

An empirical study of the convergence area and convergence speed of Agarwal et al. fixed point iteration procedure

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ABSTRACT. We present an empirical study of the convergence area and speed of Agarwal et al. fixed point iterative procedure in the particular case of the Newton's method associated to the complex polynomials $p_3(z) = z^3 - 1$ and $p_8(z) = z^8 - 1$. In order to obtain an analytical expression for the experimental data related to the mean number of iterations (MNI) and convergence area index (CAI), we use regression analysis and find some linear and nonlinear bi-variable models with good correlation coefficients.

1. INTRODUCTION

It is well-known that the problem of solving the equation $f(x) = 0$ can be equivalently transformed into a fixed point problem $T(x) = x$, where T is some operator. Then, applying different iterations procedures for fixed points approximation, one can approximate the solution of the given equation.

Let (X, d) be a complete metric space and $T : X \rightarrow X$ a selfmap on X . The set $F_T = \{x \in X / T(x) = x\}$ is the set of all fixed points of T . The operator T is usually called *iteration function*.

Many iterative processes for the approximation of fixed points have been described in the literature [3-6]. In the following, assume that each iteration process starts from any initial point $x_0 \in X$.

2. ITERATION PROCEDURES FOR FIXED POINTS APPROXIMATION

The *Picard iteration* [7] (E. Picard, 1890), was defined as:

$$x_{n+1} = T(x_n), \quad n = 0, 1, 2, \dots \quad (2.1)$$

As an example, if the operator T is defined as

$$T(x) = x - \frac{f(x)}{f'(x)} \quad (2.2)$$

then the Picard iteration becomes

$$x_{n+1} = T(x_n) = x_n - \frac{f(x_n)}{f'(x_n)}, \quad n = 0, 1, 2, \dots \quad (2.3)$$

which is the well-known Newton method for solving the nonlinear equation $f(x) = 0$.

The *Agarwal et al. iteration* [1] (Agarwal et al., 2007), was defined as the following two-step process:

$$\begin{aligned} x_{n+1} &= (1 - \alpha_n)T(x_n) + \alpha_n T(y_n), \\ y_n &= (1 - \beta_n)x_n + \beta_n T(x_n), \quad n = 0, 1, 2, \dots \end{aligned} \quad (2.4)$$

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where $\alpha_n \in [0, 1]$ and $\beta_n \in [0, 1]$ for all $n \in \mathbb{N}$.

3. NUMERICAL RESULTS

Newton's method for finding the roots of a complex polynomial $p(z)$ is given by the formula

$$z_{n+1} = N(z_n), \quad n = 0, 1, 2, \dots \quad (3.5)$$

$$N(z) = z - \frac{p(z)}{p'(z)}$$

where $p'(z)$ is the first derivative of p and $z_0 \in \mathbb{C}$ is a starting point.

If the sequence $\{z_n\}_{n=0}^{\infty}$ (the orbit of the point z_0) converges to a root z^* of the polynomial p then we say that z_0 is attracted by z^* . The attraction basin of the root z^* of the polynomial p is the set of all starting points z_0 which are attracted by z^* . We consider that the iteration method converges if the residual is less than 10^{-5} in a maximum of 13 iterations.

In [2] we present a comparative study of some iterative methods for solving nonlinear equations by using the basins of attraction.

To empirical evaluate the behaviors of Agarwal et al. iteration process, we considered the complex polynomials

$$P_3(z) = z^3 - 1 \text{ and } P_8(z) = z^8 - 1, \text{ respectively} \quad (3.6)$$

In this case, the fixed point problems are

$$z = \frac{2z^3 + 1}{3z^2}, \text{ and } z = \frac{7z^8 + 1}{8z^7}, \text{ respectively, } z \in \mathbb{C} \quad (3.7)$$

so

$$N_3(z) = \frac{2z^3 + 1}{3z^2} \text{ and } N_8(z) = \frac{7z^8 + 1}{8z^7}, z \in \mathbb{C}.$$

We take the rectangle in the complex plane $D = [-1.5, 1.5] \times [-1.5, 1.5]$, and then divide the rectangle into 250×250 grids. We apply the simplified Agarwal et al. iteration process for the Newton operators defined in (3.7)

$$z_{n+1} = (1 - \alpha)N(z_n) + \alpha N(y_n), \quad (3.8)$$

$$y_n = (1 - \beta)z_n + \beta N(z_n), \quad n = 0, 1, 2, \dots$$

starting at each grid point.

We compute the mean number of iterations denoted *MNI*, and converging area index denoted *CAI* (i.e. $CAI = \frac{\text{Number of converging points}}{\text{Total number of grid points}}$) for each pair

$$(\alpha, \beta) \in \{0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9\} \times \{0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9\}.$$

The results are presented in Tables 1-4.

All numerical results were computed by using *Matlab R2012a* [8] mathematical software package from MathWorks.

Table 1. MNI for Agarwal et al. iteration process and the polynomial

$$P_3(z) = z^3 - 1$$

$\alpha \backslash \beta$	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
0.1	5.6722	5.6220	5.6063	5.5165	5.4362	5.3166	5.2516	5.1763	5.1421
0.2	5.6167	5.5668	5.3985	5.2566	5.1722	5.0400	4.9628	4.8334	4.7946
0.3	5.6158	5.4081	5.0812	5.0812	4.9600	4.8011	4.6995	4.5931	4.4369
0.4	5.5542	5.2799	5.0718	4.9022	4.7770	4.6212	4.5068	4.3461	4.2674
0.5	5.4872	5.2002	4.9816	4.8047	4.6734	4.4938	4.3388	4.1257	3.9904
0.6	5.4032	5.1051	4.8763	4.7115	4.5225	4.3384	4.2610	3.756	6.424
0.7	5.3452	5.0535	4.719	4.6388	4.4371	4.2807	4.1631	4.0375	3.8080
0.8	5.3035	4.9707	4.7309	4.5623	4.3974	4.2313	4.0980	3.9745	3.7447
0.9	5.2412	4.9681	4.7135	4.5287	4.3106	4.2131	4.0859	3.8706	3.7230

Table 2. CAI for Agarwal et al. iteration process and the polynomial

$$P_3(z) = z^3 - 1$$

$\alpha \backslash \beta$	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
0.1	0.9419	0.9458	0.9658	0.9685	0.9758	0.9750	0.9781	0.9815	0.9877
0.2	0.9408	0.9673	0.9735	0.9754	0.9846	0.9812	0.9919	0.9923	0.9938
0.3	0.9546	0.9685	0.9685	0.9847	0.9896	0.9858	0.9904	0.9912	0.9927
0.4	0.9608	0.9696	0.9796	0.983	0.9912	0.9896	0.9923	0.9919	0.9935
0.5	0.9731	0.9797	0.9804	0.9885	0.9935	0.9942	0.9931	0.9942	0.9958
0.6	0.9715	0.9800	0.9885	0.9915	0.9912	0.9908	0.9958	0.9942	0.9965
0.7	0.9735	0.9846	0.9873	0.9919	0.9938	0.9942	0.9946	0.9958	0.9950
0.8	0.9792	0.9831	0.9915	0.9908	0.9946	0.9938	0.9963	0.9958	0.9969
0.9	0.9789	0.9981	0.9904	0.9912	0.9938	0.9958	0.9981	0.9954	0.9981

Figures 1-4 present the analytical expression for MNI and CAI, respectively, depending on α and β , generated by using the *TableCurve3D v4.0* [9] program of SYSTAT Software Inc. for surface fitting.

Surface fitting refers to finding an appropriate mathematical model that expresses the relationship between a dependent variable z and two independent variables α and β , and estimating the values of its parameters using nonlinear regression.

In Figure 1

$$z = MNI(\alpha, \beta) = a + b\alpha + c\beta$$

with $a = 6.144234$, $b = -1.2116667$ and $c = -1.5233074$, the mean number of iterations $MNI(\alpha, \beta)$ is approximated as a linear function on α and β (by using Least Squares Method for surface fitting). The value of the *accuracy indicator* $r^2 = 0.93160209$ shows that the approximation is a very good one.

In Figure 2

$$z = CAI(\alpha, \beta) = a + b\alpha + c/\beta + d\alpha^2 + e/\beta^2 + f\alpha/\beta$$

with $a = 0.98837987$, $b = 0.048021787$, $c = -0.0093104925$, $d = -0.035891053$, $e = 0.00032273351$ and $f = 0.0047416604$, the Convergence Area Index $CAI(\alpha, \beta)$ is approximated as a rational function on α and β . The value of the *accuracy indicator* $r^2 = 0.93304197$ shows that the approximation is very good too.

Table 3. MNI for Agarwal et al. iteration process and the polynomial $P_8(z) = z^8 - 1$

$\alpha \backslash \beta$	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
0.1	5.6722	5.6220	5.6063	5.5165	5.4362	5.3166	5.2516	5.1763	5.1421
0.2	5.6167	5.5668	5.3985	5.2566	5.1722	5.0400	4.9628	4.8334	4.7946
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0.5	5.4872	5.2002	4.9816	4.8047	4.6734	4.4938	4.3388	4.1257	3.9904
0.6	5.4032	5.1051	4.8763	4.7115	4.5225	4.3384	4.2610	3.756	6.424
0.7	5.3452	5.0535	4.719	4.6388	4.4371	4.2807	4.1631	4.0375	3.8080
0.8	5.3035	4.9707	4.7309	4.5623	4.3974	4.2313	4.0980	3.9745	3.7447
0.9	5.2412	4.9681	4.7135	4.5287	4.3106	4.2131	4.0859	3.8706	3.7230

Table 4. CAI for Agarwal et al. iteration process and the polynomial $P_8(z) = z^8 - 1$

$\alpha \backslash \beta$	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
0.1	0.9419	0.9458	0.9658	0.9685	0.9758	0.9750	0.9781	0.9815	0.9877
0.2	0.9408	0.9673	0.9735	0.9754	0.9846	0.9812	0.9919	0.9923	0.9938
0.3	0.9546	0.9685	0.9685	0.9847	0.9896	0.9858	0.9904	0.9912	0.9927
0.4	0.9608	0.9696	0.9796	0.983	0.9912	0.9896	0.9923	0.9919	0.9935
0.5	0.9731	0.9797	0.9804	0.9885	0.9935	0.9942	0.9931	0.9942	0.9958
0.6	0.9715	0.9800	0.9885	0.9915	0.9912	0.9908	0.9958	0.9942	0.9965
0.7	0.9735	0.9846	0.9873	0.9919	0.9938	0.9942	0.9946	0.9958	0.9950
0.8	0.9792	0.9831	0.9915	0.9908	0.9946	0.9938	0.9963	0.9958	0.9969
0.9	0.9789	0.9981	0.9904	0.9912	0.9938	0.9958	0.9981	0.9954	0.9981

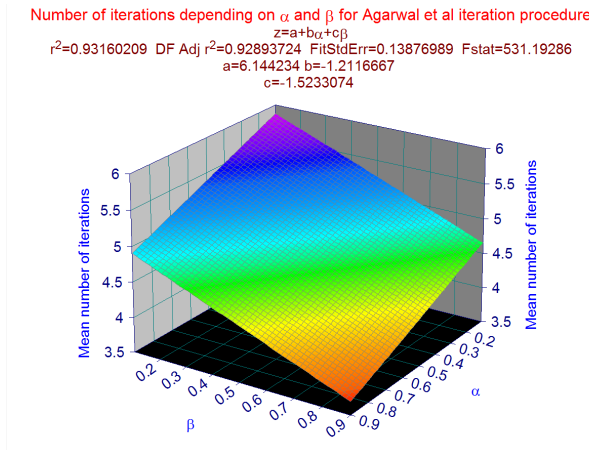


Figure 1. Mean number of iterations (MNI) for the polynomial $P_3(z)$

Convergence area index depending on α and β for Agarwal et al iteration procedure

$$z = a + b\alpha + c/\beta + d\alpha^2 + e/\beta^2 + f\alpha/\beta$$

$r^2=0.93304197$ DF Adj $r^2=0.92761294$ FitStdErr=0.0034341674 Fstat=209.02091
 $a=0.98837987$ $b=0.048021787$ $c=-0.0093104925$
 $d=-0.035891053$ $e=0.00032273351$ $f=0.0047416604$

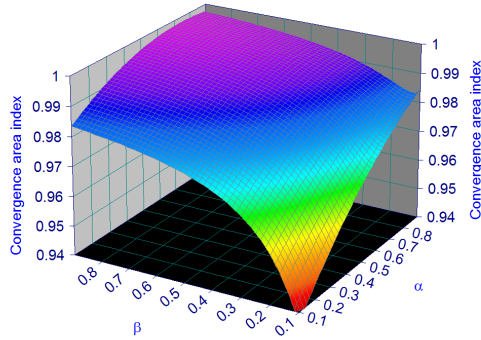


Figure 2. Convergence area index (CAI) for the polynomial $P_3(z)$

Number of iterations depending on α and β for Agarwal et al. iteration procedure

$$z = a + b\alpha + c\beta$$

$r^2=0.88273947$ DF Adj $r^2=0.87817088$ FitStdErr=0.2507507 Fstat=293.59273
 $a=8.9229849$ $b=-1.759463$
 $c=-1.9342426$

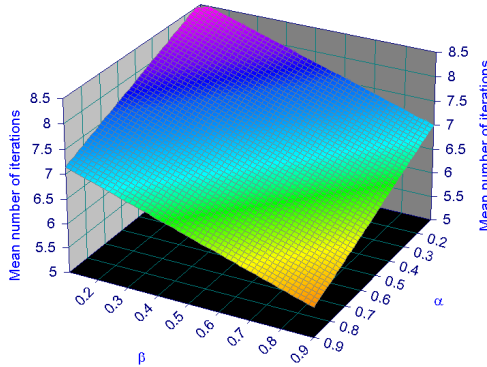


Figure 3. Mean number of iterations (MNI) for the polynomial $P_8(z)$

Convergence area index depending on α and β for Agarwal et al. iteration procedure

$$z = a + b\alpha + c/\beta + d\alpha^2 + e/\beta^2 + f\alpha/\beta$$

$r^2=0.85165672$ DF Adj $r^2=0.83962889$ FitStdErr=0.014548878 Fstat=86.116815
 $a=0.81701034$ $b=0.16695584$ $c=-0.025118619$
 $d=-0.11066258$ $e=0.0011408488$ $f=0.0072185557$

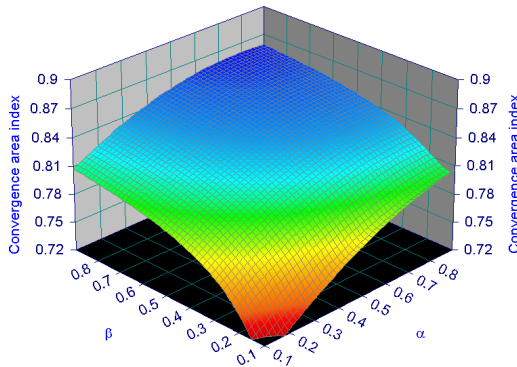


Figure 4. Convergence area index (CAI) for the polynomial $P_8(z)$

4. CONCLUSIONS

The numerical results in Table 1 and Table 3 and the graphs in Figure 1 and Figure 3 shows that, for fixed point problems (3.7), the Mean Number of Iterations is decreasing while the values of α and β are increasing (the minimum number of iterations is obtained for $\alpha = 0.9$ and $\beta = 0.9$).

The numerical results in Table 2 and Table 4 and the graphs in Figure 2 and Figure 4 shows that, for fixed point problems (3.7), the Convergence Area Index is increasing while the values of α and β are increasing (the maximum Convergence Area Index is obtained for $\alpha = 0.9$ and $\beta = 0.9$).

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