CREAT. MATH. INFORM.
Volume **25** (2016), No. 1,
Pages 63 - 70

Online version at `https://creative-mathematics.cunbm.utcluj.ro/`
Print Edition: ISSN 1584 - 286X; Online Edition: ISSN 1843 - 441X
DOI: https://doi.org/10.37193/CMI.2016.01.08

Solutions of a system of integral equations with deviating argument

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ABSTRACT. In this paper we establish two existence and uniqueness results for the solutions of a system of integral equations with deviating argument of the form

$$y_1(x) = f_1(x) + \int_a^b K_1(x, y_1(y_1(s)), y_2(y_1(s)))ds.$$

The solutions are searched in the set $C_L([a,b];[a,b]^2)$ and the main tool used in our study is the Perov's fixed point theorem.

1. INTRODUCTION

The study of integral equations with deviating arguments as well as of systems of integral equations with deviating arguments constitutes the subject for a large number of a physical, biological and economical mathematical models. A class of integral equations with modified argument are the iterative functional-integral equations, such as the equation

$$x(t) = \int_a^b K(t; s; x(s); x(g(s)))ds + f(t). \tag{1.1}$$

This kind of integral equations has been studied by several authors but we refer in the following to the ones considered by Dobriţoiu (see [9]), where the author uses the technique of Picard operators, see [2], [9], [19].
Here we consider $t \in [a,b]$, $K \in C([a,b] \times [a;b] \times \mathbb{R}^m \times \mathbb{R}^m; \mathbb{R}^m)$, $f \in C([a,b];\mathbb{R}^m)$, $g \in C([a,b];[a,b])$. Before we present our main results, we will give some useful definitions and theorems.

Definition 1.1. ([3]) Let (X,d) be a generalized metric space ($d(x,y) \in \mathbb{R}^n_+$). The operator $H : X \to X$ is Q-Lipschitz, if there exists a matrix $Q \in M_{mm}(\mathbb{R}_+)$ such that

$$d(Hx, Hy) \le Q \cdot d(x,y), \quad \forall x, y \in X.$$

Definition 1.2. ([3]) A matrix $Q \in M_{mm}(\mathbb{R}_+)$ is called convergent to zero if Q^k converges to the zero matrix as $k \to \infty$.

Theorem 1.1. ([25]) *Let $Q \in M_{mm}(\mathbb{R}_+)$. The following statements are equivalent:*

i) *$Q^k \to 0$, when $k \to \infty$;*
ii) *the eigenvalues of the matrix Q lie in the open unit disc of the complex plane;*
iii) *the matrix $I - Q$ is nonsingular and*

$$(I - Q)^{-1} = I + Q + ... + Q^k + ...,$$

where $I \in M_{mm}(\mathbb{R})$ denotes the identity matrix.

Received: 15.03.2015. In revised form: 28.09.2015. Accepted: 01.02.2016
2010 *Mathematics Subject Classification.* 45B05, 45D05, 47H10.
Key words and phrases. *System of integral equations, existence of solutions, Perov's theorem.*

The following theorem is a generalization of the Banach's fixed point theorem, in the case of the generalized metric spaces, with the metric taking values in $\mathbb{R}^n$.

Theorem 1.2. (Perov [2]) *Let (X, d) be a complete generalized metric space and let $H : X \to X$ be an operator which is Q-Lipschitz with Q a matrix convergent to zero. Then*

i) $F_H = x^*$;

ii) the sequence of successive approximations $x_k = H^k(x_0)$ converges to $x^* \in X$, for any $x_0 \in X$;

iii) the estimation $d(x_k, x^*) \leq Q^k(I - Q)^{-1} \cdot d(x_0, x_1)$ holds, for each $k \in \mathbb{N}^*$;

iv) if $T : X \to X$ satisfies the condition $d(Hx, Tx) \leq \eta$, $\forall x \in X,\ \eta \in \mathbb{R}^n$ and we consider the sequence $y_k = T^k(x_0)$, then $d(y_k, x^*) \leq (I - Q)^{-1}\eta + Q^k(I - Q)^{-1}d(x_0, x_1)$.

For some applications of Perov's fixed point theorem to the study of integral equations we refer to [10], [11], [12], [16], [21], [20], [25]. An existence result in $C_L([a, b], [a, b]^2)$ appeared in [17].

The aim of this paper is to study a system of two integral equations of Fredholm type with modified argument by using Perov's fixed point theorem.

A system of two iterative integral equations of Fredholm type is also studied by the same technique.

2. Main results

We consider a system of integral equations of Fredholm type of the form:

$$\begin{cases} y_1(x) = f_1(x) + \int\limits_a^b K_1(x, y_1(y_1(s)), y_2(y_1(s)))ds \\ y_2(x) = f_2(x) + \int\limits_a^b K_2(x, y_1(y_2(s)), y_2(y_2(s)))ds, \end{cases} \tag{2.2}$$

where $x \in [a, b], y_i : [a, b] \to [a, b], f_i \in C([a, b], [a, b]^2), i = \overline{1, 2}, K_i \in C([a, b]^3; [a, b]^2)$.

We try to fit in the conditions of Perov's fixed point theorem, in order to prove that the system (2.2) has in $C([a, b], [a, b]^2)$ a unique solution.

Let $L > 0$ be given and consider the set

$$C_L([a, b], [a, b]^2) = \{(y_1, y_2) \in C([a, b], [a, b]) : |y_i(t_1) - y_i(t_2)| \leq \\ \leq L \cdot |t_1 - t_2|\,,\ \forall t_1, t_2 \in [a, b], i = 1, 2\}.$$

The space $(C_L([a, b], [a, b]^2), d_C)$ is a complete metric space, where
$d_C : C_L([a, b], [a, b]^2) \times C_L([a, b], [a, b]^2) \to \mathbb{R}^2$ is defined by

$$d_C((y_1, y_2), (z_1, z_2)) = \begin{pmatrix} \|y_1 - z_1\|_C \\ \|y_2 - z_2\|_C \end{pmatrix} = \begin{pmatrix} \max\limits_{t\in[a,b]} |y_1(t) - z_1(t)| \\ \max\limits_{t\in[a,b]} |y_2(t) - z_2(t)| \end{pmatrix}$$

for $(y_1, y_2), (z_1, z_2) \in C_L([a, b], [a, b]^2)$.
Denote $C_x = \max\{x - a, b - x\}$, $\forall x \in [a, b]$.

Theorem 2.3. *Assume that*

i) $f \in C([a, b], [a, b]^2), K \in C([a, b]^3, [a, b]^2)$,

ii) $\exists\ L_{1j}, L_{2j} > 0$, $j = 1, 2$ *such that*

$$|K_1(t, u_1, u_2) - K_1(t, v_1, v_2)| \leq \sum_{j=1}^{2} L_{1j}\,|u_j - v_j|,$$

$$|K_2(t,u_1,u_2)-K_2(t,v_1,v_2)| \le \sum_{j=1}^{2} L_{2j}\,|u_j-v_j|$$

for all and
$\exists\, L_K>0$ *such that* $|K_i(t_1,u,v)-K_i(t,u,v)| \le L_K\cdot|t_1-t_2|$, $\forall t_1,t_2\in[a,b],\ i=1,2$
iii) $\exists\, L_f>0$ *such that* $|f_i(t_1)-f_i(t_2)|\le L_f\cdot|t_1-t_2|$, $\forall t_1,t_2\in[a,b]$, $i=1,2$,
iv) $L_f+L_K\cdot(b-a)\le L$;
v) For $x_0\in[a,b]$ *with* $|f_i(x)|\le|f_i(x_0)|,\forall x\in[a,b]$, $i=1,2$, *one of the following conditions holds:*
a) $M\cdot C_{x_0}\le C_{y_0}$, $y_0=\max(|f_1(x_0)|,|f_2(x_0)|)$ *and* $M=\max\{|K_i(t,u,v)|: t,u,v\in[a,b]\}$;
b) $x_0=a$, $M(b-a)\le b-C_{y_0}$, $K(t,u_i,v_i)\ge 0,\forall s,u_i,v_i\in[a,b], i=1,2$;
c) $x_0=b$, $M(b-a)\le C_{y_0}-a$, $K(s,u_i,v_i)\ge 0,\forall s,u_i,v_i\in[a,b], i=1,2$;
vi) the eigenvalues of the matrix

$$Q=\begin{pmatrix} (b-a)(L_{11}+L_{12}\cdot L) & (b-a)L_{12} \\ (b-a)L_{21} & (b-a)(L_{21}\cdot L+L_{22}) \end{pmatrix}$$

lie in the open unit disc of the complex plane.
Then the system (2.2) has in $C([a,b],[a,b]^2)$ *a unique solution.*

Proof. Consider the operator $H: C_L([a,b],[a,b]^2)\to C([a,b],[a,b]^2)$ defined by

$$(Hy)(t)=\begin{pmatrix}(H_1y)(t)\\(H_2y)(t)\end{pmatrix}=\begin{pmatrix} f_1(t)+\int_a^b K_1(t,y_1(y_1(s)),y_2(y_1(s)))ds \\ f_2(t)+\int_a^b K_2(t,y_1(y_2(s)),y_2(y_2(s)))ds\end{pmatrix},\quad t\in[a,b]$$

First, we prove the invariance property, i.e., the fact that

$$H(C([a,b],[a,b]^2))\subset C([a,b],[a,b]^2),$$

which means that for any $y\in C([a,b],[a,b]^2)$ we have

$$Hy\in C([a,b],[a,b]^2),(H_iy)(t)\in C([a,b],[a,b]^2),\forall t\in[a,b],\ i=1,2.$$

For $t\in[a,b]$, we have

$$|(H_1y)(t)|\le|f_1(t)|+\int_a^b |K_1(t,y_1(y_1(s)),y_2(y_1(s)))|\,ds\le|f_1(x_0)|+M(b-a)\le b,$$

and

$$|(H_1y)(t)|\ge|f_1(t)|-\int_a^b |K_1(t,y_1(y_1(s)),y_2(y_1(s)))|\,ds\ge|f_1(x_0)|-C_{y_0}\ge a.$$

So, $(H_1y)(t)\in[a,b]$ and in a similar manner we prove that $(H_2y)(t)\in[a,b]$ and by hypothesis (i) we have $Hy\in C([a,b],[a,b]^2)$.
Moreover, for $t_1,t_2\in[a,b], t_1\le t_2$ we will prove that

$$H(C_L([a,b],[a,b]^2))\subset C_L([a,b],[a,b]^2).$$

This follows in the following way

$$\|(Hy)(t_1)-(Hy)(t_2)\|_{\mathbb{R}^2}=\|((H_1y)(t_1)-(H_1y)(t_2),(H_2y)(t_1)-(H_2y)(t_2))\|_{\mathbb{R}^2}$$

$$
\begin{aligned}
|(H_1y)(t_1)-(H_1y)(t_2)| &\le |f_1(t_1)-f_1(t_2)|+\\
+\int_a^b |K_1(t_1,y_1(y_1(s)),y_2(y_1(s)))-K_1(t_2,y_1(y_1(s)),y_2(y_1(s)))|\,ds &\le\\
\le [L_f+L_K(b-a)]\,|t_1-t_2| \le L\cdot|t_1-t_2|\,.
\end{aligned}
$$

Similarly,

$$|(H_2y)(t_1)-(H_2y)(t_2)| \le L\cdot|t_1-t_2|\,.$$

We thus obtain that

$$\|(Hy)(t_1)-(Hy)(t_2)\|_{\mathbb{R}^2} \le \|(L\cdot|t_1-t_2|\,,L\cdot|t_1-t_2|)\|_{\mathbb{R}^2} = L\cdot|t_1-t_2|\,.$$

Therefore, the operator H is L-Lipschitz, which means that indeed $Hy\in C_L([a,b],[a,b]^2)$.

We will demonstrate that the operator H is Q-Lipschitz.

Accordingly, for $t\in[a,b]$ and $y,z\in C_L([a,b],[a,b]^2)$ we have:

$$d_C(Hy,Hz) = \begin{pmatrix} \|H_1y-H_1z\|_C \\ \|H_2y-H_2z\|_C \end{pmatrix}$$

and

$$
\begin{aligned}
|(H_1y)(t)-(H_1z)(t)| &\le \int_a^b |K_1(t,y_1(y_1(s)),y_2(y_1(s)))-K_1(t,z_1(z_1(s)),z_2(z_1(s)))|\,ds \le\\
&\le \int_a^b [L_{11}\,|y_1(y_1(s))-z_1(z_1(s))|+L_{12}\,|y_2(y_1(s))-z_2(z_1(s))|]ds \le\\
\le \int_a^b [L_{11}\,|y_1(y_1(s))-z_1(z_1(s))| &+ L_{12}\,|y_2(y_1(s))-y_2(z_1(s))|+L_{12}\,|y_2(z_1(s))-z_2(z_1(s))|]ds\\
&\le \int_a^b [L_{11}\,\|y_1-z_1\|+L_{12}\cdot L\,|y_1(s)-z_1(s)|+L_{12}\,|y_2(z_1(s))-z_2(z_1(s))|]ds\\
&\le (b-a)(L_{11}+L_{12}L)\cdot\|y_1-z_1\|_C+(b-a)L_{12}\cdot\|y_2-z_2\|_C\,.
\end{aligned}
$$

In a similar manner, we obtain:

$$|(H_2y)(t)-(H_2z)(t)| \le (b-a)L_{21}\cdot\|y_1-z_1\|_C+(b-a)(L_{21}L+L_{22})\cdot\|y_2-z_2\|_C$$

and hence have

$$d_C(Hy,Hz)\le Qd_C(y,z), \forall y,z\in C_L([a,b],[a,b]^2),$$

where

$$Q=\begin{pmatrix} (b-a)(L_{11}+L_{12}\cdot L) & (b-a)L_{12} \\ (b-a)L_{21} & (b-a)(L_{21}\cdot L+L_{22}). \end{pmatrix}$$

In accordance with condition (vi), the eigenvalues of the matrix Q lie in the open unit disc of the complex plane and hence $Q^k\to 0$, as $k\to\infty$.

By Perov's fixed point theorem, the operator $H: C_L([a,b],[a,b]^2)\to C_L([a,b],[a,b]^2)$ has a unique fixed point which is the solution of the system (2.2) in $C_L([a,b],[a,b]^2)$. □

Let us consider the following system of iterative integral equations

$$
\begin{cases}
y_1(x)=f_1(x)+\int_a^b K_1(x,y_1(s),y_2(s),y_1(y_1(s)),y_2(y_2(s)))ds,\\
y_2(x)=f_2(x)+\int_a^b K_2(x,y_1(s),y_2(s),y_1(y_1(s)),y_2(y_2(s)))ds.
\end{cases}
\tag{2.3}
$$

$x\in[a,b]$, $f\in C([a,b],[a,b]^2)$, $y_1,y_2\in C([a,b],[a,b])$,

Theorem 2.4. *Assume that*

i) $f \in C([a,b],[a,b]^2)$; $K \in C([a,b]^5,[a,b]^2)$;

ii) $\exists\, L_{1j}, L_{2j} > 0$, $j = 1,2$ *such that*

$$|K_1(t,u_1,u_2,u_3,u_4) - K_1(t,v_1,v_2,v_3,v_4)| \leq \sum_{j=1}^{4} L_{1j}\,|u_j - v_j|,$$

$$|K_2(t,u_1,u_2,u_3,u_4) - K_2(t,v_1,v_2,v_3,v_4)| \leq \sum_{j=1}^{4} L_{2j}\,|u_j - v_j|$$

for all $t, u_i, v_i \in [a,b],\ i = \overline{1,4}$;
and
$\exists\, L_K > 0$ *such that* $|K_i(t_1,u_1,u_2,u_3,u_4) - K_i(t_2,u_1,u_2,u_3,u_4)| \leq L_K \cdot |t_1 - t_2|$, $\forall t_1, t_2 \in [a,b],\ i = 1,2$

iii) $\exists\, L_f > 0$ *such that* $|f_i(t_1) - f_i(t_2)| \leq L_f \cdot |t_1 - t_2|$, $\forall t_1, t_2 \in [a,b]$, $i = 1,2$,

iv) $L_f + L_K \cdot (b-a) \leq L$;

v) For $x_0 \in [a,b]$ *with* $|f_i(x)| \leq |f_i(x_0)|\,, \forall x \in [a,b]$, $i = 1,2$, *one of the following conditions holds:*

a) $M \cdot C_{x_0} \leq C_{y_0}$, $y_0 = \max(|f_1(x_0)|\,, |f_2(x_0)|)$ *and* $M = \max\{|K_i(t,u_1,u_2,u_3,u_4)| : t, u_j \in [a,b], j = \overline{1,4}\}$;

b) $x_0 = a$, $M(b-a) \leq b - C_{y_0}$, $K(t,u_1,u_2,u_3,u_4) \geq 0$, $\forall s, u_j \in [a,b], j = \overline{1,4}$;

c) $x_0 = b$, $M(b-a) \leq C_{y_0} - a$, $K(t,u_1,u_2,u_3,u_4) \geq 0$, $\forall s, u_j \in [a,b], j = \overline{1,4}$;

vi) the eigenvalues of the matrix

$$Q = \begin{pmatrix} (b-a)(L_{11} + L_{13}(L+1)) & (b-a)(L_{12} + L_{14}(L+1)) \\ (b-a)(L_{21} + L_{23}(L+1)) & (b-a)(L_{22} + L_{24}(L+1)) \end{pmatrix}$$

lie in the open unit disc of the complex plane.
Then the system (2.3) *has in* $C_L([a,b],[a,b]^2)$ *a unique solution.*

Proof. Consider the operator $H : C_L([a,b],[a,b]^2) \to C([a,b],[a,b]^2)$ defined by

$$(Hy)(t) = ((H_1y)(t), (H_2y)(t)),\ t \in [a,b],$$

$$(H_1y)(t) = f_1(t) + \int_a^b K_1(t, y_1(s), y_2(s), y_1(y_1(s)), y_2(y_2(s)))ds,$$
$$(H_2y)(t) = f_2(2) + \int_a^b K_2(t, y_1(s), y_2(s), y_1(y_1(s)), y_2(y_2(s)))ds.$$

By conditions (i)-(v), we deduce that the operator is well defined and similarly to Theorem 2.3, we obtain that the operator H is L-Lipschitzian, hence

$$H(C_L([a,b],[a,b]^2)) \subset C_L([a,b],[a,b]^2).$$

In what follows we will demonstrate that the operator H is Q-Lipschitzian. For $t \in [a,b]$ and $y, z \in C_L([a,b],[a,b]^2)$, we have:

$$d_C(Hy, Hz) \;=\; \begin{pmatrix} \|H_1y - H_1z\|_C \\ \|H_2y - H_2z\|_C \end{pmatrix}$$

We have

$$|(H_1y)(t)-(H_1z)(t)|\leq$$

$$\leq \int_a^b |K_1(t,y_1(s),y_2(s),y_1(y_1(s)),y_2(y_2(s)))-K_1(t,z_1(s),z_2(s),z_1(z_1(s)),z_2(z_2(s)))|\,ds\leq$$

$$\leq \int_a^b [L_{11}\,|y_1(s)-z_1(s)|+L_{12}\,|y_2(s)-z_2(s)|+L_{13}\,|y_1(y_1(s))-z_1(z_1(s))|+$$
$$+L_{14}\,|y_2(y_2(s))-z_2(z_2(s))|]ds\leq$$
$$\leq \int_a^b [L_{11}\,|y_1(s)-z_1(s)|+L_{12}\,|y_2(s)-z_2(s)|+L_{13}\,|y_1(y_1(s))-y_1(z_1(s))|$$
$$+L_{13}\,|y_1(z_1(s))-z_1(z_1(s))|+L_{14}\,|y_2(y_2(s))-y_2(z_2(s))|+L_{14}\,|y_2(z_2(s))-z_2(z_2(s))|]ds$$
$$\leq (b-a)[L_{11}+L_{13}(L+1)]\cdot\|y_1-z_1\|_C+(b-a)[L_{12}+L_{14}(L+1)]\cdot\|y_2-z_2\|_C$$

In a similar manner, we obtain

$$|(H_2y)(t)-(H_2z)(t)|\leq(b-a)[L_{21}+L_{23}(L+1)]\cdot\|y_1-z_1\|_C+(b-a)[L_{22}+L_{24}(L+1)]\cdot\|y_2-z_2\|_C$$

Consequently, we infer that $d_C(Hy,Hz)\leq Q\cdot d_C(y,z)$, $\forall y,z\in C_L([a,b],[a,b]^2)$, where

$$Q=\begin{pmatrix} (b-a)[L_{11}+L_{13}(L+1)] & (b-a)[L_{12}+L_{14}(L+1)]\\ (b-a)[L_{21}+L_{23}(L+1)] & (b-a)[L_{22}+L_{24}(L+1)]\end{pmatrix}$$

By hypothesis (vi), the eigenvalues of the matrix Q lie in the open unit disc of the complex plane.

From the Theorem 1.2 it follows that Q is convergent to zero and, by Perov's fixed point theorem, the operator $H: C_L([a,b],[a,b]^2)\to C_L([a,b],[a,b]^2)$ has a unique fixed point which is the solution of (2.3) in $C_L([a,b],[a,b]^2)$. □

3. Examples

To illustrate our last results, we present the following examples.

Example 3.1. We consider the following system:

$$\begin{cases} y_1(x)=\frac{x}{2}+\int_0^1\left[\frac{x+1}{3}\cdot y_1(y_1(x))+\frac{2x+1}{6}y_2(y_1(x))\right]dx,\\ y_2(x)=\frac{x^2}{4}+\int_0^1\left[\frac{x+1}{7}\cdot y_1(y_2(x))+\frac{3x+1}{14}y_2(y_2(x))\right]dx,\end{cases} \tag{3.4}$$

and apply Theorem 2.3.

In this case we have $a=0, b=1$, $K\in C([0,1]^3,[0,1]^2)$, $K=(K_1(x,u_1,u_2);K_2(x,u_1,u_2))$,

$$K_1(x,u_1,u_2)=\frac{x+1}{3}\cdot u_1+\frac{2x+1}{6}\cdot u_2;\; K_2(x,u_1,u_2)=\frac{x+1}{7}\cdot u_1+\frac{3x+1}{14}\cdot u_2.$$

$f\in C([0,1],[0,1]^2)$, $f=(f_1(x),f_2(x))$, $f_1(x)=\dfrac{x}{2}$, $f_2(x)=\dfrac{x^2}{4}$.

The Lipschitz constants are $L_{11}=\frac{2}{3}$, $L_{12}=\frac{1}{2}$, $L_{21}=L_{22}=\frac{2}{7}$, and $L_K=\frac{1}{3}$, $L_f=\frac{1}{2}$, $M=\frac{2}{3}$. By condition (iv) we obtain $L\geq\frac{5}{6}$ and

$$Q=\begin{pmatrix}\dfrac{4+3L}{6} & \dfrac{1}{2}\\ \dfrac{2}{7} & \dfrac{2L+2}{7}\end{pmatrix}.$$

and its eigenvalues are

$$r_1 := \frac{1}{84}(33L + 40 + \sqrt{1599L^2 + 288L + 1264}),$$

$$r_2 := \frac{1}{84}(33L + 40 - \sqrt{1599L^2 + 288L + 1264})$$

The eigenvalues of the matrix Q, $r_1, r_2 \in (-1, 1)$ for $1 \leq L < 17,6394$ (the previous calculations were made by the Maple).

Under these assumptions the system (3.4) has a unique solution.

Example 3.2. We now illustrate Theorem 2.4 by means of the following system:

$$\begin{cases} y_1(x) = \frac{x}{3} + \int\limits_0^1 \left[\frac{x}{10} \cdot (y_1(x) + y_2(x)) + \frac{x+1}{20}(y_1(y_1(x)) + y_2(y_2(x)))\right] dx, \\ y_2(x) = \frac{x^2}{9} + \int\limits_0^1 \left[\frac{x+1}{9}(y_1(x) + y_2(x)) + \frac{3x+1}{18}(y_1(y_1(x)) + y_2(y_2(x)))\right] dx. \end{cases} \tag{3.5}$$

We have $a = 0, b = 1$, $K \in C([0,1]^5, [0,1]^2)$,
$K = (K_1(x, u_1, u_2, u_3, u_4), K_2(x, u_1, u_2, u_3, u_4))$;

$K_1(x, u_1, u_2, u_3, u_4) = \dfrac{x}{10} \cdot (u_1 + u_2) + \dfrac{x+1}{20} \cdot (u_3 + u_4)$

$K_2(x, u_1, u_2, u_3, u_4) = \dfrac{x+1}{9} \cdot (u_1 + u_2) + \dfrac{3x+1}{18} \cdot (u_3 + u_4)$

$f \in C([0,1], [0,1]^2)$, $f = (f_1(x), f_2(x))$, $f_1(x) = \frac{x}{3}$, $f_2(x) = \frac{x^2}{9}$.

The Lipschitz constants are $L_{11} = L_{12} = L_{13} = L_{14} = \dfrac{1}{10}$, $L_{21} = L_{22} = L_{23} = L_{24} = \dfrac{2}{9}$, $L_K = \dfrac{1}{10}$, $L_f = \dfrac{1}{3}$, and $M = \dfrac{4}{5}$. The matrix

$$Q = \begin{pmatrix} \dfrac{2+L}{10} & \dfrac{2+L}{10} \\ \dfrac{4+L}{9} & \dfrac{4+L}{9} \end{pmatrix},$$

has the eigenvalues

$$r_1 = 0, \ r_2 = \frac{19L + 58}{90}.$$

The matrix Q is convergent to zero if $\dfrac{2+L}{10} + \dfrac{4+L}{9} < 1$. By hypothesis (iv) of Theorem 2.4 we obtain $L \geq \dfrac{13}{30}$ and by the fact that $r_1, r_2 \in (-1, 1)$, we deduce that L must satisfy the double inequality

$$\frac{13}{30} \leq L \leq \frac{32}{19}.$$

By choosing $L = 1$, we conclude that the system (3.5) has a unique solution in $C_1([0,1], [0,1]^2)$ and this solution can be obtained by successive approximation, through the recurrence relation

$$(y_{1_{k+1}}, y_{2_{k+1}}) = H(y_{1_k}, y_{2_k}), \forall k \in \mathbb{N},$$

where H is the operator defined in Theorem 2.4.

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