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Strong convergence results for nonlinear mappings in real Banach spaces

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ABSTRACT. Let X be a real Banach space, K be a nonempty closed convex subset of X, $T : K \to K$ be a nearly uniformly L-Lipschitzian mapping with sequence $\{a_n\}$. Let $k_n \subset [1, \infty)$ and ϵ_n be sequences with $\lim_{n\to\infty} k_n = 1,\ \lim_{n\to\infty} \epsilon_n = 0$ and $F(T) = \{\rho \in K : T\rho = \rho\} \neq \emptyset$. Let $\{\alpha_n\}_{n\geq 0}$ be real sequence in $[0, 1]$ satisfying the following conditions: (i)$\sum_{n\geq 0} \alpha_n = \infty$ (ii) $\lim_{n\to\infty} \alpha_n = 0$. For arbitrary $x_0 \in K$, let $\{x_n\}_{n\geq 0}$ be iteratively defined by $x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T^n x_n, \quad n \geq 0$. If there exists a strictly increasing function $\Phi : [0, \infty) \to [0, \infty)$ with $\Phi(0) = 0$ such that

$$< T^n x - T^n \rho, j(x - \rho) >\leq k_n\|x - \rho\|^2 - \Phi(\|x - \rho\|) + \epsilon_n$$

for all $x \in K$, then, $\{x_n\}_{n\geq 0}$ converges strongly to $\rho \in F(T)$.

It is also proved that the sequence of iteration $\{x_n\}$ defined by

$$x_{n+1} = (1 - b_n - d_n)x_n + b_n T^n x_n + d_n w_n, n \geq 0,$$

where $\{w_n\}_{n\geq 0}$ is a bounded sequence in K and $\{b_n\}_{n\geq 0},\ \{d_n\}_{n\geq 0}$ are sequences in [0,1] satisfying appropriate conditions, converges strongly to a fixed point of T.

1. INTRODUCTION

We denote by J the normalized duality mapping from X into 2^{X^*} by

$$J(x) = \{f \in X^* : \langle x, f\rangle = \|x\|^2 = \|f\|^2\},$$

where X^* denotes the dual space of real Banach space X and $\langle ., .\rangle$ denotes the generalized duality pairing between elements of X and X^*. We first recall and define some concepts as follows (see [1-5]):
Let K be a nonempty subset of real Banach space X. In 1972, Goebel and Kirk [5] introduced the class of asymptotically nonexpansive mappings as follows.

Definition 1.1. A mapping T is called asymptotically nonexpansive if for each $x, y \in K$

$$\|T^n x - T^n y\| \leq k\|x - y\| \leq k_n\|x - y\|^2, \forall n \geq 1,$$

where $(k_n) \subset [1, \infty)$ with $lim_{n\to\infty} k_n = 1$.
Schu [17], introduced the concept of asymptotically pseudocontractive mappings and proved the correlation results.

Definition 1.2. A mapping T is called asymptotically pseudocontractive with the sequence $(k_n) \subset [1, \infty)$ if and only if $lim_{n\to\infty} k_n = 1$, and for all $n \in N$ and all $x, y \in K$, there exists $j(x - y) \in J(x - y)$ such that

$$< T^n x - T^n y, j(x - y) >\leq k_n\|x - y\|^2, \forall n \geq 1.$$

It is easy to see that every asymptotically nonexpansive mapping is asymptotically pseudocontractive. However, the converse is not true in general (see [2,16]).

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In recent years, many authors have given much attention to iterative methods for approximating fixed points of Lipschitz asymptotically type of some nonlinear mappings (see [1-4, 7, 13-18]).
In [1], Chang extended the results of Schu [17] to a real uniformly smooth Banach space and proved the following theorem:

Theorem 1.1. ([1]). *Let E be a real uniformly smooth Banach space, K be a nonempty bounded closed convex subset of E, $T : K \to K$ be an asymptotically pseudocontractive mapping with a sequence $k_n \subset [1, \infty)$ with $k_n \to 1$ and $F(T) \neq \emptyset$, where $F(T)$ is the set of fixed points of T in K. Let $\{\alpha_n\}_{n=0}^{\infty}$ be a sequence in $[0, 1]$ satisfying the following conditions: (i) $\lim_{n\to\infty} \alpha_n = 0$ (ii)$\sum_{n=0}^{\infty} \alpha_n = \infty$. For any $x_0 \in K$, let $\{x_n\}_{n=0}^{\infty}$ be the iterative sequence defined by*

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T^n x_n, \quad n \geq 0.$$

If there exists a strictly increasing function $\Phi : [0, \infty) \to [0, \infty)$ with $\Phi(0) = 0$ such that

$$< T^n x_n - \rho, j(x_n - \rho) > \leq k_n \|x_n - \rho\|^2 - \Phi(\|x_n - \rho\|), \quad n \geq 0 \tag{1.1}$$

where $\rho \in F(T)$ is some fixed point of T in K, then $x_n \to \rho$ as $n \to \infty$.
The iteration process of Theorem 1.1 is a modification of the well-known Mann iteration process (see, e.g., [9]).

Remark 1.1. Theorem 1.1, as stated is a modification of Theorem 2.1 of Chang [1] who actually included error terms in his iteration process.
Ofoedu [13] used the modified Mann iteration process (1.1) introduced by Schu [17] to obtain a strong convergence theorem for uniformly Lipschitzian asymptotically pseudocontractive mapping in real Banach space setting. He proved the following theorem:

Theorem 1.2. ([13]). *Let E be a real Banach space, K be a nonempty closed convex subset of E, $T : K \to K$, be a uniformly L-Lipschitzian asymptotically mappings with a sequence $k_n \subset [1, \infty)$, $k_n \to 1$ such that $\rho \in F(T)$, where $F(T)$ is the set of fixed points of T in K. Let $\{\alpha_n\}_{n=0}^{\infty}$ be a sequence in $[0, 1]$ satisfying the following conditions: (i) $\sum_{n=0}^{\infty} \alpha_n = \infty$ (ii) $\sum_{n=0}^{\infty} \alpha_n^2 < \infty$ (iii) $\sum_{n=0}^{\infty} \beta_n < \infty$ (iv) $\sum_{n=0}^{\infty} \alpha_n(k_n - 1) < \infty$. For any $x_0 \in K$, let $\{x_n\}_{n=0}^{\infty}$ be the iterative sequence defined by (1.1).*
If there exists a strictly increasing function $\Phi : [0, \infty) \to [0, \infty)$ with $\Phi(0) = 0$ such that

$$< T^n x_n - \rho, j(x_n - \rho) > \leq k_n \|x_n - \rho\|^2 - \Phi(\|x_n - \rho\|)$$

for all $x \in K$, then $\{x_n\}_{n=0}^{\infty}$ converges strongly to ρ.
Obviously, this result extends Theorem 1.2 of Chang [1] from a real uniformly smooth Banach space to an arbitrary real Banach space and removes the boundedness condition imposed on K.
It is important to note the following remark:

Remark 1.2. (Remark 2, p. 567, of Rafiq [15]). One can see that, with $\sum_{n=0}^{\infty} \alpha_n = \infty$ the conditions $\sum_{n=0}^{\infty} \alpha_n^2 < \infty$ and $\sum_{n=0}^{\infty} \alpha_n(k_n - 1) < \infty$ are not always true. Let us take $\alpha_n = \frac{1}{\sqrt{n}}$ and $k_n = 1 + \frac{1}{\sqrt{n}}$, then obviously $\sum_{n=0}^{\infty} \alpha_n = \infty$, but $\sum_{n=0}^{\infty} \alpha_n^2 = \infty = \sum_{n=0}^{\infty} \alpha_n(k_n - 1)$.
Hence the results of Ofoedu [13] and Chang et al. [3] need to be improved.
Sahu [18] recently introduced the following new class of nonlinear map which is more general than the class of generalized Lipschitzian mappings and the class of uniformly L-Lipschitzian mappings. In fact, he introduced the following class of nearly Lipschitzian:
Let K be a subset of a normed space X and let $\{a_n\}_{n\geq 0}$ be a sequence in $[0, \infty)$ such that $lim_{n\to\infty} a_n = 0$.
A mapping $T : K \to K$ is called nearly Lipschitzian with respect to $\{a_n\}$ if for each $n \in N$, there exists a constant $k_n \geq 0$ such that

$$\|T^n x - T^n y\| \leq k_n(\|x - y\| + a_n), \quad \forall \quad x, y \in K. \tag{1.2}$$

Define

$$\mu(T^n) = sup\{\frac{||T^n x - T^n y||}{||x-y|| + a_n} : x, y \in K, x \neq y\}.$$

Observe that for any sequence $\{k_n\}_n \geq 1$ satisfying (1.1) $\mu(T^n) \leq k_n \ \forall n \in N$ and that

$$\|T^n x - T^n y\| \leq \mu(T^n)(\|x-y\| + a_n), \quad \forall \quad x, y \in K$$

$\mu(T^n)$ is called the nearly Lipschitz constant of the mapping T. A nearly Lipschitzian mapping T is said to be
(i) nearly contraction if $\mu(T^n) < 1$ for all $n \in N$;
(ii) nearly nonexpansive if $\mu(T^n) = 1$ for all $n \in N$;
(iii) nearly asymptotically nonexpansive if $\mu(T^n) \geq 1$ for all $n \in N$ and $\lim_{n\to\infty} \mu(T^n) = 1$;
(iv) nearly uniformly $L-$ Lipschitzian if $\mu(T^n) \leq L$ for all $n \in N$;
(v) nearly uniformly $k-$ contraction if $\mu(T^n) \leq k < 1$ for all $n \in N$.
A nearly Lipschitzian mapping T with sequence $\{a_n\}$ is said to be nearly uniformly $L-$ Lipschitzian if $k_n = L$ for all $n \in N$.
Observe that the class of nearly uniformly $L-$ Lipschitzian mapping is more general than the class of uniformly $L-$ Lipschitzian mappings.

Example 1.1. (see Sahu [18]). Let $E = R, \ K = [0, 1]$.
Define $T : K \to K$ by

$$Tx = \begin{cases} 1/2, & x \in [0, 1/2), \\ 0, & x \in (1/2, 1]. \end{cases}$$

It is obvious that T is not continuous, and thus not Lipschitz. However, T is nearly nonexpansive. In fact, for a real sequence $\{a_n\}_n \geq 1$ with $a_1 = \frac{1}{2}$ and $a_n \to 0$ as $n \to \infty$, we have

$$||Tx - Ty|| \leq ||x - y|| + a_1, \ \forall x, y \in K$$

and

$$||T^n x - T^n y|| \leq ||x - y|| + a_n, \ \forall x, y \in K, \ n \geq 2.$$

This is because $T^n x = \frac{1}{2}, \forall x \in [0, 1], \ n \geq 2$.

Remark 1.3. The class of nearly uniformly $L-$ Lipschitzian is not necessarily continuous. In 1995, Liu [8] introduced what he called the Ishikawa and Mann iteration processes with errors. In [19], Xu objected the definition given by Liu [8] on the ground that the conditions $||u_n|| < \infty$ and $||v_n|| < \infty$ (as imposed by Liu) are not compatible with the randomness of the occurrence of errors (since they imply in particular, that the errors tend to zero as n tends to infinity). He then modified the definitions of Liu by assuming that the sequences $\{u_n\}$ and $\{v_n\}$ (in his own definitions) are only bounded. It is easy to observe that the most reasonable error term is the one introduced by Xu [19]. In particular, if T is self-mapping of a convex bounded set, then the boundedness requirement for the error term is automatically satisfied.
Inspired and motivated by the work of Xu [19] and the iteration above, we discuss the following Modified Mann iteration with errors:

$$x_{n+1} = (1 - b_n - d_n)x_n + b_n T^n x_n + d_n w_n, \ n \geq 0, \tag{1.3}$$

where $\{b_n\}_{n=0}^{\infty}$ and $\{d_n\}_{n=0}^{\infty}$ are sequences in $[0, 1]$ with $b_n + d_n \leq 1$ and $\{w_n\}$ is a bounded sequence of K.
We observe that the iteration process (1.3) is well defined and is a generalization of the

modified Mann iteration (1.1). This is evident by specialising some of the parameters. Indeed, when $d_n = 0$ and $\alpha_n = b_n$, then (1.3) reduces to (1.1) which is used by several authors working in this area of research.

It is our purpose in this paper to significantly generalise Theorem 1.3. of Ofoedu [13] which itself is a generalization of several results to more general class of nonlinear mappings. A related result involving bounded sequence of error terms is also included.

Lemma 1.1. ([1, 4]). *Let X be real Banach Space and $J : X \to 2^{X^*}$ be the normalized duality mapping. Then, for any $x, y \in X$*

$$\|x+y\|^2 \le \|x\|^2 + 2 < y, j(x+y) >, \forall j(x+y) \in J(x+y).$$

Lemma 1.2. ([10]). *Let $\Phi : [0,\infty) \to [0,\infty)$ be an increasing function with $\Phi(x) = 0 \Leftrightarrow x = 0$ and let $\{b_n\}_{n=0}^{\infty}$ be a positive real sequence satisfying*

$$\sum_{n=0}^{\infty} b_n = +\infty \quad \textit{and} \quad \lim_{n\to\infty} b_n = 0.$$

Suppose that $\{a_n\}_{n=0}^{\infty}$ is a nonnegative real sequence. If there exists an integer $N_0 > 0$ satisfying

$$a_{n+1}^2 < a_n^2 + o(b_n) - b_n\Phi(a_{n+1}), \quad \forall n \ge N_0$$

where $\lim_{n\to\infty} \frac{o(b_n)}{b_n} = 0$, then $\lim_{n\to\infty} a_n = 0$.

2. Main results

Theorem 2.3. *Let X be a real Banach space, K be a nonempty closed convex subset of X, $T : K \to K$ be a nearly uniformly L-Lipschitzian mapping with sequence $\{a_n\}$. Let $k_n \subset [1,\infty)$ and ϵ_n be sequences with $\lim_{n\to\infty} k_n = 1$, $\lim_{n\to\infty} \epsilon_n = 0$ and $F(T) = \{\rho \in K : T\rho = \rho\} \ne \emptyset$. Let $\{\alpha_n\}_{n\ge 0}$ be real sequence in $[0,1]$ satisfying the following conditions: (i)$\sum_{n\ge 0} \alpha_n = \infty$, (ii) $\lim_{n\to\infty} \alpha_n = 0$. For arbitrary $x_0 \in K$, let $\{x_n\}_{n\ge 0}$ be iteratively defined by (1.1). If there exists a strictly increasing function $\Phi : [0,\infty) \to [0,\infty)$ with $\Phi(0) = 0$ such that*

$$< T^n x - T^n\rho, j(x-\rho) > \le k_n\|x-\rho\|^2 - \Phi(\|x-\rho\|) + \epsilon_n$$

for all $x \in K$, then, $\{x_n\}_{n\ge 0}$ converges strongly to $\rho \in F(T)$.

Proof. Since there exists a strictly increasing continuous function $\Phi : [0,\infty) \to [0,\infty)$ with $\Phi(0) = 0$ such that

$$\langle T^n x - T^n\rho, j(x-\rho)\rangle \le k_n\|x-\rho\|^2 - \Phi(\|x-\rho\|) + \epsilon_n, \tag{2.4}$$

for $x \in K,\ \rho \in F(T)$, that is

$$\epsilon_n + \langle k_n(x-\rho) - (T^n x - \rho), j(x-\rho)\rangle \ge \Phi(\|x-\rho\|). \tag{2.5}$$

To ensure that $\Phi^{-1}(a_0)$ is well defined, choose some $x_0 \in K$ and $x_0 \ne Tx_0$ such that

$$\epsilon_n + (k_n + L)\|x_0-\rho\|^2 + L\|x_0-\rho\|^2 \in R(\Phi)$$

and denote

$$a_0 = \epsilon_n + (k_n + L)\|x_0-\rho\|^2 + L\|x_0-\rho\|^2,$$

where $R(\Phi)$ is the range of Φ.

Indeed, if $\Phi(a) \to +\infty$ as $a \to \infty$, then $a_0 \in R(\Phi)$; if $\sup\{\Phi(a) : a \in [0,\infty]\} = a_1 < +\infty$ with $a_1 < a_0$, then for $\rho \in K$, there exists a sequence $\{u_n\}$ in K such that $u_n \to \rho$ as $n \to \infty$ with $u_n \ne \rho$.

Clearly, $Tu_n \to T\rho$ as $n \to \infty$ thus $\{u_n - Tu_n\}$ is a bounded sequence.

Therefore, there exists a natural number n_0 such that

$$\epsilon_n + (k_n + L)\|u_n-\rho\|^2 + L\|u_n-\rho\|^2 < \frac{a_1}{2}$$

for $n \geq n_0$, and then we redefine $x_0 = u_{n_0}$ and

$$\epsilon_n + (k_n + L)\|x_0 - \rho\|^2 + L\|x_0 - \rho\|^2 \in R(\Phi).$$

Step 1. We first show that $\{x_n\}_{n=0}^{\infty}$ is a bounded sequence.
Set $R = \Phi^{-1}(a_0)$, then from above (2.5), we obtain that $\|x_0 - \rho\| \leq R$.
Denote

$$B_1 = \{x \in K : \|x - \rho\| \leq R\}, \quad B_2 = \{x \in K : \|x - \rho\| \leq 2R\}. \tag{2.6}$$

Now, we want to prove that $x_n \in B_1$. If $n = 0$, then $x_0 \in B_1$.

Now assume that it holds for some n, that is, $x_n \in B_1$. Suppose that, it is not the case, then $\|x_{n+1} - \rho\| > R > \frac{R}{2}$.

Since $\{a_n\} \in [0, \infty]$ with $a_n \to 0$ as $n \to \infty$, set $M = sup\{a_n : n \in N\}$. Denote

$$\tau_0 = \min\left\{1, \frac{R}{L(2R+M)}, \frac{R}{(L(2R+M)+R)}, \frac{\Phi(\frac{R}{2})}{32R^2}, \frac{\Phi(\frac{R}{2})}{16R[2(L(2R+M)+R)+M]}, \frac{\Phi(\frac{R}{2})}{16R[L(2R+M)+R]}, \frac{\Phi(\frac{R}{2})}{8}\right\}. \tag{2.7}$$

Since $\lim_{n\to\infty} \alpha_n = 0$ and $\lim_{n\to\infty} k_n = 1$, without loss of generality, let
$0 \leq \alpha_n, k_n - 1, \epsilon_n \leq \tau_0$ for any $n \geq 0$.
We get

$$\begin{aligned} \|x_{n+1} - \rho\| &\leq (1 - \alpha_n)\|x_n - \rho\| + \alpha_n\|T^n x_n - \rho\| \\ &\leq R + \tau_0 L(2R + M) \\ &\leq 2R. \end{aligned}$$

Therefore, we have

$$\begin{aligned} \|x_{n+1} - x_n\| &\leq \alpha_n\|T^n x_n - x_n\| \\ &\leq \alpha_n(\|T^n x_n - \rho\| + \|x_n - \rho\|) \\ &\leq \tau_0(L(2R + M) + R). \end{aligned} \tag{2.8}$$

Using Lemma 1.1 and the above estimates, we have

$$\begin{aligned} \|x_{n+1} - \rho\|^2 &\leq \|x_n - \rho\|^2 + 2\alpha_n < T^n x_n - x_n, j(x_{n+1} - \rho) > \\ &= \|x_n - \rho\|^2 + 2\alpha_n < T^n x_{n+1} - x_{n+1}, j(x_{n+1} - \rho) > \\ &\quad +2\alpha_n < x_{n+1} - x_n, j(x_{n+1} - \rho) > \\ &\quad +2\alpha_n < T^n x_n - T^n x_{n+1}, j(x_{n+1} - \rho) > \\ &\leq \|x_n - \rho\|^2 + 2\alpha_n(k_n\|x_{n+1} - \rho\|^2 - \Phi(\|x_{n+1} - \rho\|) + \epsilon_n) \\ &\quad -2\alpha_n\|x_{n+1} - \rho\|^2 + 2\alpha_n L(\|x_n - x_{n+1}\| + a_n)\|x_{n+1} - \rho\| \\ &\quad +2\alpha_n\|x_{n+1} - x_n\|\|x_{n+1} - \rho\| \\ &\leq \|x_n - \rho\|^2 - 2\alpha_n\Phi(\tfrac{R}{2}) + 2\alpha_n\frac{\Phi(\frac{R}{2})}{32R^2}4R^2 + 2\alpha_n\frac{\Phi(\frac{R}{2})}{8} \\ &\quad +2\alpha_n L\frac{\Phi(\frac{R}{2})}{16R[2(L(2R+M)+R)+M]}2R[2(L(2R + M) + R) + M] \\ &\quad +2\alpha_n\frac{\Phi(\frac{R}{2})}{16R[L(2R+M)+R]}2R[L(2R + M) + R] \\ &\leq \|x_n - \rho\|^2 - \alpha_n\Phi(\tfrac{R}{2}) \\ &\leq R^2. \end{aligned} \tag{2.9}$$

which is a contradiction. Hence $\{x_n\}_{n=0}^{\infty}$ is a bounded sequence.

Step 2. We want to prove that $\|x_n - \rho\| \to 0$ as $n \to \infty$.
Since $\lim_{n\to\infty} \alpha_n = 0$, $\lim_{n\to\infty} k_n = 1$ and $\{x_n\}_{n=0}^{\infty}$ is bounded. From 2.9, we observed that

$$\lim_{n\to\infty} L\|x_n - x_{n+1}\| = 0. \tag{2.10}$$

So from (1.3), we have

$$\begin{aligned}
\|x_{n+1}-\rho\|^2 &\leq \|x_n-\rho\|^2+2\alpha_n < T^n x_n - x_n, j(x_{n+1}-\rho) > \\
&= \|x_n-\rho\|^2+2\alpha_n < T^n x_{n+1} - x_{n+1}, j(x_{n+1}-\rho) > \\
&\quad + < x_{n+1}-x_n, j(x_{n+1}-\rho) > \\
&\quad + < T^n x_n - T^n x_{n+1}, j(x_{n+1}-\rho) > \\
&\leq \|x_n-\rho\|^2+2\alpha_n(k_n\|x_{n+1}-\rho\|^2-\Phi(\|x_{n+1}-\rho\|)+\epsilon_n) \\
&\quad -2\alpha_n\|x_{n+1}-\rho\|^2+2\alpha_n L(\|x_n-x_{n+1}\|+a_n)\|x_{n+1}-\rho\| \\
&\quad +2\alpha_n\|x_{n+1}-x_n\|\|x_{n+1}-\rho\| \\
&\leq \|x_n-\rho\|^2+2\alpha_n(k_n-1)\|x_{n+1}-\rho\|^2 \\
&\quad -2\alpha_n\Phi(\|x_{n+1}-\rho\|)+2\alpha_n\epsilon_n \\
&\quad +2\alpha_n L(\|x_n-x_{n+1}\|+a_n)\|x_{n+1}-\rho\| \\
&\quad +2\alpha_n\|x_{n+1}-x_n\|\|x_{n+1}-\rho\| \\
&= \|x_n-\rho\|^2-2\alpha_n\Phi(\|x_{n+1}-\rho\|)+o(\alpha_n),
\end{aligned} \tag{2.11}$$

where

$$\begin{aligned}
&2\alpha_n(k_n-1)\|x_{n+1}-\rho\|^2++2\alpha_n L(\|x_n-x_{n+1}\|+a_n)\|x_{n+1}-\rho\| \\
&+2\alpha_n\|x_{n+1}-x_n\|\|x_{n+1}-\rho\|+2\alpha_n\epsilon_n \\
&=o(\alpha_n).
\end{aligned}$$

By Lemma 1.2, we obtain that

$$\lim_{n\to\infty}\|x_n-\rho\|=0.$$

This completes the proof. □

Now, we give an example to illustrate the validity of our Theorem 2.3.

Example 2.2. Let $X=R, K=[0,1]$ and $T:K\to K$ be a map defined by

$$Tx=\frac{x}{1+x}, \forall x\in[0,1)$$

Clearly, T is nearly uniformly Lipschitzian ($a_n=\frac{1}{2^n}$) with $F(0)=0$.
Define $\Phi:[0,\infty)\to[0,\infty)$ by

$$\Phi(t)=\frac{t^2}{1+nt}$$

then, Φ is a strictly increasing function with $\Phi(0)=0$. For all $x\in K, \rho\in F(T)$, we have that operator T in Theorem 2.3 satisfies

$$< T^n x - T^n\rho, j(x-\rho) >\leq k_n\|x-\rho\|^2-\Phi(\|x-\rho\|)+\epsilon_n$$

with the sequences $k_n=1$ and $\epsilon_n=\frac{x^2}{1+nx}$. Set $\alpha_n=\frac{1}{2+n}\ \forall n\geq 0$.
It is well known that whenever a theorem is proved using Mann iteration (without error terms), the method of proof carries over easily to the case of Mann iteration process with error terms. Thus, we have the following theorem:

Theorem 2.4. *Let X be a real Banach space, K be a nonempty closed convex subset of X, $T:K\to K$ be a nearly uniformly L-Lipschitzian mapping with sequence $\{a_n\}$. Let $k_n\subset[1,\infty)$ and ϵ_n be sequences with $\lim_{n\to\infty}k_n=1,\ \lim_{n\to\infty}\epsilon_n=0$ and $F(T)=\{\rho\in K: T\rho=\rho\}\neq\emptyset$. Let $\{b_n\}_{n\geq 0}$ and $\{d_n\}_{n\geq 0}$ be real sequences in $[0,1]$ satisfying the following conditions: (i) $b_n+d_n=1$, (ii)$\sum_{n\geq 0}(b_n+d_n)=\infty$ (iii) $\lim_{n\to\infty}(b_n+d_n)=0$. For arbitrary $x_0\in K$, let $\{x_n\}_{n\geq 0}$ be iteratively defined by $x_{n+1}=(1-b_n-d_n)x_n+b_nT^nx_n+d_nw_n, n\geq 0$. If there* *exists a strictly increasing function* $\Phi : [0, \infty) \to [0, \infty)$ *with* $\Phi(0) = 0$ *such that*

$$< T^n x - T^n \rho, j(x - \rho) >\leq k_n \|x - \rho\|^2 - \Phi(\|x - \rho\|) + \epsilon_n$$

for all $x \in K$*, then,* $\{x_n\}_{n\geq 0}$ *converges strongly to* $\rho \in F(T)$*.*

Proof. Define $\alpha_n = b_n + d_n$. Then (1.3) becomes

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T^n x_n + d_n(w_n - T^n x_n), n \geq 0.$$

The boundedness of $\{x_n\}_{n\geq 0}$ follows as in the step 1 of the proof of Theorem 2.3 and the rest of the result follows as in the step 2 of the proof of Theorem 2.3. This completes the proof. □

Remark 2.4. Theorem 2.4 remains true for the so-called modified Ishikawa-type iteration scheme. This is a modification of the scheme introduced by Ishikawa in [6]. There is no further generality obtained in using the cumbersome-Ishikawa iteration process, rather than the iteration process considered in this paper.

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