0$,

$$\omega_{\vartheta}(f,\delta):=\sup\left\{\frac{|f(x)-f(y)|}{\varphi(x)+\varphi(y)}\ :\ x,y\geq0,\ |\vartheta(x)-\vartheta(y)|\leq\delta\right\}.$$

Two important properties of this modulus read as follows (see [12]):

$$\lim_{\delta\to0}\omega_{\vartheta}(f;\delta)=0,\quad f\in U_{\varphi}(R^{+}),$$

$$\omega_{\vartheta}(f;\lambda\delta)\leq(2+\lambda)\omega_{\vartheta}(f;\delta),\quad \delta,\lambda\geq0. \tag{3.8}$$

The following results were applied in [4] to the generalized Szász-Mirakyan operators. Here we follow the same pattern and state without proofs both illustrative qualitative and quantitative results for our sequence of operators.

Theorem 3.1. *For $f\in C_{\varphi}^{k}(\mathbb{R}^{+})$,*

$$\lim_{n\to\infty}\left\|A_{n}^{\vartheta}(f)-f\right\|_{\varphi}=0.$$

Theorem 3.2. *For $f\in C_{\varphi}(\mathbb{R}^{+})$,*

$$\left\|A_{n}^{\vartheta}(f)-f\right\|_{\varphi^{\frac{3}{2}}}\leq\left(7+\frac{4}{n}\right)\omega_{\vartheta}\left(f;2\sqrt{2/n}+18/n\right).$$

4. Quantitative Type Theorems

In recent years, the classical Voronovskaya formulae have been studied in quantitative form intensively. This kind of theorems are more interesting and important because they show the error of approximation and the rate of convergence simultaneously. For details we refer the readers to [10], [11]. Now we state a result of this type for our sequence of operators. Firstly, we prove a lemma that extends [12, Lemma 3]

Lemma 4.3. *Let $f \in C_{\varphi}(\mathbb{R}^{+})$ and $x \in \mathbb{R}^{+}$. Then, for $\delta > 0$, $y \in \mathbb{R}^{+}$ and ξ lying between x and y,*

$$|f(\xi) - f(x)| \leq 2(\varphi(y) + \varphi(x)) \left(2 + \frac{|\vartheta(y) - \vartheta(x)|}{\delta}\right) \omega_{\vartheta}(f;\delta). \tag{4.9}$$

Proof. Assume first that $x \leq \xi \leq y$. As φ is increasing and $|\vartheta(\xi) - \vartheta(x)| \leq |\vartheta(y) - \vartheta(x)|$, then

$$\begin{aligned}
\frac{|f(\xi) - f(x)|}{\varphi(y) + \varphi(x)} &\leq \frac{|f(\xi) - f(x)|}{\varphi(\xi) + \varphi(x)} \\
&\leq \sup\left\{\frac{|f(s) - f(t)|}{\varphi(s) + \varphi(t)} : s, t \geq 0,\ |\vartheta(s) - \vartheta(t)| \leq |\vartheta(y) - \vartheta(x)|\right\} \\
&= \omega_{\vartheta}(f; |\vartheta(x) - \vartheta(y)|) = \omega_{\vartheta}(f; \frac{|\vartheta(x) - \vartheta(y)|}{\delta}\delta) \\
&\leq \left(2 + \frac{|\vartheta(x) - \vartheta(y)|}{\delta}\right) \omega_{\vartheta}(f;\delta),
\end{aligned}$$

where in the last inequality we have used (3.8).

Assume now that $y \leq \xi \leq x$. Using the triangular inequality and the monotonicity of φ,

$$\begin{aligned}
\frac{|f(\xi) - f(x)|}{\varphi(y) + \varphi(x)} &\leq \frac{|f(\xi) - f(y)|}{\varphi(y) + \varphi(x)} + \frac{|f(y) - f(x)|}{\varphi(y) + \varphi(x)} \\
&\leq \frac{|f(\xi) - f(y)|}{\varphi(\xi) + \varphi(y)} + \frac{|f(y) - f(x)|}{\varphi(y) + \varphi(x)},
\end{aligned}$$

from where the proof is over proceeding twice as above. □

Theorem 4.3. *Let us assume that the functions $f''/(\vartheta')^2$ and $f'\vartheta''/(\vartheta')^3$ belong to the space $U_{\varphi}(\mathbb{R}^{+})$. Then, for $x \in \mathbb{R}^{+}$,*

$$\begin{aligned}
&\left| n\Big(A_n^{\vartheta}(f;x) - f(x)\Big) - \frac{\vartheta(x)(1 + \vartheta(x))}{2} \left(f \circ \vartheta^{-1}\right)''(\vartheta(x)) \right| \\
\leq\ & O(1)\left(1 + \vartheta^2(x)\right) \left\{ \omega_{\vartheta}\left(\frac{f''}{(\vartheta')^2}; n^{-1/2}\right) + \omega_{\vartheta}\left(\frac{f'\vartheta''}{(\vartheta')^3}; n^{-1/2}\right) \right\}.
\end{aligned}$$

Proof. By using the Taylor expansion of $f \circ \vartheta^{-1}$ at the point $x \in \mathbb{R}^{+}$ we can write, for $u \in \mathbb{R}^{+}$,

$$\begin{aligned}
f(u) &= \left(f \circ \vartheta^{-1}\right)(\vartheta(u)) = \left(f \circ \vartheta^{-1}\right)(\vartheta(x)) + \left(f \circ \vartheta^{-1}\right)'(\vartheta(x))(\vartheta(u) - \vartheta(x)) \\
&+ \frac{\left(f \circ \vartheta^{-1}\right)''(\vartheta(x))(\vartheta(u) - \vartheta(x))^2}{2} + h_x(u)(\vartheta(u) - \vartheta(x))^2,
\end{aligned}$$

where

$$h_x(u) = \frac{\left(f \circ \vartheta^{-1}\right)''(\vartheta(\xi)) - \left(f \circ \vartheta^{-1}\right)''(\vartheta(x))}{2}$$

and ξ is a number between x and u. Equivalently, we can write the functional identity

$$f = f(x)e_0 + \left(f \circ \vartheta^{-1}\right)'(\vartheta(x))\,\vartheta_x^1 + \frac{\left(f \circ \vartheta^{-1}\right)''(\vartheta(x))}{2}\vartheta_x^2 + h_x\vartheta_x^2.$$

If we apply the operator to both sides of this last equality and evaluate at the point x, we immediately have

$$\left| A_n^{\vartheta}(f;x) - f(x) - \frac{\left(f \circ \vartheta^{-1}\right)''(\vartheta(x))}{2} M_{n,2}^{\vartheta}(x) \right| \leq A_n^{\vartheta}\left(|h_x|\,\vartheta_x^2;x\right),$$

and using Lemma 2.2 we can write

$$\left| A_n^{\vartheta}(f;x) - f(x) - \frac{\vartheta(x)(1+\vartheta(x))}{2n}\left(f \circ \vartheta^{-1}\right)''(\vartheta(x)) \right| \leq A_n^{\vartheta}\left(|h_x|\,\vartheta_x^2;x\right).$$

In order to complete the proof, we estimate $A_n^{\vartheta}\left(|h_x|\,\vartheta_x^2;x\right)$. Using Lemma 4.3, we can write, for any $\delta > 0$,

$$\begin{aligned}
h_x(u) &= \frac{\left(f \circ \vartheta^{-1}\right)''(\vartheta(\xi)) - \left(f \circ \vartheta^{-1}\right)''(\vartheta(x))}{2} \\
&= \frac{1}{2}\left\{ \frac{f''(\xi)}{(\vartheta'(\xi))^2} - \frac{f''(x)}{(\vartheta'(x))^2} + f'(x)\frac{\vartheta''(x)}{(\vartheta'(x))^3} - f'(\xi)\frac{\vartheta''(\xi)}{(\vartheta'(\xi))^3} \right\} \\
&\leq (\varphi(u)+\varphi(x))\left(2 + \frac{|\vartheta(u)-\vartheta(x)|}{\delta}\right)\left\{ \omega_{\vartheta}\left(\frac{f''}{(\vartheta')^2};\delta\right) + \omega_{\vartheta}\left(\frac{f'\vartheta''}{(\vartheta')^3};\delta\right) \right\} \\
&\leq 2(\varphi(u)+\varphi(x))\left(1 + \frac{|\vartheta(u)-\vartheta(x)|}{\delta}\right)\left\{ \omega_{\vartheta}\left(\frac{f''}{(\vartheta')^2};\delta\right) + \omega_{\vartheta}\left(\frac{f'.\vartheta''}{(\vartheta')^3};\delta\right) \right\},
\end{aligned}$$

where in the last inequality we have used that for all $\alpha, \beta \geq 0$, $\alpha(2+\beta) \leq 2\alpha(1+\beta)$. By using Cauchy-Schwarz inequality, we have

$$\begin{aligned}
& A_n^{\vartheta}\left(|h_x|\,\vartheta_x^2;x\right) \\
\leq\ & 2\left\{ \omega_{\vartheta}\left(\frac{f''}{(\vartheta')^2};\delta\right) + \omega_{\vartheta}\left(\frac{f'\vartheta''}{(\vartheta')^3};\delta\right) \right\} A_n^{\vartheta}\left((\varphi+\varphi(x)e_0)\left(e_0 + \frac{|\vartheta_x^1|}{\delta}\right)\vartheta_x^2;x\right) \\
=\ & 2\left\{ \omega_{\vartheta}\left(\frac{f''}{(\vartheta')^2};\delta\right) + \omega_{\vartheta}\left(\frac{f'\vartheta''}{(\vartheta')^3};\delta\right) \right\} A_n^{\vartheta}\left((\varphi+\varphi(x)e_0)\,\vartheta_x^2;x\right) \\
& +2\left\{ \omega_{\vartheta}\left(\frac{f''}{(\vartheta')^2};\delta\right) + \omega_{\vartheta}\left(\frac{f'\vartheta''}{(\vartheta')^3};\delta\right) \right\} A_n^{\vartheta}\left((\varphi+\varphi(x)e_0)\frac{|\vartheta_x^1|}{\delta}\vartheta_x^2;x\right) \\
\leq\ & 2\left\{ \omega_{\vartheta}\left(\frac{f''}{(\vartheta')^2};\delta\right) + \omega_{\vartheta}\left(\frac{f'\vartheta''}{(\vartheta')^3};\delta\right) \right\} \left(A_n^{\vartheta}\left(\varphi^2;x\right)\right)^{1/2}\left(M_{n,4}^{\vartheta}(x)\right)^{1/2} \\
& +2\varphi(x)\left\{ \omega_{\vartheta}\left(\frac{f''}{(\vartheta')^2};\delta\right) + \omega_{\vartheta}\left(\frac{f'\vartheta''}{(\vartheta')^3};\delta\right) \right\} M_{n,2}^{\vartheta}(x) \\
& +2\left\{ \omega_{\vartheta}\left(\frac{f''}{(\vartheta')^2};\delta\right) + \omega_{\vartheta}\left(\frac{f'\vartheta''}{(\vartheta')^3};\delta\right) \right\} \frac{1}{\delta}\left(A_n^{\vartheta}\left(\varphi^2;x\right)\right)^{1/2}\left(M_{n,6}^{\vartheta}(x)\right)^{1/2} \\
& +2\varphi(x)\left\{ \omega_{\vartheta}\left(\frac{f''}{(\vartheta')^2};\delta\right) + \omega_{\vartheta}\left(\frac{f'\vartheta''}{(\vartheta')^3};\delta\right) \right\} \frac{1}{\delta}\left(M_{n,6}^{\vartheta}(x)\right)^{1/2},
\end{aligned}$$

that is,

$$\begin{aligned}
&A_n^{\vartheta}\left(|h_x|\vartheta_x^2;x\right)\\
&\le 2\left\{\omega_{\vartheta}\left(\frac{f''}{(\vartheta')^2};\delta\right)+\omega_{\vartheta}\left(\frac{f'\vartheta''}{(\vartheta')^3};\delta\right)\right\}\left(\left(A_n^{\vartheta}\left(\varphi^2;x\right)\right)^{1/2}\left(M_{n,4}^{\vartheta}(x)\right)^{1/2}+\varphi(x)\,M_{n,2}^{\vartheta}(x)\right)\\
&\quad+2\left\{\omega_{\vartheta}\left(\frac{f''}{(\vartheta')^2};\delta\right)+\omega_{\vartheta}\left(\frac{f'\vartheta''}{(\vartheta')^3};\delta\right)\right\}\left(M_{n,6}^{\vartheta}(x)\right)^{1/2}\frac{1}{\delta}\left(\left(A_n^{\vartheta}\left(\varphi^2;x\right)\right)^{1/2}+\varphi(x)\right).
\end{aligned}$$

Since $\left(A_n^{\vartheta}\left(\varphi^2;x\right)\right)^{1/2}=O(1)$, $\left(M_{n,4}^{\vartheta}(x)\right)^{1/2}=O\left(n^{-1}\right)$, $M_{n,2}^{\vartheta}(x)=O\left(n^{-1}\right)$, $\left(M_{n,6}^{\vartheta}(x)\right)^{1/2}=O\left(n^{-3/2}\right)$, we can write

$$\begin{aligned}
&A_n^{\vartheta}\left(|h_x|\vartheta_x^2;x\right)\\
\le\;& 2\left(1+\vartheta^2(x)\right)\left\{\omega_{\vartheta}\left(\frac{f''}{(\vartheta')^2};\delta\right)+\omega_{\vartheta}\left(\frac{f'\vartheta''}{(\vartheta')^3};\delta\right)\right\}O\left(n^{-1}\right)\\
&+2\left\{\omega_{\vartheta}\left(\frac{f''}{(\vartheta')^2};\delta\right)+\omega_{\vartheta}\left(\frac{f'\vartheta''}{(\vartheta')^3};\delta\right)\right\}O\left(n^{-3/2}\right)\frac{1}{\delta}.
\end{aligned}$$

Choosing $\delta=n^{-1/2}$, we get

$$nA_n^{\vartheta}\left(|h_x|\vartheta_x^2;x\right)\le O(1)\left(1+\vartheta^2(x)\right)\left\{\omega_{\vartheta}\left(\frac{f''}{(\vartheta')^2};n^{-1/2}\right)+\omega_{\vartheta}\left(\frac{f'\vartheta''}{(\vartheta')^3};n^{-1/2}\right)\right\},$$

which completes the proof. □

Next corollary shows the classical Voronovskaya's formula:

Corollary 4.1. *Let $f\in U_{\varphi}(\mathbb{R}^+)$, $x\in R^+$ and suppose that the first and second derivatives of $f\circ\vartheta^{-1}$ exist at $\vartheta(x)$. If the second derivative of $f\circ\vartheta^{-1}$ is bounded on R^+, then*

$$\lim_{n\to\infty}n\left(A_n^{\vartheta}(f;x)-f(x)\right)=\frac{\vartheta^2(x)+\vartheta(x)}{2}\left(f\circ\vartheta^{-1}\right)''(\vartheta(x)).$$

Next result deals with the approximation of the first derivative of a function by using the first derivatives of our operator A_n^{ϑ}.

Theorem 4.4. *Assume that $f'/\vartheta'\in U_{\varphi}(\mathbb{R}^+)$, then*

$$\sup_{x\in\mathbb{R}^+}\frac{\left|A_n^{\vartheta}(f;x)'-f'(x)\right|}{\vartheta'(x)\left(1+\vartheta^2(x)\right)}\le\left\{3+\frac{4}{n}+\frac{(n+1)(2n+6)}{n^2}\right\}\omega_{\vartheta}\left(\frac{f'}{\vartheta'};\frac{1}{n}\right).$$

Proof. It is easily seen that

$$\begin{aligned}
A_n^{\vartheta}(f;x)' &= n\vartheta'(x)\sum_{k=0}^{\infty}\binom{n+k}{k}\frac{\vartheta^k(x)}{(1+\vartheta(x))^{n+k+1}}\left[\left(f\circ\vartheta^{-1}\right)\left(\frac{k+1}{n}\right)-\left(f\circ\vartheta^{-1}\right)\left(\frac{k}{n}\right)\right]\\
&= n\vartheta'(x)\sum_{k=0}^{\infty}\binom{n+k}{k}\frac{\vartheta^k(x)}{(1+\vartheta(x))^{n+k+1}}\Delta\left(f\circ\vartheta^{-1}\right)\left(\frac{k}{n}\right) \qquad (4.10)
\end{aligned}$$

and

$$\begin{aligned}
&A_n^{\vartheta}(f;x)' - f'(x)\\
=\;& \vartheta'(x)\sum_{k=0}^{\infty}\binom{n+k}{k}\frac{\vartheta^k(x)}{(1+\vartheta(x))^{n+k+1}}\left[n\Delta(f\circ\vartheta^{-1})\left(\frac{k}{n}\right)-\frac{f'(x)}{\vartheta'(x)}\right]\\
=\;& \vartheta'(x)\sum_{k=0}^{\infty}\binom{n+k}{k}\frac{\vartheta^k(x)}{(1+\vartheta(x))^{n+k+1}}\left[n\Delta(f\circ\vartheta^{-1})\left(\frac{k}{n}\right)-(f\circ\vartheta^{-1})'(\vartheta(x))\right].
\end{aligned}$$

Using the relation between the forward difference and the derivative of the corresponding function (See, [16, p.34, eq.(1.80)]), we obtain

$$n\Delta(f\circ\vartheta^{-1})\left(\frac{k}{n}\right)=(f\circ\vartheta^{-1})'(\vartheta(\xi_t))$$

for $t\in\mathbb{N}_0$ and $\xi_t\in\left[t,t+\frac{1}{n}\right)$. Furthermore we get

$$\begin{aligned}
\left|n\Delta(f\circ\vartheta^{-1})\left(\frac{k}{n}\right)-(f\circ\vartheta^{-1})'(\vartheta(x))\right| &= \left|(f\circ\vartheta^{-1})'(\vartheta(\xi_t))-(f\circ\vartheta^{-1})'(\vartheta(x))\right|\\
&= \left|\frac{f'(\xi_t)}{\vartheta'(\xi_t)}-\frac{f'(x)}{\vartheta'(x)}\right|.
\end{aligned}$$

Using Lemma 4.3, we can write

$$\begin{aligned}
&\left|(f\circ\vartheta^{-1})'(\vartheta(\xi_t))-(f\circ\vartheta^{-1})'(\vartheta(x))\right|\\
\le\;& \left(\varphi\left(\frac{k+1}{n}\right)+\varphi(x)\right)\left(2+\frac{1}{\delta}\left|\vartheta\left(\frac{k+1}{n}\right)-\vartheta(x)\right|\right)\omega_{\vartheta}\left(\frac{f'}{\vartheta'};\delta\right).
\end{aligned}$$

For the convenient notation let us consider the operators

$$A_{n,1}^{\vartheta}(f;x):=\sum_{k=0}^{\infty}\binom{n+k}{k}\frac{\vartheta^k(x)}{(1+\vartheta(x))^{n+k+1}}f\left(\frac{k}{n}\right)$$

which are linear and positive. So we can write

$$\begin{aligned}
&\left|A_n^{\vartheta}(f;x)'-f'(x)\right|\\
\le\;& \Bigg(2\vartheta'(x)A_{n,1}^{\vartheta}\left(\left(\varphi\left(\frac{k+1}{n}\right)+\varphi(x)\right);x\right)\\
&+\frac{\vartheta'(x)}{\delta}\left\{A_{n,1}^{\vartheta}\left(\varphi^2\left(\frac{k+1}{n}\right);x\right)\right\}^{1/2}\left\{A_{n,1}^{\vartheta}\left(\left(\vartheta\left(\frac{k+1}{n}\right)-\vartheta(x)\right)^2;x\right)\right\}^{1/2}\\
&+\frac{\vartheta'(x)}{\delta}\varphi(x)\left\{A_{n,1}^{\vartheta}\left(\left(\vartheta\left(\frac{k+1}{n}\right)-\vartheta(x)\right)^2;x\right)\right\}^{1/2}\Bigg)\omega_{\vartheta}\left(\frac{f'}{\vartheta'};\delta\right).
\end{aligned}$$

Since

$$A_{n,1}^{\vartheta}\left(\varphi\left(\frac{k+1}{n}\right);x\right)=\left(1+\frac{1}{n}\right)\vartheta(x)+\frac{1}{n}$$

and

$$A_{n,1}^{\vartheta}\left(\varphi^2\left(\frac{k+1}{n}\right);x\right)=\frac{(n+1)(n+2)}{n^2}\vartheta^2(x)+\frac{3(n+1)}{n^2}\vartheta(x)+\frac{1}{n^2},$$

if we choose

$$\delta=\frac{1}{n}$$

we have the desired result. □

5. Some further properties of $A_n^{\vartheta}(f;x)$

In this section we firstly show that if we assume the ϑ-convexity of the function f, then the convergence of $A_n^{\vartheta}(f;x)$ to its limit is monotone. The first result is a very direct consequence of Corollary 4.1, whose proof follows from the convexity of $f\circ\vartheta^{-1}$, while the second one requires certain calculations.

Corollary 5.2. *If f is ϑ-convex on $\mathbb{R}^+$, then*

$$A_n^{\vartheta}(f,x)\geq f(x).$$

Theorem 5.5. *If f is ϑ-convex on $\mathbb{R}^+$, then*

$$A_n^{\vartheta}\left(f;\ x\right)\geq A_{n+1}^{\vartheta}\left(f;\ x\right)$$

for all $n\geq 0$ and $x\in\mathbb{R}^+$ such that $\vartheta(x)\neq\frac{k}{n}$, $(k=0,1,2,\dots)$. If $f\circ\vartheta^{-1}$ is linear then $A_n^{\vartheta}\left(f;\ x\right)=A_{n+1}^{\vartheta}\left(f;\ x\right)$.

Proof. Firstly, we arrange the operators $A_n^{\vartheta}\left(f;\ x\right)$ and $A_{n+1}^{\vartheta}\left(f;\ x\right)$. We can write

$$\begin{aligned}
A_n^{\vartheta}\left(f;\ x\right) &= \frac{\left(f\circ\vartheta^{-1}\right)(0)}{\left(1+\vartheta(x)\right)^n}+\sum_{k=1}^{\infty}\binom{n+k-1}{k}\left(f\circ\vartheta^{-1}\right)\left(\frac{k}{n}\right)\frac{\vartheta^k(x)}{\left(1+\vartheta(x)\right)^{n+k}}\\
&= \frac{\left(f\circ\vartheta^{-1}\right)(0)}{\left(1+\vartheta(x)\right)^n}+\sum_{k=0}^{\infty}\binom{n+k}{k+1}\left(f\circ\vartheta^{-1}\right)\left(\frac{k+1}{n}\right)\frac{\vartheta^{k+1}(x)}{\left(1+\vartheta(x)\right)^{n+k+1}}
\end{aligned}$$

and

$$\begin{aligned}
&A_{n+1}^{\vartheta}\left(f;\ x\right)\\
=\ &\sum_{k=0}^{\infty}\binom{n+k}{k}\left(f\circ\vartheta^{-1}\right)\left(\frac{k}{n+1}\right)\frac{\vartheta^k(x)}{\left(1+\vartheta(x)\right)^{n+k}}\\
&-\sum_{k=0}^{\infty}\binom{n+k}{k}\left(f\circ\vartheta^{-1}\right)\left(\frac{k}{n+1}\right)\frac{\vartheta^{k+1}(x)}{\left(1+\vartheta(x)\right)^{n+k+1}}\\
=\ &\frac{\left(f\circ\vartheta^{-1}\right)(0)}{\left(1+\vartheta(x)\right)^n}+\sum_{k=1}^{\infty}\binom{n+k}{k}\left(f\circ\vartheta^{-1}\right)\left(\frac{k}{n+1}\right)\frac{\vartheta^k(x)}{\left(1+\vartheta(x)\right)^{n+k}}\\
&-\sum_{k=0}^{\infty}\binom{n+k}{k}\left(f\circ\vartheta^{-1}\right)\left(\frac{k}{n+1}\right)\frac{\vartheta^{k+1}(x)}{\left(1+\vartheta(x)\right)^{n+k+1}}.
\end{aligned}$$

Now using the equations above, we can write the difference of the operators $A_{n+1}^{\vartheta}\left(f;\ x\right)$ and $A_n^{\vartheta}\left(f;\ x\right)$:

$$\begin{aligned}
&A_{n+1}^{\vartheta}\left(f;\ x\right)-A_n^{\vartheta}\left(f;\ x\right)\\
=\ &\sum_{k=0}^{\infty}\frac{\vartheta^{k+1}(x)}{\left(1+\vartheta(x)\right)^{n+k+1}}\left[\left(f\circ\vartheta^{-1}\right)\left(\frac{k+1}{n+1}\right)\binom{n+k+1}{k+1}\right.\\
&\left.-\left(f\circ\vartheta^{-1}\right)\left(\frac{k}{n+1}\right)\binom{n+k}{k}-\left(f\circ\vartheta^{-1}\right)\left(\frac{k+1}{n}\right)\binom{n+k}{k+1}\right].
\end{aligned}$$

Then using the following equalities

$$\binom{n+k+1}{k+1}=\binom{n+k}{k}\frac{n+k+1}{k+1},\qquad\binom{n+k}{k+1}=\binom{n+k}{k}\frac{n}{k+1}$$

we have

$$\begin{aligned} & A_{n+1}^{\vartheta}(f;\ x) - A_n^{\vartheta}(f;\ x) \\ = \; & -\sum_{k=0}^{\infty} \frac{\vartheta^{k+1}(x)}{(1+\vartheta(x))^{n+k+1}} \binom{n+k}{k} \\ & \left[\left(f\circ\vartheta^{-1}\right)\left(\frac{k}{n+1}\right) - \frac{n+k+1}{k+1}\left(f\circ\vartheta^{-1}\right)\left(\frac{k+1}{n+1}\right) + \frac{n}{k+1}\left(f\circ\vartheta^{-1}\right)\left(\frac{k+1}{n}\right)\right]. \end{aligned}$$

From the divided differences of $\left(f\circ\vartheta^{-1}\right)$ on the knots $\frac{k}{n+1}, \frac{k+1}{n+1}$ and $\frac{k+1}{n}$, we get

$$\begin{aligned} & \left(f\circ\vartheta^{-1}\right)\left[\frac{k}{n+1}, \frac{k+1}{n+1}, \frac{k+1}{n}\right] \\ = \; & \frac{\left(f\circ\vartheta^{-1}\right)\left[\frac{k+1}{n+1}, \frac{k+1}{n}\right] - \left(f\circ\vartheta^{-1}\right)\left[\frac{k}{n+1}, \frac{k+1}{n+1}\right]}{\frac{k+1}{n} - \frac{k}{n+1}} \\ = \; & \frac{n(n+1)^2}{n+k+1}\left[\left(f\circ\vartheta^{-1}\right)\left(\frac{k}{n+1}\right)\right. \\ & \left. -\frac{n+k+1}{k+1}\left(f\circ\vartheta^{-1}\right)\left(\frac{k+1}{n+1}\right) + \frac{n}{k+1}\left(f\circ\vartheta^{-1}\right)\left(\frac{k+1}{n}\right)\right]. \end{aligned}$$

This completes the proof. □

Now we show a shape preserving property of A_n^{ϑ}.

Theorem 5.6. *Assume that $f(x) \geq 0$ for $x > 0$. If $\left(f\circ\vartheta^{-1}\right)(x)/x$ is decreasing for $x > 0$, then $\frac{A_n^{\vartheta}(f;x)}{\vartheta(x)}$ is also decreasing for $x > 0$.*

Proof. From the definition of the operator $A_n^{\vartheta}(f;\ x)$, we can write

$$\frac{A_n^{\vartheta}(f;\ x)}{\vartheta(x)} = \frac{f(0)}{\vartheta(x)(1+\vartheta(x))^n} + \sum_{k=1}^{\infty}\binom{n+k-1}{k}\left(f\circ\vartheta^{-1}\right)\left(\frac{k}{n}\right)\vartheta(x)^{k-1}(1+\vartheta(x))^{-n-k}.$$

Thus we have

$$\begin{aligned} \left(\frac{A_n^{\vartheta}(f;x)}{\vartheta(x)}\right)' = \; & -\frac{\vartheta'(x) f(0)}{\vartheta^2(x)(1+\vartheta(x))^n} - n\frac{\vartheta'(x) f(0)}{\vartheta(x)(1+\vartheta(x))^{n+1}} \\ & +\vartheta'(x)\sum_{k=1}^{\infty}\left(f\circ\vartheta^{-1}\right)\left(\frac{k+1}{n}\right)\binom{n+k}{k+1} k\vartheta(x)^{k-1}(1+\vartheta(x))^{-n-1-k} \\ & -\vartheta'(x)\sum_{k=1}^{\infty}\left(f\circ\vartheta^{-1}\right)\left(\frac{k}{n}\right)\binom{n+k-1}{k}(n+k)\vartheta(x)^{k-1}(1+\vartheta(x))^{-n-1-k}. \end{aligned}$$

Using the inequalities

$$\binom{n+k}{k+1} = \binom{n+k}{k}\frac{n}{k+1}, \qquad \binom{n+k-1}{k}(n+k) = \binom{n+k}{k} n$$

we obtain

$$\begin{aligned}\left(\frac{A_n^{\vartheta}(f;x)}{\vartheta(x)}\right)' &= -\frac{\vartheta'(x) f(0)}{\vartheta^2(x)(1+\vartheta(x))^n} - n\frac{\vartheta'(x) f(0)}{\vartheta(x)(1+\vartheta(x))^{n+1}} \\ &\quad +\vartheta'(x)\sum_{k=1}^{\infty}\left(f\circ\vartheta^{-1}\right)\left(\frac{k+1}{n}\right)\binom{n+k}{k+1}k\vartheta(x)^{k-1}(1+\vartheta(x))^{-n-1-k} \\ &\quad -\vartheta'(x)\sum_{k=1}^{\infty}\left(f\circ\vartheta^{-1}\right)\left(\frac{k}{n}\right)\binom{n+k-1}{k}(n+)\,\vartheta(x)^{k-1}(1+\vartheta(x))^{-n-1-k}.\end{aligned}$$

and

$$\left(\frac{A_n^{\vartheta}(f;x)}{\vartheta(x)}\right)' = -\frac{\vartheta'(x) f(0)}{\vartheta^2(x)(1+\vartheta(x))^n} - n\frac{\vartheta'(x) f(0)}{\vartheta(x)(1+\vartheta(x))^{n+1}} \tag{5.11}$$

$$+\vartheta'(x)\sum_{k=1}^{\infty}\binom{n+k}{k}\vartheta(x)^{k-1}(1+\vartheta(x))^{-n-1-k}\,k\left(\left(f\circ\vartheta^{-1}\right)\left(\frac{k+1}{n}\right)\frac{n}{k+1}-\left(f\circ\vartheta^{-1}\right)\left(\frac{k}{n}\right)\frac{n}{k}\right).$$

Since $\left(f\circ\vartheta^{-1}\right)(x)/x$ is decreasing for $x\in(0,\infty)$, (5.11) completes the proof. □

Now, we introduce a Lipschitz space related to ϑ, analogous to the classical space $Lip_M\alpha$. Then we prove a direct result for functions from this space.

Definition 5.1. Let $0<\alpha\le 1$ and $M>0$. We denote by $Lip_M(\vartheta;\alpha)$ the set of all functions f satisfying the inequality

$$|f(t)-f(x)|\le M|\vartheta(t)-\vartheta(x)|^{\alpha},\quad x,t\ge 0.$$

It is obvious that for $f\in Lip_M(\vartheta;\alpha)$

$$\omega_{\vartheta}(f,\delta)\le M\delta^{\alpha}.$$

Theorem 5.7. *For any $f\in Lip_M(\vartheta;\alpha)$, $\alpha\in(0,1]$ and for every $x>0$, we have*

$$\left|A_n^{\vartheta}(f;x)-f(x)\right|\le M\left(\frac{\vartheta^2(x)+\vartheta(x)}{n}\right)^{\alpha/2}.$$

Proof. Here we extend the notation of ϑ_x^k and consider $\vartheta_x^{\alpha}(t)=(\vartheta(t)-\vartheta(x))^{\alpha}$ for any $\alpha>0$.

First assume $\alpha=1$. Then we have

$$\begin{aligned}\left|A_n^{\vartheta}(f;x)-f(x)\right| &\le A_n^{\vartheta}(|f-f(x)e_0|;x) \\ &\le MA_n^{\vartheta}\left(|\vartheta_x^1|;x\right)\end{aligned} \tag{5.12}$$

for $f\in Lip_M(\vartheta;1)$ and $x>0$. Applying Cauchy-Schwarz inequality in (5.12), we get

$$\begin{aligned}\left|A_n^{\vartheta}(f;x)-f(x)\right| &\le M\left[A_n^{\vartheta}\left(\vartheta_x^2;x\right)\right]^{1/2} \\ &= M\sqrt{\frac{\vartheta^2(x)+\vartheta(x)}{n}}.\end{aligned}$$

Now assume that $\alpha\in(0,1)$. Then we have

$$\begin{aligned}\left|A_n^{\vartheta}(f;x)-f(x)\right| &\le A_n^{\vartheta}(|f-f(x)e_0|;x) \\ &\le MA_n^{\vartheta}\left(|\vartheta_x^{\alpha}|;x\right)\end{aligned} \tag{5.13}$$

for $f \in Lip_M(\vartheta;\alpha)$ and $x>0$. Taking $p=1/\alpha$ and $q=1/(1-\alpha)$, for any $f \in Lip_M(\vartheta;\alpha)$ and applying the Hölder inequality in (5.13), we obtain

$$\left|A_n^{\vartheta}(f;x)-f(x)\right| \leq M\left[A_n^{\vartheta}\left(|\vartheta_x^1|;x\right)\right]^{\alpha}. \tag{5.14}$$

Using (5.14) and Cauchy-Schwarz inequality, we get

$$A_n^{\vartheta}(f;x)-f(x) \leq M\left(\frac{\vartheta^2(x)+\vartheta(x)}{n}\right)^{\alpha/2}.$$

This completes the proof. □

Finally, we define the following kind of generalized Lipschitz-type maximal function of order α

$$\tilde{\omega}_{\vartheta}^{\alpha}(f,x)=\sup_{\substack{x\neq t\\ t\in[0,\infty)}}\frac{|f(t)-f(x)|}{|\vartheta(t)-\vartheta(x)|^{\alpha}}, \tag{5.15}$$

for $f\in C_B[0,\infty)$, $\alpha\in(0,1]$.

Theorem 5.8. *Assume $f\in C_B[0,\infty)$ and $0<\alpha\leq 1$. Then*

$$\left|A_n^{\vartheta}(f;x)-f(x)\right| \leq \tilde{\omega}_{\vartheta}^{\alpha}(f,x)\left(\frac{\vartheta(x)+\vartheta^2(x)}{n}\right)^{\alpha/2}, \quad x>0.$$

Proof. By using (5.15), we have

$$\left|A_n^{\vartheta}(f;x)-f(x)\right| \leq \tilde{\omega}_{\vartheta}^{\alpha}(f,x)\,A_n^{\vartheta}\left(\left|\vartheta_x^1\right|^{\alpha};x\right). \tag{5.16}$$

If we use the Hölder inequality for $p=2/\alpha$ and $1/q=1-1/p$ in (5.16), we can write

$$\begin{aligned}\left|A_n^{\vartheta}(f;x)-f(x)\right| &\leq \tilde{\omega}_{\vartheta}^{\alpha}(f,x)\left[A_n^{\vartheta}\left(\vartheta_x^2;x\right)\right]^{\alpha/2}\\ &\leq \tilde{\omega}_{\vartheta}^{\alpha}(f,x)\left(\frac{\vartheta^2(x)+\vartheta(x)}{n}\right)^{\alpha/2}.\end{aligned}$$

□

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