CREAT. MATH. INFORM.
Volume **25** (2016), No. 1,
Pages 51 - 55

Online version at `https://creative-mathematics.cunbm.utcluj.ro/`
Print Edition: ISSN 1584 - 286X; Online Edition: ISSN 1843 - 441X
DOI: https://doi.org/10.37193/CMI.2016.01.06

Generalized connectedness and separation axioms of Minkowski space

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ABSTRACT. In this study, we investigate the generalized connectedness of Minkowski space endowed with $s-$topology and obtain that M^S is $\alpha-$connected but not semi, β and $b-$connected. Moreover, we study separation axioms in Minkowski space and show that M^S is α, semi, b, β and pre$-T_i (i = 0, 1/2, 1, 2, 5/2)$ space.

1. INTRODUCTION

Zeeman suggested a new topology instead of Euclidean topology on Minkowski space M, the $4-$dimensional space- time continuum of special relativity, since M is not locally homogeneous [29]. He called this topology as fine topology and defined it such that the induced topology on every spacelike hyperplane and every timelike line is $3-$dimensional and $1-$dimensional Euclidean topology, respectively [29, 30]. The structure of Minkowski space M is given by

$$M = \{x = (x_0, x_1, x_2, x_3) : x_0, x_1, x_2, x_3 \in \mathbb{R}\}$$

where its quadratic form Q is

$$Q(x) = x_0^2 - x_1^2 - x_3^2 - x_4^2.$$

An event $x \in M$ is called spacelike, timelike or null (lightlike) vector if $Q(x)$ is negative, positive or zero, respectively.

The sets

$$C^S(x) = \{y : y = x \text{ or } Q(y - x) < 0\},$$
$$C^T(x) = \{y : y = x \text{ or } Q(y - x) > 0\},$$

and

$$C^L(x) = \{y : Q(y - x) = 0\}$$

are called space cone, time cone and null (light) cone, respectively [29]. Nanda defined space topology ($s-$topology) and time topology ($t-$topology) that are weaker version of Zeeman's fine topology [20, 21]. The $s-$topology was defined by a countable local base $B(x) = \left\{N_\varepsilon^S(x) : \varepsilon > 0, \varepsilon \in \mathbb{Q}\right\}$ at each point of M where $N_\varepsilon^S(x) = N_\varepsilon^E(x) \cap C^S(x)$ and $N_\varepsilon^E(x)$ is the Euclidean $\varepsilon-$neighborhood of the point x [21]. Also, $t-$topology on M was defined in the same aspect [21]. Agrawal and Shrivastava proved that $n-$dimensional Minkowski space is connected, Hausdorff, not normal and not regular with respect to $s-$topology and $t-$topology in the recent works, [2] and [3]. Generalized $s-$topology of Minkowski space was introduced in [11].

Received: 29.07.2015. In revised form: 17.11.2015. Accepted: 01.02.2016
2010 *Mathematics Subject Classification.* 54A05, 53B30.
Key words and phrases. *Minkowski space, generalized connectedness, separation axioms.*
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The purpose of the present study is to investigate the connectedness and separation axioms of Minkowski space with respect to generalized $s-$topology. In the literature, generalized topology has been studied widely. The collection μ of subsets of X is generalized topology on X iff $\emptyset \in \mu$ and any union of elements of μ belongs to μ [9]. The elements of μ are called $\mu-$open sets and the complement of $\mu-$open sets are called $\mu-$closed sets. Several kinds of more and less nearly open sets have been introduced; e.g. preopen [19], semi-open [16], $\alpha-$open [22], $\beta-$open [1, 4] and $b-$open [5] sets. The classes of these sets constitute generalized topologies.

2. Preliminaries

Let (X, τ) be a topological space. A set $A \subset X$ is said to be preopen [19] (resp., semi-open [16], $\alpha-$open [22], $\beta-$open [1, 4] and $b-$open [5]) set iff $A \subset ic(A)$ (resp., $A \subset ci(A)$, $A \subset ici(A)$, $A \subset cic(A)$ and $A \subset ic(A) \cup ci(A)$) where $c(A)$ and $i(A)$ denote the closure and interior of A. The family of all preopen (resp., semi-open, $\alpha-$open, $\beta-$open and $b-$open) sets in (X, τ) is denoted by $PO\,(X)$ $(SO\,(X)\,,\ \alpha O\,(X)\,,\ \beta O\,(X)\,,\text{and}\ BO\,(X))$ and there are the relations $\tau \subset \alpha O\,(X) \subset SO\,(X) \subset BO\,(X) \subset \beta O\,(X)$, $\alpha O\,(X) \subset PO\,(X) \subset BO\,(X)$ and $\alpha O\,(X) = SO\,(X) \cap PO\,(X)$ for these families [8].

Definition 2.1. Let (X, τ) be a topological space. Then (X, τ) is called pre-connected [23] (resp., semi-connected [24], $\alpha-$connected [13], $\beta-$connected [14] and $b-$connected [10]) if X can not be expressed as the union of two nonempty disjoint preopen (resp., semi-open, $\alpha-$open, $\beta-$open and $b-$open) sets of X.

Lemma 2.1. [13] *Let (X, τ) be a topological space. Then (X, τ) is $\alpha-$connected if and only if (X, τ) is connected.*

It is known that $\tau \subset \alpha O\,(X) \subset SO\,(X) \subset BO\,(X) \subset \beta O\,(X)$ for semi-open, $\alpha-$open, $\beta-$open and $b-$open sets. Also, by Lemma 2.1, we can say that
$\beta-$connected $\Rightarrow$ $b-$connected $\Rightarrow$ semi-connected $\Rightarrow$ $\alpha-$connected $\Leftrightarrow$ connected.

Definition 2.2. [17] Let (X, τ) be a topological space. (X, τ) is irreducible if and only if $U \cap V \neq \emptyset$ for every nonempty $U \in \tau$ and $V \in \tau$.

Definition 2.3. [22] Let (X, τ) be a topological space. Then (X, τ) is extremally disconnected if the closure of each open set of X is open in X.

Definition 2.4. [15] Let (X, τ) be a topological space, $A \subset X$ is $g-$closed if $c(A) \subseteq U$ when $A \subseteq U$ and U is open.

Definition 2.5. [15] A space (X, τ) is a $T_{1/2}$ space if every $g-$closed subset of X is closed.

α (β, semi, pre and $b-$)$T_{1/2}$ spaces are defined with the same aspect.

Definition 2.6. [28] Let (X, τ) be a topological space. Then (X, τ) is $T_{5/2}$ space if any two distinct point in X are separated by closed neighbourhoods.

By replacing open sets by pre (semi, α, β and b)-open sets, respectively, the fundamental separation axioms are studied in literature, widely [18, 12, 6, 7, 26, 25].

Definition 2.7. Let (X, τ) be a topological space. If $(X, \alpha O\,(X))$ (resp., $(X, \beta O\,(X))$, $(X, SO\,(X))$, $(X, PO\,(X))$ and $(X, BO\,(X))$) is $T_i (i = 0, 1/2, 1, 2, 5/2)$ space. Then (X, τ) is $\alpha - T_i$ (resp., $\beta - T_i$, semi$-T_i$, pre$-T_i$ and $b - T_i$) $(i = 0, 1/2, 1, 2, 5/2)$ space.

3. GENERALIZED CONNECTEDNESS OF MINKOWSKI SPACE

The $s-$topology on $n-$dimensional Minkowski space M was studied by Agrawal and Shrivastava in [3] and they were proved that the $s-$topology on $n-$dimensional Minkowski space M is strictly finer than the Euclidean topology on M.

Let M^S denotes $n-$dimensional Minkowski space endowed with $s-$topology. M^S is connected, since M^S is path-connected [3].

The family of all $s-$open (resp., $\alpha - s-$open, $\beta - s-$open, semi$-s-$open, pre$-s-$open and $b - s-$open) sets on $n-$dimensional Minkowski space M is denoted by $\tau(s)$ (resp., $\alpha O\left(M^S\right)$, $\beta O\left(M^S\right)$, $SO\left(M^S\right)$, $PO\left(M^S\right)$ and $BO\left(M^S\right)$).

In this section, we investigate generalized connectedness of M^S with respect to aforesaid generalized topologies. On the other hand, the obtained relationships can be given for $t-$topology on $n-$dimensional Minkowski space M.

Proposition 3.1. *Let M^S be $n-$dimensional Minkowski space endowed with $s-$topology. Then M^S is not an irreducible space.*

Proof. $C^T(x) - \{x\}$ and $C^S(x) - \{x\}$ are two open subsets of $n-$dimensional Minkowski space endowed with Euclidean topology [3]. Also, these sets are open with respect to $s-$topology on is strictly finer than Euclidean topology on M [3]. On the other hand $C^T(x) - \{x\} \cap C^S(x) - \{x\} = \emptyset$. Considering Definition 2.2 completes the proof. □

The relations $\tau(s) \subset \alpha O\left(M^S\right) \subset SO\left(M^S\right) \subset BO\left(M^S\right) \subset \beta O\left(M^S\right)$ and $\alpha O\left(M^S\right) \subset PO\left(M^S\right) \subset BO\left(M^S\right)$ for topological space M^S give the following corollary.

Corollary 3.1. *The topological spaces M endowed with $\alpha O\left(M^S\right)$, $SO\left(M^S\right)$, $BO\left(M^S\right)$, $\beta O\left(M^S\right)$ and $PO\left(M^S\right)$ are not irreducible.*

Proposition 3.2. *Let M^S be $n-$dimensional Minkowski space endowed with $s-$topology. Then M^S is not extremally disconnected.*

Proof. $C^S(x)$ is an open subset of M^S [3]. The $s-$closure of $C^S(x)$ is $C^S(x) \cup C^L(x)$ [11]. Also, it is proved that $C^S(x) \cup C^L(x)$ is not open subset of the $n-$dimensional Minkowski space endowed with $s-$topology [11]. Since we found an open subset of M^S which has a not open $s-$closure, M^S is not extremally disconnected. □

Proposition 3.3. *Let M^S be $n-$dimensional Minkowski space endowed with $s-$topology. Then M^S is $\alpha-$connected.*

Proof. Since M^S is connected [3] and from Lemma 2.1, M^S is $\alpha-$connected. □

Proposition 3.4. *Let M^S be $n-$dimensional Minkowski space endowed with $s-$topology. Then M^S is not semi-connected.*

Proof. We find that M^S is not irreducible space. So M^S is not semi connected [27]. □

Corollary 3.2. *Let M^S be $n-$dimensional Minkowski space endowed with $s-$topology. Then M^S is not $b, \beta-$connected.*

4. GENERALIZED SEPARATION AXIOMS IN MINKOWSKI SPACE

It is known that $n-$dimensional Minkowski space endowed with Euclidean topology is Hausdorff. Also, the $s-$topology on M is strictly finer than the Euclidean topology on M [3]. Thus M^S is Hausdorff.

Proposition 4.5. *Let M^S be $n-$dimensional Minkowski space endowed with $s-$topology. Then M^S is α, semi, b, β and pre$-T_2$ space, respectively.*

Proof. Since $\tau(s) \subset \alpha O(M^S) \subset SO(M^S) \subset BO(M^S) \subset \beta O(M^S)$ and $\alpha O(M^S) \subset PO(M^S) \subset BO(M^S)$ and M^S is T_2-space, $(M, \alpha O(M^S))$, $(M, SO(M^S))$, $(M, BO(M^S))$, $(M, \beta O(M^S))$ and $(M, PO(M^S))$ are T_2-space, too. Thus, we obtain that M^S is α, semi, b, β and pre$-T_2$ space. □

Since $T_2 \Rightarrow T_1 \Rightarrow T_{1/2} \Rightarrow T_0$, the following corollary is obvious.

Corollary 4.3. *Let M^S be $n-$dimensional Minkowski space endowed with $s-$topology. Then M^S is α, semi, b, β and pre$-T_i (i = 0, 1/2, 1)$ space, respectively.*

Agrawal and Shrivastava showed that $n-$dimensional Minkowski space endowed with $s-$topology is not regular [3]. So M^S is not T_3, and so on. But we have to verify it to say M^S is, either $T_{5/2}$ or not.

Theorem 4.1. *Let M^S be $n-$dimensional Minkowski space endowed with $s-$topology. Then M^S is $T_{5/2}-$space.*

Proof. Let $x, y \in M$ and $x \neq y$. Then there is a positive real number ε such that $d_E(x, y) = \varepsilon$. Let us consider the $s-$neighborhoods of x and y, respectively, $N^S_{\varepsilon/3}(x)$ and $N^S_{\varepsilon/3}(y)$. Suppose that
$cl_s\left(N^S_{\varepsilon/3}(x)\right) \cap cl_s\left(N^S_{\varepsilon/3}(y)\right) \neq \emptyset$. Thus there is any $z \in cl_s\left(N^S_{\varepsilon/3}(x)\right) \cap cl_s\left(N^S_{\varepsilon/3}(y)\right)$, that is, $z \in cl_s\left(N^S_{\varepsilon/3}(x)\right)$ and $z \in cl_s\left(N^S_{\varepsilon/3}(y)\right)$. Since $N^S_{\varepsilon/3}(x) = N^E_{\varepsilon/3}(x) \cap C^S(x)$ and $N^S_{\varepsilon/3}(y) = N^E_{\varepsilon/3}(y) \cap C^S(y)$ from the definition of the $s-$topology [21], we know $cl_s\left(N^S_{\varepsilon/3}(x)\right) \subset cl_s\left(N^E_{\varepsilon/3}(x)\right)$ and $cl_s\left(N^S_{\varepsilon/3}(y)\right) \subset cl_s\left(N^E_{\varepsilon/3}(y)\right)$. Thus we have $z \in cl_s\left(N^E_{\varepsilon/3}(x)\right)$ and $z \in cl_s\left(N^E_{\varepsilon/3}(y)\right)$. If $z \in cl_s\left(N^E_{\varepsilon/3}(x)\right)$ and $z \in cl_s\left(N^E_{\varepsilon/3}(y)\right)$, there are the inequalities such that $d_E(z, x) \leq \varepsilon/3$ and $d_E(z, y) \leq \varepsilon/3$ where d_E denotes the Euclidean distance function on M. Also, from the triangle inequality $d_E(x, y) \leq d_E(z, x) + d_E(z, y)$ we obtain $\varepsilon \leq 2\varepsilon/3$ and this is a contradiction. Hence $cl_s\left(N^S_{\varepsilon/3}(x)\right) \cap cl_s\left(N^S_{\varepsilon/3}(y)\right) = \emptyset$. This completes the proof. □

As a consequence of this theorem the following assertion is evident.

Corollary 4.4. *M^S is α, semi, b, β and pre$-T_{5/2}$ space.*

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