

On a notion of rapidity of convergence used in the study of fixed point iterative methods

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ABSTRACT. There exist many papers written in the last 50 years or so which are using a concept of rapidity of convergence for two comparable sequences. It appears that the original source of this notion is not known to most of the authors of those papers, since for this notion they are refereeing to different sources or simply do not refer to any other publication like this notion would be "folklore" or even like they would be the ones who introduced for the first time such a notion.

Starting from this fact, the aim of the present note is to try to find out who was the true author of that notion of rapidity of convergence and, secondly, to illustrate how it is used in the particular case of the study of fixed point iterative methods.

1. INTRODUCTION

One of the most effective ways to solve a nonlinear functional equation of the form

$$F(x) = 0, \tag{1.1}$$

where F maps a topological space E into a linear space Y , is to convert it equivalently into a fixed point problem

$$x = Tx, \tag{1.2}$$

where T is a self map of a set X endowed with a certain structure (topological, metric, linear etc.).

Now, in order to establish an existence (and uniqueness) result for equation (1.1), the standard approach is to apply an appropriate fixed point theorem to its equivalent form (1.2). Already in possession of such an existence (and uniqueness) result for equation (1.2), at the next step, we are interested to compute the solution (s) of (1.2), if possible.

In general, there are no exact methods to do this and, therefore, it is possible only to approximate the solution (s) of (1.2) by means of a certain iterative method. The simplest method used in such situations is the well known *method of successive approximations* (also called *Picard iteration*), defined by an initial value $x_0 \in X$, and

$$x_{n+1} = Tx_n, n \geq 0. \tag{1.3}$$

But, in order to ensure the convergence of the above iterative method, one needs to guarantee various strong contractive properties for the mapping T on X or on a subset C of X .

If T obeys only weaker contractive conditions, then Picard iteration (1.3) does not converge or, even if it converges, its limit is not a fixed point of T . Hence, a convergence result cannot be obtained for the iterative method (1.3) under weak contractive conditions.

There exist more reliable fixed point iterative methods that could still be applied to (1.2) under the above mentioned weak contractive conditions on T , but these methods could be defined only in the presence of a linear structure in the set X . The most familiar fixed

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point iterative methods of this kind are the following ones, see [17] for more details and a rich bibliographical list.

Let X be a linear space and $T : X \rightarrow X$ be a given mapping and let $x_0 \in X$ be arbitrary and $\{\alpha_n\} \subset [0, 1]$ be a sequence of real numbers. The sequence $\{x_n\}_{n=0}^\infty \subset X$ defined by

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T x_n, \quad n = 0, 1, 2, \dots \quad (1.4)$$

is called the *Mann iteration* or *Mann iterative procedure*, see [17].

In the particular case $\alpha_n = \lambda$ (constant), $n = 0, 1, 2, \dots$, the sequence $\{x_n\}_{n=0}^\infty \subset X$ obtained from (1.4),

$$x_{n+1} = (1 - \lambda)x_n + \lambda T x_n, \quad n = 0, 1, 2, \dots \quad (1.5)$$

is usually called the *Krasnoselskij iteration* or *Krasnoselskij iterative procedure*, see [17].

The sequence $\{x_n\}_{n=0}^\infty \subset X$ defined by

$$\begin{cases} x_{n+1} &= (1 - \alpha_n)x_n + \alpha_n T y_n, & n = 0, 1, 2, \dots \\ y_n &= (1 - \beta_n)x_n + \beta_n T x_n, & n = 0, 1, 2, \dots, \end{cases} \quad (1.6)$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences of positive numbers in $[0, 1]$, and $x_0 \in X$ is arbitrary, is called the *Ishikawa iteration* or *Ishikawa iterative procedure*, see [17].

Assume now that we are in the situation that for equation (1.2) we know the following two facts:

- (1) equation (1.2) possesses a (unique) solution $p \in X$;
- (2) at least two iterative methods amongst (1.3)-(1.6) converge to p .

In such a case (and there are plenty in literature, see [17] and references therein), we are interested to know which is the best method - from a numerical point of view - amongst the methods (1.3)-(1.6). In order to take objectively such a decision, one needs an appropriate concept of *rate of convergence* for the considered fixed point iterative methods.

In [14], see also [15] and [17], the author introduced and studied a concept of rate of convergence for comparing two fixed point iterative methods. This was done in the following way.

Let $(X, \|\cdot\|)$ be a linear normed space and let $T : X \rightarrow X$ be a mapping. Let $\{u_n\}_{n=0}^\infty$ and $\{v_n\}_{n=0}^\infty$ be two fixed point iteration procedures associated to the fixed point problem (1.2), both of them converging to the same fixed point p of T .

Assume further that the following error estimates

$$\|u_n - p\| \leq a_n, \quad n = 0, 1, 2, \dots \quad (1.7)$$

and

$$\|v_n - p\| \leq b_n, \quad n = 0, 1, 2, \dots, \quad (1.8)$$

are available (and these estimates are the best ones available), where $\{a_n\}_{n=0}^\infty$ and $\{b_n\}_{n=0}^\infty$ are two sequences of positive numbers (converging to zero).

Now, in order to compare the two sequences $\{u_n\}_{n=0}^\infty$ and $\{v_n\}_{n=0}^\infty$ in X , it suffices to compare the two sequences of real numbers $\{a_n\}_{n=0}^\infty$, and $\{b_n\}_{n=0}^\infty$ converging to 0 appearing in the right hand side of (1.7) and (1.8), respectively.

To this end, we have used the following concept of rate of convergence, for which we referred, for convenience, to our paper [13], where it has been illustrated by several concrete examples.

Definition 1.1. Assume there exists

$$l = \lim_{n \rightarrow \infty} \frac{|a_n - a|}{|b_n - b|}.$$

Then

- a) If $l = 0$, then we say that $\{a_n\}_{n=0}^{\infty}$ converges *faster* to a than $\{b_n\}_{n=0}^{\infty}$ to b ;
 b) If $0 < l < \infty$, then we say that $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$ have the same rate of convergence.

Remark 1.1. 1) In the case a) one uses the notation $a_n - a = o(b_n - b)$;
 2) If $l = \infty$, then the sequence $\{b_n\}_{n=0}^{\infty}$ converges faster than $\{a_n\}_{n=0}^{\infty}$, that is

$$b_n - b = o(a_n - a).$$

The concept of rapidity of convergence that I have introduced in [14], see the next definition, has been used to study the rate of convergence of some fixed point iterative schemes in [14] and [15], see also [17].

Definition 1.2. ([14]) Let $\{u_n\}_{n=0}^{\infty}$ and $\{v_n\}_{n=0}^{\infty}$ be two fixed point iteration procedures that converge to the same fixed point p and satisfy (1.7) and (1.8), respectively. If $\{a_n\}_{n=0}^{\infty}$ converges faster than $\{b_n\}_{n=0}^{\infty}$ (in the sense of Definition 1.1 above), then we shall say that $\{u_n\}_{n=0}^{\infty}$ converges *faster* than $\{v_n\}_{n=0}^{\infty}$ to p .

This concept turned out to be a very useful and versatile tool in studying the fixed point iterative schemes and hence various authors have used it, see [1]-[5], [18], [22], [23], [28], [32]-[34], [37]-[41], [40], [43]-[46], [55]-[57], [66],[68]-[72], [74], [78]-[81], to cite just an incomplete list.

At the moment I have written [14], [15] and, later, [17], I did not care much about the original source of the concept of rapidity of convergence of two sequences of real numbers in Definition 1.1 and so I have quoted my own paper [13] for convenience, as no other more authoritative source has been easily available to me.

In fact, Definition 1.1 was considered as a kind of "folklore" in Romanian literature, and perhaps this fact has motivated me to publish the popularization paper [13].

But, very recently, I have been informed ([34]) about the paper by Bařarir [10] (where Definition 1.1 was attributed to Bajraktarević [6]) and I have been politely asked to compare the definition of rapidity of convergence in [10], on the one hand, and the corresponding definition in [13], [14], [15], [17], on the other hand, in order to decide whether they are or are not different.

This enabled me to go back from [10] to [6] and thus I have been pointed to a series of papers by Bajraktarević [6]-[9], published in the eighties, where he introduced and studied the same concept used by me in Definition 1.1 from [13].

Note that some other authors that worked in quite the same period with Bajraktarević have also credited him as the author of the concept of rate of convergence of two numerical sequences and, naturally, referred to Bajraktarević's papers, see the paper by Bařarir [10], and Miller and collaborators [58]-[65].

On the other hand, almost all the authors of the papers [1]-[5], [18], [22], [23], [28], [32]-[34], [37]-[41], [40], [43]-[46], [55]-[57], [66],[68]-[72], [74], [78]-[81], who have used further the concept of order of convergence of two fixed point iterations are referring correctly in their works to the papers [14], [15] or [17] for the concept I have given in Definition 1.2 [14].

But most of them are also referring to the papers [14], [15], [17] and/ or [13] for the concept of rapidity of convergence of two sequences of real numbers, as given in Definition 1.1, thus implicitly attributing this concept to Berinde [13].

Hence, in view of the whole story on this topic, I concluded that it is my duty to enlighten this issue and try to find out who was the true author of the concept of rapidity of convergence given by Definition 1.1.

Going back with the documentation on the concept given in Definition 1.1, we actually found out that the original source is back away than the year 1973 and that for sure this is not due to Bajraktarević [6]. This is the main aim of this note.

Subsidiary, I shall discuss some aspects related to other concepts of rapidity of convergence and point out the differences between them and I shall also illustrate how they are used to study fixed point iterative methods.

2. VARIOUS CONCEPTS OF RATE OF CONVERGENCE

There exist a few distinct concepts of rate of convergence / rapidity of convergence used in pure and applied mathematics. We briefly present them in the following.

Let (X, d) be a metric space and let $\{x_n\} \subset X$ be a convergent sequence with the limit \bar{x} .

One of the most used concepts of rate of convergence in computational and applied mathematics is based on an estimate of the form

$$d(x_n, \bar{x}) \leq C\alpha_n, \quad n = 0, 1, 2, \dots, \quad (2.9)$$

where $C > 0$ is a constant and $\{\alpha_n\}$ is a sequence of positive real numbers (convergent to 0).

In this case, (2.9) expresses the fact that the sequence $\{x_n\}$ converges to its limit \bar{x} at least as fast as the sequence of positive terms $\{\alpha_n\}$ converges to 0.

For such a situation, it is in use the notation with "big O":

$$d(x_n, \bar{x}) = O(\alpha_n),$$

that goes back as early as the nineteenth century, see for example [73], [35].

This concept of rapidity of convergence is clearly conditioned by the *a priori* knowledge of an estimate of the form (2.9).

There exists also another concept of rate of convergence which is very useful for studying the convergence of numerical methods, in particular of Newton type numerical methods, see for example [67], where another form of such a concept is defined and studied.

Let, like previously, $\{x_n\}_{n \geq 0} \subset X$ be a convergent sequence with limit \bar{x} . If, for some r , we have

$$\lim_{n \rightarrow \infty} \frac{d(x_{n+1}, x^*)}{[d(x_n, x^*)]^r} = \lambda < +\infty, \quad (2.10)$$

then r is called the *rate of convergence* of $\{x_n\}_{n \geq 0}$, while λ is termed as its *asymptotic error*, see [67] for more details.

If $r = 1$, we say that the convergence of $\{x_n\}_{n \geq 0}$ is *linear*, if $r = 2$, we say that the convergence is *quadratic*, while, for $0 < r < 1$, we say that the convergence is *superlinear*.

Note that, while the rate of convergence defined by (2.10) is an *absolute* one, in the sense that it involves only the sequence $\{x_n\}_{n \geq 0}$, the notion of rapidity of convergence given by means of (2.9) is a *relative* one since it involves two sequences, see [13] for more details and illustrative examples.

Another concept of rate of convergence, which also involves two (comparable) sequences, corresponds to Definition 1.1. We re-consider it here in the setting of a metric space.

Let $\{x_n\}_{n \geq 0}, \{y_n\}_{n \geq 0} \subset X$ be two convergent sequences with their limits denoted \bar{x} and \bar{y} , respectively.

If there exists

$$\lim_{n \rightarrow \infty} \frac{d(x_n, \bar{x})}{d(y_n, \bar{y})} = \beta, \quad (2.11)$$

then, if $\beta = 0$, we say that $\{x_n\}_{n \geq 0}$ converges to \bar{x} *faster* than $\{y_n\}_{n \geq 0}$ to \bar{y} , and if $\beta \neq 0$, we say that $\{x_n\}_{n \geq 0}$ and $\{y_n\}_{n \geq 0}$ have the same speed (rapidity) of convergence.

Clearly, if $\beta = \infty$, then $\{y_n\}_{n \geq 0}$ converges to \bar{x} *faster* than $\{x_n\}_{n \geq 0}$ to \bar{y} .

For the case of (2.11) with $\beta = 0$, one uses the old notation with "small o":

$$d(x_n, \bar{x}) = o(d(y_n, \bar{y})),$$

see [36], [51] etc.

The next example, taken from [18], illustrates in part the difference between the last two concepts of rapidity of convergence. For other related examples, see [13].

Example 2.1. If we consider the sequences $\{a_n\}_{n \geq 0}$, $\{b_n\}_{n \geq 0}$, $\{c_n\}_{n \geq 0}$ given by

$$a_n = \frac{1}{n+1}, \quad b_n = \frac{1}{2^n}, \quad c_n = 2^{-2^n},$$

then, obviously, $a_n \rightarrow 0$, $b_n \rightarrow 0$ and $c_n \rightarrow 0$, as $n \rightarrow \infty$, and since

$$a) \lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = 1, \quad b) \lim_{n \rightarrow \infty} \frac{b_{n+1}}{b_n} = \frac{1}{2}, \quad c) \lim_{n \rightarrow \infty} \frac{b_n}{a_n} = 0,$$

it follows that $\{a_n\}_{n \geq 0}$ and $\{b_n\}_{n \geq 0}$ have the same rate of convergence (linear) but however, $\{b_n\}_{n \geq 0}$ converges faster than $\{a_n\}_{n \geq 0}$ to 0.

Moreover, since

$$\lim_{n \rightarrow \infty} \frac{c_{n+1}}{(c_n)^2} = 1,$$

it follows that the sequence $\{c_n\}_{n \geq 0}$ has quadratic rate of convergence and, as an immediate consequence, it converges faster than both $\{a_n\}_{n \geq 0}$ and $\{b_n\}_{n \geq 0}$.

Let us also mention that, in connection to the study of convergent and divergent series, various authors have used a fourth concept of order of convergence.

According to Hadamard [35], Pringsheim [73] and even Abel were interested in studying a series that "diverges at least as rapidly as" another series, or "diverges less rapidly than" another series.

As inferred from the work of Hadamard [35], where it is not given an explicit definition, Hadamard [35], Pringsheim [73] and their predecessors have used in this context the following simple concept of rapidity of convergence.

Let $\sum_{n=0}^{\infty} u_n = S$ and $\sum_{n=0}^{\infty} v_n = T$ be two convergent series of positive terms (which implies that $u_n \rightarrow 0$, $v_n \rightarrow 0$, as $n \rightarrow \infty$). If

$$u_n \leq v_n, \quad n = 0, 1, 2, \dots, \quad (2.12)$$

then one says that the series $\sum_{n=0}^{\infty} u_n$ converges more rapidly than $\sum_{n=0}^{\infty} v_n$ or that the series

$$\sum_{n=0}^{\infty} v_n \text{ converges less rapidly than } \sum_{n=0}^{\infty} u_n.$$

(Inequalities (2.12) imply that the sequences u_n and $S_n := u_0 + u_1 + \dots + u_n$ approaches faster their limits, 0 and S , respectively, than the sequences v_n and $T_n := v_0 + v_1 + \dots + v_n$ approaches their limits, 0 and T , respectively).

In a metric space setting, this concept would be formulated as follows.

Let $\{x_n\}_{n \geq 0}$, $\{y_n\}_{n \geq 0} \subset X$ be two convergent sequences with the limits denoted by \bar{x} and \bar{y} , respectively. If

$$d(x_n, \bar{x}) \leq d(y_n, \bar{y}), \quad n \geq n_0, \quad (2.13)$$

then we say that $\{x_n\}_{n \geq 0}$ converges to \bar{x} *more rapidly* than $\{y_n\}_{n \geq 0}$ to \bar{y} or that $\{y_n\}_{n \geq 0}$ converges to \bar{y} *less rapidly* than $\{x_n\}_{n \geq 0}$ to \bar{x} .

Now, going back to the question of priority concerning the notion given in Definition 1.1, we may report that, after checking some hundreds of reviews in Zentralblatt MATH and MathScinet, as well as some old monographs and papers, we found out that this concept is actually due to Konrad Knopp (1887-1957).

Knopp's book *Theorie und Anwendung der unendlichen Reihen* (*Theory and Applications of Infinite Series*) has been very successful as shown by the printing of six German editions: first edition in 1922 [47], second edition (no information on it available in Zentralblatt MATH), third edition in 1931 [48], fourth edition in 1947 [49], fifth edition in 1964 [53], and sixth edition in 1996 [54]. The English version [51] has been published in 1956, the same year when the Polish edition [52] has also been printed.

The concept of rapidity of convergence whose ultimate origin we are looking for appears even in the first edition of *Theorie und Anwendung der unendlichen Reihen* [47], in paragraph 37, at pages 271-272 (in the English version [51], it appears on page 279).

It is formulated in Definition 1 for two convergent series and reads as follows (we give the English version here, for the convenience of the reader).

Definition 2.3. "Given two series $\sum c_n = s$ and $\sum c'_n = s'$ of positive terms whose partial sums are denoted by s_n and s'_n , the corresponding remainders by $s - s_n = r_n$, $s' - s'_n = r'_n$, we say that the second converges *more or less rapidly* (or *better or less well*) than the first, according as

$$\lim \frac{r'_n}{r_n} = 0 \text{ or } \lim \frac{r'_n}{r_n} = \infty. \quad (2.14)$$

If the limit of this ratio exists and has a finite positive value, or if be known merely that its lower limit > 0 and its upper limit $< +\infty$, then the convergence of the two series will be said to be of the same kind.

In any other case a comparison of the rapidity of convergence of the two series is impracticable."

It is easy to see that (2.14) actually expresses the fact that

$$\lim \frac{s' - s'_n}{s - s_n} = 0 \text{ or } \lim \frac{s' - s'_n}{s - s_n} = \infty,$$

which are exactly like the limit involved in Definition 1.1.

However, we have to point out that there were some authors that published their work earlier than the papers by Bajraktarević [6]-[9], Bařarir [10], Miller and collaborators [58]-[65], Berinde [13] etc., and who cited correctly Knopp's book.

We first mention here Dawson who, in a paper from 1964 [25], explicitly mentions that the definition of rate of convergence he is using in his paper is taken from Knopp's book [51]:

"The basis for comparing convergence rates of sequences used here is as follows: If $\{x_p\}$ converges to X and $\{y_p\}$ converges to Y , then " $\{x_p\}$ converges faster than $\{y_p\}$ " means that

$$\lim_{n \rightarrow \infty} \frac{x_n - X}{y_n - Y} = 0." \quad (2.15)$$

Note also that Bařarir [10], after attributing the priority in 1988 to Bajraktarević [6], only one year later [11] he already cited correctly Knopp's book [50], which appears to be an English edition from 1947 that cannot be found in MathScinet.

Miller and Zanelli [64] have used "the standard term-by-term definition for comparing rates of convergence: z converges faster than t provided that

$$z_m - \lim z_m = o(t_m - \lim t_m),"$$

which is just Definition 1.1.

Even later in 2004, Miller and Orhan [65] still credited Bajraktarević [6] as the author of this concept, although, as we have shown above, it has been introduced by Knopp more than 80 years ago.

To close this section, we mention that Fridy [30] considered a slightly different "concept of comparison of rates, namely, the sequence $z = \{z_n\}$ converges faster than the sequence $t = \{t_n\}$ provided that

$$\rho_m z = o(\rho_m t),$$

where $\rho_m x$ is defined by

$$\rho_m x := \max_{n>m} |x_n - \lim x_n|."$$

Therefore, without the intention to blame those many authors who ignored Knopp's priority (I am amongst them), the conclusion of this section is that, in order to be rigorous, it is the duty of any author dealing with the concept of rapidity of convergence in Definition 1.1 to cite Knopp's monograph, regardless of edition, as the original source of this important and useful notion.

3. RATE OF CONVERGENCE FOR FIXED POINT ITERATIVE METHODS

Of all concepts of rapidity of convergence presented above for numerical sequences, the one introduced by us in Definition 1.2 [14] appears to be the most suitable in the study of fixed point iterative methods.

As mentioned in Introduction, the first approach on this topic has been reported in [14] and [15], see also [17], and has been afterwards continued by some other authors: [1]-[5], [18], [22], [23], [28], [32], [33], [39], [40], [43]-[45], [55], [66],[68]-[72], [78]-[81],...

Note also that, in order to compare two fixed point iteration procedures $\{u_n\}_{n=0}^{\infty}$ and $\{v_n\}_{n=0}^{\infty}$ that converge to a certain fixed point p of a given operator T , Rhoades [75] considered another concept of rapidity of convergence, corresponding to that given by (2.13), i.e., he said that $\{u_n\}$ is *better* than $\{v_n\}$ if

$$\|u_n - p\| \leq \|v_n - p\|, \quad \text{for all } n.$$

For a recent study involving Rhoades' concept of rapidity of convergence, see [19] and [46]. The next example clearly shows that our concept of rapidity of convergence for fixed point iteration schemes given by Definition 1.2 is more sensitive than the one due to Rhoades [75].

Example 3.2. If we take $p = 0$, $u_n = \frac{1}{n+1}$, $v_n = \frac{1}{n}$, $n \geq 1$, then $\{u_n\}$ is better than $\{v_n\}$, but $\{u_n\}$ does not converge faster than $\{v_n\}$. Indeed, we have

$$\lim_{n \rightarrow \infty} \frac{u_n}{v_n} = 1,$$

and hence $\{u_n\}$ and $\{v_n\}$ have the same rate of convergence in the sense of Definition 1.2.

We close this section by stating the first comparison result established in [14] concerning the rapidity of convergence of two fixed point iteration procedures.

Theorem 3.1. *Let E be an arbitrary Banach space, K a closed convex subset of E , and $T : K \rightarrow K$ an operator satisfying Zamfirescu's contractive conditions, i.e., there exist the real numbers a, b and c satisfying $0 \leq a < 1$, $0 < b, c < 1/2$ such that for each pair x, y in X , at least one of the following is true:*

$$(z_1) \|Tx - Ty\| \leq a \|x - y\|;$$

$$(z_2) \|Tx - Ty\| \leq b[\|x - Tx\| + \|y - Ty\|];$$

$$(z_3) \|Tx - Ty\| \leq c[\|x - Ty\| + \|y - Tx\|].$$

Let $\{x_n\}_{n=0}^{\infty}$ be the Picard iteration defined by

$$x_{n+1} = Tx_n, \quad n = 0, 1, 2, \dots \quad (3.16)$$

and $x_0 \in K$ and let $\{y_n\}_{n=0}^{\infty}$ be the Mann iteration defined by

$$y_{n+1} = (1 - \alpha_n)y_n + \alpha_nTy_n, \quad n = 0, 1, 2, \dots \quad (3.17)$$

and $y_0 \in K$, with $\{\alpha_n\} \subset [0, 1]$ satisfying

$$(iv) \quad \sum_{n=0}^{\infty} \alpha_n = \infty.$$

Then $\{x_n\}_{n=0}^{\infty}$ and $\{y_n\}_{n=0}^{\infty}$ converge strongly to the unique fixed point of T and, moreover, Picard iteration converges faster than the Mann iteration.

Remark 3.2. As previously mentioned, the concept of rapidity of convergence introduced in Definition 1.2 is essentially dependent on the accuracy of the estimates (1.7) and (1.8). So, we tacitly admitted in Definition 1.2 that the estimates (1.7) and (1.8) taken into consideration are the best possible. Of course, this is not always possible in practice, except for some particular cases, when we could in fact compute directly the limit

$$\lim \frac{\|u_n - p\|}{\|v_n - p\|}. \quad (3.18)$$

Some authors, see for example Qing and Rhoades [74], noticed the inconsistency of Definition 1.2 due to inaccurate estimates (1.7) and (1.8) and have shown that in some cases the conclusion following by the direct evaluation of the limit in (3.18) could provide a different conclusion than the one resulting by comparing the sequences $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$ from the estimates (1.7) and (1.8).

Despite this drawback, our concept of rapidity of convergence has been used successfully by various authors in various contexts. We summarise a few of them in the following.

In [14], we asked whether the result in Theorem 3.1 can be extended to the class of quasi-contraction maps, which is more general than the class of Zamfirescu operators.

In 2007, Olaleru [66] answered this question in the affirmative, i.e., he has shown that Picard iteration converges faster than the Mann iteration scheme for quasi-contraction maps.

Next, Babu and Vara Prasad [5] have compared Mann iteration and Ishikawa iteration and have shown that, in the class of Zamfirescu mappings, Mann iteration converges faster than Ishikawa iteration. Note that this result has been previously obtained in [16].

Therefore, by Theorem 3.1 and this result, it follows that, in the class of Zamfirescu operators, Picard iteration converges faster than Mann and Ishikawa iteration.

Xue [80] established a direct comparison of the rate of convergence of Picard and Mann iterations in the class of Zamfirescu operators (Theorem 2.1), and a similar result for Krasnoselskij and Ishikawa iterations (Theorem 2.2), with the price of imposing some additional assumptions on the parameters that are involved in these fixed point iterations.

However, Xue [80] did not succeed in obtaining a direct comparison result of the rate of convergence in the case of Mann and Ishikawa iterations for the same class of mappings.

Popescu [71], [72], compared Picard iteration and Mann iteration in the class of so-called quasi- φ -contractions, thus extending significantly the results in [14], [16], [5] and [80].

Rhoades and Xue [76] performed a comparison of the rate of convergence of Picard, Mann, Ishikawa, and Noor iterations for the class of quasicontractive maps.

So, the best result for these fixed point iterations remains the one established by Babu and Vara Prasad [5], obtained by means of the comparison sequences $\{a_n\}$ and $\{b_n\}$ and not in a direct way.

For other papers that used the concept of the rate of convergence introduced in [14] and [15] and studied it in the presence of various contractive conditions, we refer to [1]-[5], [18], [22], [23], [28], [32]-[34], [37]-[41], [40], [43]-[46], [55]-[57], [66],[68]-[72], [74], [78]-[81].

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