CREAT. MATH. INFORM.
Volume **25** (2016), No. 1,
Pages 77 - 84

Online version at https://creative-mathematics.cunbm.utcluj.ro/
Print Edition: ISSN 1584 - 286X; Online Edition: ISSN 1843 - 441X
DOI: https://doi.org/10.37193/CMI.2016.01.10

Integral inequalities concerning polynomials with polar derivatives

ABDULLAH MIR and SHAHISTA BASHIR

ABSTRACT. Let $P(z)$ be a polynomial of degree n and for any complex number α, let

$$D_\alpha P(z) = nP(z) + (\alpha - z)P'(z)$$

denote the polar derivative of $P(z)$ with respect to a complex number α.

In this paper, we present an integral inequality for the polar derivative of a polynomial $P(z)$. Our result includes as special cases several interesting generalizations of some Zygmund type inequalities for polynomials.

1. INTRODUCTION

Let $\mathbb{P}_n$ be the class of polynomials $P(z) = \sum_{v=0}^{n} a_v z^v$ of degree at most n and $P'(z)$ its derivative. For a complex number α and for $P \in \mathbb{P}_n$, let

$$D_\alpha P(z) := nP(z) + (\alpha - z)P'(z).$$

Note that $D_\alpha P(z)$ is a polynomial of degree at most $n-1$. This is the so-called polar derivative of $P(z)$ with respect to point α ([12]). It generalizes the ordinary derivative in the following sense:

$$\lim_{\alpha \to \infty} \frac{D_\alpha P(z)}{\alpha} = P'(z).$$

Now corresponding to a given n^{th} degree polynomial $P(z)$, we construct a sequence of polar derivatives

$$\begin{aligned} D_{\alpha_1} P(z) &= nP(z) + (\alpha_1 - z)P'(z) \\ &\vdots \\ D_{\alpha_1} D_{\alpha_2} \dots D_{\alpha_k} P(z) &= (n-k+1) D_{\alpha_1} D_{\alpha_2} \dots D_{\alpha_{k-1}} P(z) \\ &\quad + (\alpha_k - z)(D_{\alpha_1} D_{\alpha_2} \dots D_{\alpha_{k-1}} P(z))', k = 2, 3, \dots, n. \end{aligned}$$

The points $\alpha_1, \alpha_2, \dots \alpha_k, k = 1, 2, 3, \dots, n$ may or may not be distinct. Like the k^{th} ordinary derivative $P^{(k)}(z)$ of $P(z)$, the k^{th} polar derivative $D_{\alpha_1} D_{\alpha_2} \dots D_{\alpha_k} P(z)$ of $P(z)$ is a polynomial of degree at most $n-k$. We shall write $P_k(z) = D_{\alpha_k} D_{\alpha_{k-1}} \dots D_{\alpha_1} P(z)$, so that

$$\begin{aligned} &P_t(z) = (n-t+1)P_{t-1}(z) + (\alpha_t - z)P'_{t-1}(z), t = 1, 2, \dots n, \\ &P_0(z) = P(z). \end{aligned}$$

Received: 12.11.2015. In revised form: 13.01.2016. Accepted: 01.02.2016
2010 *Mathematics Subject Classification.* 30A10, 30C10, 30C15.
Key words and phrases. *Polynomial, polar derivative, integral inequality.*
Corresponding author: Abdullah Mir; mabdullah_mir@yahoo.co.in

For $P \in \mathbb{P}_n$, we have

$$\max_{|z|=1} |P'(z)| \leq n \max_{|z|=1} |P(z)| \tag{1.1}$$

and for every $r \geq 1$,

$$\left\{ \int_0^{2\pi} \left|P'(e^{i\theta})\right|^r d\theta \right\}^{\frac{1}{r}} \leq n \left\{ \int_0^{2\pi} \left|P(e^{i\theta})\right|^r d\theta \right\}^{\frac{1}{r}}. \tag{1.2}$$

The inequality (1.1) is a classical result of Bernstein [9] whereas the inequality (1.2) is due to Zygmund [17] who proved it for all trigonometric polynomials of degree n and not only for those of the form $P(e^{i\theta})$. Arestov [1] proved that (1.2) remains true for $0 < r < 1$ as well. If we let $r \to \infty$ in (1.2) we get (1.1).

The above two inequalities (1.1) and (1.2) can be sharpened if we restrict ourselves to the class of polynomials having no zeros in $|z| < 1$. In fact, if $P(z) \neq 0$ in $|z| < 1$, then (1.1) and (1.2) can be respectively replaced by

$$\max_{|z|=1} |P'(z)| \leq \frac{n}{2} \max_{|z|=1} |P(z)| \tag{1.3}$$

and

$$\left\{ \int_0^{2\pi} \left|P'(e^{i\theta})\right|^r d\theta \right\}^{\frac{1}{r}} \leq nB_r \left\{ \int_0^{2\pi} \left|P(e^{i\theta})\right|^r d\theta \right\}^{\frac{1}{r}}, \tag{1.4}$$

where

$$B_r = \left\{ \frac{1}{2\pi} \int_0^{2\pi} \left|1 + e^{i\alpha}\right|^r d\alpha \right\}^{\frac{-1}{r}}.$$

The inequality (1.3) was conjectured by Erdös and later verified by Lax [7], whereas (1.4) was proved by De-Bruijn [6] for $r \geq 1$. Further, Rahman and Schmeisser [13] have shown that (1.4) holds for $0 < r < 1$ as well. If we let $r \to \infty$ in inequality (1.4), we get (1.3).

Aziz was among the first to extend some of the above inequalities by replacing the derivative with the polar derivatives of polynomials. Recently several papers were devoted by different authors to the same topic (for example see [8], [11], [15] and [16]). In fact in 1988, Aziz [2] extended (1.3) to the polar derivative of a polynomial and proved that if $P \in \mathbb{P}_n$ and $P(z) \neq 0$ in $|z| < 1$, then for every complex number α with $|\alpha| \geq 1$,

$$\max_{|z|=1} |D_\alpha P(z)| \leq \frac{n}{2}(|\alpha| + 1) \max_{|z|=1} |P(z)|. \tag{1.5}$$

As an L_r analogue of (1.5) and a generalization of (1.4) Aziz and Rather [4] proved that, if $P \in \mathbb{P}_n$ and $P(z) \neq 0$ in $|z| < 1$, then for every complex number α with $|\alpha| \geq 1$ and $r \geq 1$,

$$\left\{ \int_0^{2\pi} \left|D_\alpha P(e^{i\theta})\right|^r d\theta \right\}^{\frac{1}{r}} \leq n(|\alpha| + 1)C_r \left\{ \int_0^{2\pi} \left|P(e^{i\theta})\right|^r d\theta \right\}^{\frac{1}{r}}, \tag{1.6}$$

where

$$C_r = \left\{ \frac{1}{2\pi} \int_0^{2\pi} \left|1 + e^{i\gamma}\right|^r d\gamma \right\}^{\frac{-1}{r}}. \tag{1.7}$$

Dividing both sides of (1.5) and (1.6) by $|\alpha|$ and letting $\alpha \to \infty$ yields (1.3) and (1.4) respectively.

In this paper, we shall prove the following more general result which as special case gives interesting generalizations of (1.5) and (1.6).

2. Main results

Theorem 2.1. *If $P \in \mathbb{P}_n$ and $P(z) \neq 0$ in $|z| < 1$, then for all β, α_i with $|\beta| \leq 1, |\alpha_i| \geq 1$ for $1 \leq i \leq t, t \leq n-1$ and $r \geq 1$,*

$$\left\{ \int_0^{2\pi} \left| e^{it\theta} P_t(e^{i\theta}) + \beta \frac{n_t A_{\alpha_t}}{2^t} P(e^{i\theta}) \right|^r d\theta \right\}^{\frac{1}{r}} \leq n_t \left\{ B_{\alpha_t} + \frac{|\beta| A_{\alpha_t}}{2^{t-1}} \right\} C_r \times \left\{ \int_0^{2\pi} \left| P(e^{i\theta}) \right|^r d\theta \right\}^{\frac{1}{r}}, \tag{2.8}$$

where

$$\begin{aligned} A_{\alpha_t} &= (|\alpha_1| - 1)(|\alpha_2| - 1) \ldots (|\alpha_t| - 1), \\ B_{\alpha_t} &= (|\alpha_1| + 1)(|\alpha_2| + 1) \ldots (|\alpha_t| + 1), \\ n_t &= n(n-1) \ldots (n-t+1), \end{aligned} \tag{2.9}$$

and C_r is defined by (1.7).

If we take $\alpha_1 = \alpha_2 = \ldots = \alpha_t = \alpha$, then divide both sides of (2.8) by $|\alpha|^t$ and let $|\alpha| \to \infty$, we get the following result.

Corollary 2.1. *If $P \in \mathbb{P}_n$ and $P(z) \neq 0$ in $|z| < 1$, then for all β with $|\beta| \leq 1, 1 \leq t \leq n-1$ and $r \geq 1$,*

$$\left\{ \int_0^{2\pi} \left| e^{it\theta} P^{(t)}(e^{i\theta}) + \beta \frac{n_t}{2^t} P(e^{i\theta}) \right|^r d\theta \right\}^{\frac{1}{r}} \leq n_t \left\{ 1 + \frac{|\beta|}{2^{t-1}} \right\} C_r \left\{ \int_0^{2\pi} \left| P(e^{i\theta}) \right|^r d\theta \right\}^{\frac{1}{r}},$$

where n_t is defined in (2.9) *and C_r in* (1.7).

If we put $t = 1$ in (2.8) *, we get the following result.*

Corollary 2.2. *If $P \in \mathbb{P}_n$ and $P(z) \neq 0$ in $|z| < 1$, then for every complex numbers α, β with $|\alpha| \geq 1, |\beta| \leq 1$ and $r \geq 1$,*

$$\left\{ \int_0^{2\pi} \left| e^{i\theta} D_\alpha P(e^{i\theta}) + n\beta \frac{(|\alpha| - 1)}{2} P(e^{i\theta}) \right|^r d\theta \right\}^{\frac{1}{r}} \leq n \left\{ (|\alpha| + 1) + |\beta|(|\alpha| - 1) \right\} C_r \times \left\{ \int_0^{2\pi} \left| P(e^{i\theta}) \right|^r d\theta \right\}^{\frac{1}{r}}, \tag{2.10}$$

where C_r is defined in (1.7).

Remark 2.1. *For $\beta = 0,$* (2.10) *reduces to* (1.6). *Further, making $r \to \infty$ in* (2.10), *we get for $|\alpha| \geq 1$ and $|\beta| \leq 1$,*

$$\max_{|z|=1} \left| z D_\alpha P(z) + n\beta \frac{(|\alpha| - 1)}{2} P(z) \right| \leq \frac{n}{2} \left\{ (|\alpha| + 1) + |\beta|(|\alpha| - 1) \right\} \max_{|z|=1} |P(z)|. \tag{2.11}$$

If we take $\beta = 0$ in (2.11), *we get* (1.5).

3. Lemmas

For the proof of the Theorem 2.1, we need the following Lemmas:

Lemma 3.1. *Let $Q(z)$ be a polynomial of degree n having all its zeros in $|z| \leq 1$ and $P(z)$ be a polynomial of degree at most n. If $|P(z)| \leq |Q(z)|$ for $|z| = 1$, then for all β, α_i with $|\beta| \leq 1, |\alpha_i| \geq 1$, $i = 1, 2, \dots, t$ and $t \leq n-1$,*

$$\left|z^t P_t(z) + \beta \frac{n_t A_{\alpha_t}}{2^t} P(z)\right| \leq \left|z^t Q_t(z) + \beta \frac{n_t A_{\alpha_t}}{2^t} Q(z)\right| \text{ for } |z| \geq 1,$$

where n_t and A_{α_t} are defined in (2.9).

The above lemma is due to Bidkham and Soleiman Mezerji [5].

Lemma 3.2. *If $P(z)$ is a polynomial of degree n, then for every complex α and $r > 0$,*

$$\left\{\int_0^{2\pi} \left|D_\alpha P(e^{i\theta})\right|^r d\theta\right\}^{\frac{1}{r}} \leq n(|\alpha| + 1)\left\{\int_0^{2\pi} \left|P(e^{i\theta})\right|^r d\theta\right\}^{\frac{1}{r}}.$$

The above Lemma is due to Rather [14].

Lemma 3.3. *If $P \in \mathbb{P}_n$ and $Q(z) = z^n \overline{P(\frac{1}{\bar{z}})}$, then for every $r > 0$ and γ real,*

$$\int_0^{2\pi} \int_0^{2\pi} \left|P'(e^{i\theta}) + e^{i\gamma} Q'(e^{i\theta})\right|^r d\theta d\gamma \leq 2\pi n^r \int_0^{2\pi} \left|P(e^{i\theta})\right|^r d\theta.$$

The above Lemma is due to Aziz and Rather [3].

4. Proof of theorem

Proof of Theorem 2.1. Since $P \in \mathbb{P}_n$ and $P(z) \neq 0$ in $|z| < 1$, then the polynomial $Q(z) = z^n \overline{P(\frac{1}{\bar{z}})} \in \mathbb{P}_n$ and $Q(z) \neq 0$ in $|z| \geq 1$.
By Lemma 3.1, we have for all β, α_i with $|\beta| \leq 1, |\alpha_i| \geq 1, 1 \leq i \leq t, t \leq n-1$

$$\left|z^t P_t(z) + \beta \frac{n_t A_{\alpha_t}}{2^t} P(z)\right| \leq \left|z^t Q_t(z) + \beta \frac{n_t A_{\alpha_t}}{2^t} Q(z)\right| \text{ for } |z| = 1, \tag{4.12}$$

where n_t and A_{α_t} are defined in (2.9).
Now for every real γ and $s \geq 1$, it can be easily verified that

$$\left|s + e^{i\gamma}\right| \geq \left|1 + e^{i\gamma}\right|,$$

which implies for each $r \geq 1$,

$$\int_0^{2\pi} \left|s + e^{i\gamma}\right|^r d\gamma \geq \int_0^{2\pi} \left|1 + e^{i\gamma}\right|^r d\gamma.$$

If $e^{it\theta} P_t(e^{i\theta}) + \beta \frac{n_t A_{\alpha_t}}{2^t} P(e^{i\theta}) \neq 0$, we take

$$s = \left|\frac{e^{it\theta} Q_t(e^{i\theta}) + \beta \frac{n_t A_{\alpha_t}}{2^t} Q(e^{i\theta})}{e^{it\theta} P_t(e^{i\theta}) + \beta \frac{n_t A_{\alpha_t}}{2^t} P(e^{i\theta})}\right|,$$

then by (4.12), $s \geq 1$ and we get

$$\int_0^{2\pi} \left|\left\{e^{it\theta}P_t(e^{i\theta}) + \beta\frac{n_t A_{\alpha_t}}{2^t}P(e^{i\theta})\right\} + e^{i\gamma}\left\{e^{it\theta}Q_t(e^{i\theta}) + \beta\frac{n_t A_{\alpha_t}}{2^t}Q(e^{i\theta})\right\}\right|^r d\gamma$$

$$= \left|e^{it\theta}P_t(e^{i\theta}) + \beta\frac{n_t A_{\alpha_t}}{2^t}P(e^{i\theta})\right|^r \int_0^{2\pi} \left|1 + e^{i\gamma}\left\{\frac{e^{it\theta}Q_t(e^{i\theta}) + \beta\frac{n_t A_{\alpha_t}}{2^t}Q(e^{i\theta})}{e^{it\theta}P_t(e^{i\theta}) + \beta\frac{n_t A_{\alpha_t}}{2^t}P(e^{i\theta})}\right\}\right|^r d\gamma$$

$$= \left|e^{it\theta}P_t(e^{i\theta}) + \beta\frac{n_t A_{\alpha_t}}{2^t}P(e^{i\theta})\right|^r \int_0^{2\pi} \left|1 + e^{i\gamma}\left|\frac{e^{it\theta}Q_t(e^{i\theta}) + \beta\frac{n_t A_{\alpha_t}}{2^t}Q(e^{i\theta})}{e^{it\theta}P_t(e^{i\theta}) + \beta\frac{n_t A_{\alpha_t}}{2^t}P(e^{i\theta})}\right|\right|^r d\gamma$$

$$= \left|e^{it\theta}P_t(e^{i\theta}) + \beta\frac{n_t A_{\alpha_t}}{2^t}P(e^{i\theta})\right|^r \int_0^{2\pi} \left|\left|\frac{e^{it\theta}Q_t(e^{i\theta}) + \beta\frac{n_t A_{\alpha_t}}{2^t}Q(e^{i\theta})}{e^{it\theta}P_t(e^{i\theta}) + \beta\frac{n_t A_{\alpha_t}}{2^t}P(e^{i\theta})}\right| + e^{i\gamma}\right|^r d\gamma$$

$$\geq \left|e^{it\theta}P_t(e^{i\theta}) + \beta\frac{n_t A_{\alpha_t}}{2^t}P(e^{i\theta})\right|^r \int_0^{2\pi} \left|1 + e^{i\gamma}\right|^r d\gamma. \tag{4.13}$$

This inequality is trivially true if

$$e^{it\theta}P_t(e^{i\theta}) + \beta\frac{n_t A_{\alpha_t}}{2^t}P(e^{i\theta}) = 0.$$

Integrating both sides of (4.13) with respect to θ from 0 to 2π, we get

$$\int_0^{2\pi}\int_0^{2\pi} \left|\left\{e^{it\theta}P_t(e^{i\theta}) + \beta\frac{n_t A_{\alpha_t}}{2^t}P(e^{i\theta})\right\} + e^{i\gamma}\left\{e^{it\theta}Q_t(e^{i\theta}) + \beta\frac{n_t A_{\alpha_t}}{2^t}Q(e^{i\theta})\right\}\right|^r d\theta d\gamma$$
$$\geq \int_0^{2\pi} \left|e^{it\theta}P_t(e^{i\theta}) + \beta\frac{n_t A_{\alpha_t}}{2^t}P(e^{i\theta})\right|^r d\theta \int_0^{2\pi} \left|1 + e^{i\gamma}\right|^r d\gamma. \tag{4.14}$$

As $Q(z) = z^n\overline{P(\frac{1}{\bar{z}})}$, therefore $P(z) = z^n\overline{Q(\frac{1}{\bar{z}})}$ and it can be easily verified that for $0 \leq \theta < 2\pi$,

$$nP(e^{i\theta}) - e^{i\theta}P'(e^{i\theta}) = e^{i(n-1)\theta}\overline{Q'(e^{i\theta})}$$

and

$$nQ(e^{i\theta}) - e^{i\theta}Q'(e^{i\theta}) = e^{i(n-1)\theta}\overline{P'(e^{i\theta})}.$$

Hence

$$nP(e^{i\theta}) + e^{i\gamma}nQ(e^{i\theta}) = e^{i\theta}P'(e^{i\theta}) + e^{i(n-1)\theta}\overline{Q'(e^{i\theta})} + e^{i\gamma}\Big(e^{i\theta}Q'(e^{i\theta}) + e^{i(n-1)\theta}\overline{P'(e^{i\theta})}\Big)$$
$$= e^{i\theta}\Big(P'(e^{i\theta}) + e^{i\gamma}Q'(e^{i\theta})\Big) + e^{i(n-1)\theta}\Big(\overline{Q'(e^{i\theta})} + e^{i\gamma}\overline{P'(e^{i\theta})}\Big),$$

which gives

$$n\Big|P(e^{i\theta})+e^{i\gamma}Q(e^{i\theta})\Big| \leq \Big|P'(e^{i\theta})+e^{i\gamma}Q'(e^{i\theta})\Big|+\Big|\overline{Q'(e^{i\theta})}+e^{i\gamma}\overline{P'(e^{i\theta})}\Big|$$
$$=2\Big|P'(e^{i\theta})+e^{i\gamma}Q'(e^{i\theta})\Big|. \tag{4.15}$$

Also, we have

$$\begin{aligned}\Big|D_\alpha P(e^{i\theta})+e^{i\gamma}D_\alpha Q(e^{i\theta})\Big| &= \Big|nP(e^{i\theta})+(\alpha-e^{i\theta})P'(e^{i\theta})+e^{i\gamma}\Big(nQ(e^{i\theta})+(\alpha-e^{i\theta})Q'(e^{i\theta})\Big)\Big|\\ &= \Big|\Big(nP(e^{i\theta})-e^{i\theta}P'(e^{i\theta})\Big)+e^{i\gamma}\Big(nQ(e^{i\theta})-e^{i\theta}Q'(e^{i\theta})\Big)\\ &\qquad +\alpha\Big(P'(e^{i\theta})+e^{i\gamma}Q'(e^{i\theta})\Big)\Big|\\ &= \Big|\Big(\overline{Q'(e^{i\theta})}+e^{i\gamma}\overline{P'(e^{i\theta})}\Big)e^{i(n-1)\theta}+\alpha\Big(P'(e^{i\theta})+e^{i\gamma}Q'(e^{i\theta})\Big)\Big|\\ &\leq \Big|\overline{P'(e^{i\theta})+e^{i\gamma}Q'(e^{i\theta})}\Big|+|\alpha|\Big|P'(e^{i\theta})+e^{i\gamma}Q'(e^{i\theta})\Big|\\ &= \big(|\alpha|+1\big)\Big|P'(e^{i\theta})+e^{i\gamma}Q'(e^{i\theta})\Big|. \end{aligned} \tag{4.16}$$

Further, since $F(z)=P(z)+e^{i\gamma}Q(z)$ is a polynomial of degree n so that $F_t(z)=P_t(z)+e^{i\gamma}Q_t(z)$ is a polynomial of degree $n-t, t\leq n-1$, we have by the repeated application of Lemma 3.2, for $r\geq 1$,

$$\int_0^{2\pi}\Big|D_{\alpha_1}D_{\alpha_2}\dots D_{\alpha_t}F(e^{i\theta})\Big|^r d\theta \leq (n-t+1)^r(|\alpha_t|+1)^r\int_0^{2\pi}\Big|D_{\alpha_1}D_{\alpha_2}\dots D_{\alpha_{t-1}}F(e^{i\theta})\Big|^r d\theta.$$

Equivalently,

$$\begin{aligned}&\int_0^{2\pi}\Big|D_{\alpha_1}D_{\alpha_2}\dots D_{\alpha_t}P(e^{i\theta})+e^{i\gamma}D_{\alpha_1}D_{\alpha_2}\dots D_{\alpha_t}Q(e^{i\theta})\Big|^r d\theta\\ &\leq (n-t+1)^r(|\alpha_t|+1)^r\int_0^{2\pi}\Big|D_{\alpha_1}D_{\alpha_2}\dots D_{\alpha_{t-1}}P(e^{i\theta})+e^{i\gamma}D_{\alpha_1}D_{\alpha_2}\dots D_{\alpha_{t-1}}Q(e^{i\theta})\Big|^r d\theta\\ &\vdots\\ &\leq (n-t+1)^r(n-t+2)^r\dots(n-1)^r(|\alpha_t|+1)^r(|\alpha_{t-1}|+1)^r\dots(|\alpha_2|+1)^r\\ &\qquad\times\int_0^{2\pi}\Big|D_{\alpha_1}P(e^{i\theta})+e^{i\gamma}D_{\alpha_1}Q(e^{i\theta})\Big|^r d\theta. \end{aligned} \tag{4.17}$$

Integrating both sides of (4.17) with respect to γ from 0 to 2π, we get with the help of Lemma 3.3 and inequality (4.16) that for each $r\geq 1$,

$$\begin{aligned}\int_0^{2\pi}\int_0^{2\pi}\Big|P_t(e^{i\theta})+e^{i\gamma}Q_t(e^{i\theta})\Big|^r d\theta d\gamma &\leq 2\pi(n-t+1)^r(n-t+2)^r\dots(n-1)^r\\ &\times n^r(|\alpha_t|+1)^r(|\alpha_{t-1}|+1)^r\dots(|\alpha_2|+1)^r(|\alpha_1|+1)^r\int_0^{2\pi}\Big|P(e^{i\theta})\Big|^r d\theta. \end{aligned} \tag{4.18}$$

From (4.14), we have for $r \geq 1$, it follows by Minkowski's inequality that

$$\left\{\int_0^{2\pi} |1+e^{i\gamma}|^r d\gamma\right\}^{\frac{1}{r}} \left\{\int_0^{2\pi} \left|e^{it\theta}P_t(e^{i\theta}) + \beta\frac{n_t A_{\alpha_t}}{2^t}P(e^{i\theta})\right|^r d\theta\right\}^{\frac{1}{r}}$$

$$\leq \left\{\int_0^{2\pi}\int_0^{2\pi} \left|e^{it\theta}\Big(P_t(e^{i\theta}) + e^{i\gamma}Q_t(e^{i\theta})\Big) + \beta\frac{n_t A_{\alpha_t}}{2^t}\Big(P(e^{i\theta}) + e^{i\gamma}Q(e^{i\theta})\Big)\right|^r d\theta d\gamma\right\}^{\frac{1}{r}}$$

$$\leq \left\{\int_0^{2\pi}\int_0^{2\pi} \left|P_t(e^{i\theta}) + e^{i\gamma}Q_t(e^{i\theta})\right|^r d\theta d\gamma\right\}^{\frac{1}{r}} + |\beta|\frac{n_t A_{\alpha_t}}{2^t}\left\{\int_0^{2\pi}\int_0^{2\pi} \left|P(e^{i\theta}) + e^{i\gamma}Q(e^{i\theta})\right|^r d\theta d\gamma\right\}^{\frac{1}{r}},$$

which gives on using (4.15),(4.18) and Lemma 3.3 that for every β with $|\beta| \leq 1$, $r \geq 1$ and γ real,

$$\left\{\int_0^{2\pi} |1+e^{i\gamma}|^r d\gamma\right\}^{\frac{1}{r}} \left\{\int_0^{2\pi} \left|e^{it\theta}P_t(e^{i\theta}) + \beta\frac{n_t A_{\alpha_t}}{2^t}P(e^{i\theta})\right|^r d\theta\right\}^{\frac{1}{r}}$$

$$\leq (2\pi)^{\frac{1}{r}} n_t \left\{B_{\alpha_t} + |\beta|\frac{A_{\alpha_t}}{2^{t-1}}\right\}\left\{\int_0^{2\pi} \left|P(e^{i\theta})\right|^r d\theta\right\}^{\frac{1}{r}}, \tag{4.19}$$

which is equivalent to (2.8). This completes the proof of the Theorem 2.1. □

Remark 4.2. Since inequality (1.6) holds for $r \geq 1$ and it has been shown in [10] that the same inequality holds for $0 < r < 1$ as well. The authors have a feeling that the results of the present paper may be extended to $0 < r \leq \infty$.

Acknowledgements. The work is sponsored by UGC, Govt. of India under the Major Research Project Scheme vide no. MRP-MAJOR-MATH-2013-29143.

The authors are very grateful to the referees for their valuable suggestions regarding the paper.

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UNIVERSITY OF KASHMIR
DEPARTMENT OF MATHEMATICS
HAZRATBAL, 190006, SRINAGAR, INDIA
Email address: mabdullah_mir@yahoo.co.in
Email address: pandithsha@gmail.com