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On the generalized Boolean sum Schurer-Stancu approximation formula

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ABSTRACT. In this paper, the Schurer-Stancu generalized Boolean sum (GBS, for short) approximation formula is considered and it's remainder term is expressed in terms of bivariate divided differences. When the approximated function is sufficiently smooth, an upper bound estimation for the remainder term is also established. As particular cases, GBS Schurer and respectively GBS Bernstein approximation formulas are obtained and the expressions of their remainder are explicitly given.

1. INTRODUCTION

We denote $N = \{1, 2, \dots\}$ and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Let $p \in \mathbb{N}_0$ and let α, β be real parameters so that $0 \le \alpha \le \beta$. The Schurer-Stancu operator [3] is defined for any $m \in \mathbb{N}$, any function $f \in C([0, 1+p])$ and any $x \in [0, 1+p]$ by:

$$\left(\widetilde{S}^{(\alpha,\beta)}_{m,p} f\right)(x) = \sum_{k=0}^{m+p} \widetilde{p}_{mk}(x) f\left(\frac{k+\alpha}{m+\beta}\right), \tag{1.1}$$

where

$$\widetilde{p}_{m,k}(x) = \binom{m+p}{k} x^k (1-x)^{m-k} \tag{1.2}$$

are the fundamental Schurer polynomials [17].
Note that the multiparameter operator (1.1) includes as particular cases other well-known operators: for $\alpha = \beta = 0$, the operator (1.1) reduces to the Schurer operator [17]; for $p = 0$, (1.1) becomes the Stancu operator [20], while, for $\alpha = \beta = 0$ and $p = 0$, (1.1) is the classical Bernstein operator [10].

The method of "parametric extensions" is a very efficient one for constructing multivariate operators [12]. By applying the above mentioned method, in [4] the author considered the following parametric extensions of the operator (1.1):

$$\left({}_x\widetilde{S}^{(\alpha,\beta)}_{m,p} f\right)(x, y) = \sum_{k=0}^{m+p} \sum_{j=0}^{n+q} \widetilde{p}_{m,k}(x) \widetilde{p}_{n,j}(y) f\left(\frac{k+\alpha}{m+\beta}, y\right), \tag{1.3}$$

$$\left({}_y\widetilde{S}^{(\gamma,\delta)}_{n,q} f\right)(x, y) = \sum_{k=0}^{m+p} \sum_{j=0}^{n+q} \widetilde{p}_{m,k}(x) \widetilde{p}_{n,j}(y) f\left(x, \frac{j+\gamma}{n+\delta}\right). \tag{1.4}$$

Note that in (1.3) and (1.4), $p, q \in \mathbb{N}_0$, $\alpha, \beta, \gamma, \delta$ are real parameters such that $0 \le \alpha \le \beta$, $0 \le \gamma \le \delta$ and $f : [0, 1+p] \times [0, 1+q] \to \mathbb{R}$ is a given real valued function.

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The notion of "Generalized Boolean Sum" (GBS, for short) operator was introduced by C. Badea and C. Cottin [1] as the boolean sum of the parametric extensions of a univariate operator.

By using the above definition, the GBS Schurer-Stancu operator is defined for any $m, n \in \mathbb{N}$, any $p, q \in \mathbb{N}_0$ and any real parameters $\alpha, \beta, \gamma, \delta$ satisfying $0 \le \alpha \le \beta$, $0 \le \gamma \le \delta$ by:

$$\left(\widetilde{U}_{m,p,n,q}^{(\alpha,\beta,\gamma,\delta)} f\right)(x,y) \tag{1.5}$$
$$= \sum_{k=0}^{m+p} \sum_{j=0}^{n+q} \widetilde{p}_{m,k}(x) \widetilde{p}_{n,j}(y) \left\{ f\left(\frac{k+\alpha}{m+\beta}, y\right) + f\left(x, \frac{j+\gamma}{n+\delta}\right) - f\left(\frac{k+\alpha}{m+\beta}, \frac{j+\gamma}{n+\delta}\right) \right\},$$

for any bounded function $f : [0, 1+p] \times [0, 1+q] \to \mathbb{R}$.
Approximation properties of the operator (1.5) were established in [4], [7].
As particular cases of (1.5) we mention the GBS operators of Schurer, Stancu, respectively Bernstein type.

By considering the GBS Schurer-Stancu approximation formula

$$f = \widetilde{U}_{m,p,n,q}^{(\alpha,\beta,\gamma,\delta)} f + \widetilde{R}_{m,p,n,q}^{(\alpha,\beta,\gamma,\delta)} f, \tag{1.6}$$

the first focus of the paper is to express the remainder term $\widetilde{R}_{m,p,n,q}^{(\alpha,\beta,\gamma,\delta)} f$ by using the bivariate divided difference of the approximated function.

We start by recalling some results concerning the divided differences.
Suppose that $I \subset \mathbb{R}$ is an interval of the real axis, $f : I \to \mathbb{R}$ is a given function and $x_0, x_1 \in I$ $(x_0 \neq x_1)$ are given. Then, the divided difference (of first order) of f with respect the distinct knots x_0, x_1 is defined by:

$$[x_0, x_1; f] = \frac{f(x_1) - f(x_0)}{x_1 - x_0}. \tag{1.7}$$

If the knots $x_0, x_1, \ldots, x_p \in I$ are distinct, the p-th order divided difference of f with respect to the mentioned distinct knots is defined by the following recurrence formula:

$$[x_0, x_1, \ldots, x_p; f] = \frac{[x_1, \ldots, x_p; f] - [x_0, \ldots, x_{p-1}; f]}{x_p - x_0}, p \ge 2. \tag{1.8}$$

For the bivariate divided differences, several definitions were given in [13], [14], [15]. In [8], by using the method of parametric extensions we re-found the definition of the bivariate divided difference.

In the following let $I, J \subset \mathbb{R}$ be intervals, $f : I \times J \to \mathbb{R}$ be a given function, $p, q \in \mathbb{N}_0$, $x_0, x_1, \ldots, x_p \in I$ and $y_0, y_1, \ldots, y_q \in J$ be distinct knots.

If $p, q \in \mathbb{N}$, $p \ge 2$ and $q \ge 2$, the following recurrence formula

$$\begin{bmatrix} x_0, x_1, \ldots, x_p \\ y_0, y_1, \ldots, y_q \end{bmatrix}; f \Bigg] = \frac{1}{(x_p - x_0)(y_q - y_0)} \left(\begin{bmatrix} x_1, x_2, \ldots, x_p \\ y_1, y_2, \ldots, y_q \end{bmatrix}; f \Bigg] \right. \tag{1.9}$$
$$\left. - \left[\begin{matrix} x_0, x_1, \ldots, x_{p-1} \\ y_1, y_2, \ldots, y_q \end{matrix}; f\right] - \left[\begin{matrix} x_1, x_2, \ldots, x_p \\ y_0, y_1, \ldots, y_{q-1} \end{matrix}; f\right] + \left[\begin{matrix} x_0, x_1, \ldots, x_{p-1} \\ y_0, y_1, \ldots, y_{q-1} \end{matrix}; f\right] \right)$$

holds (see [3]) and

$$\left[\begin{matrix} x_0, x_1, \ldots, x_p \\ y_0, y_1, \ldots, y_q \end{matrix}; f\right] = \left[\begin{matrix} x_{i_0}, x_{i_1}, \ldots, x_{i_p} \\ y_{j_0}, y_{j_1}, \ldots, y_{j_q} \end{matrix}; f\right] \tag{1.10}$$

where $(i_0, i_1, \ldots, i_p)$, $(j_0, j_1, \ldots, j_q)$ are permutations of $(0, 1, \ldots, p)$, respectively $(0, 1, \ldots, q)$.

Theorem 1.1. *If $q \geq 2$, then*

$$\begin{bmatrix} x_0, x_1, \ldots, x_p \\ y_0, y_1, \ldots, y_q \end{bmatrix}; f \Bigg] \tag{1.11}$$

$$= \frac{1}{y_q - y_{q-1}} \left(\begin{bmatrix} x_0, x_1, \ldots, x_p & \\ y_0, y_1, \ldots, y_{q-2}, y_q & ; f \end{bmatrix} - \begin{bmatrix} x_0, x_1, \ldots, x_p & \\ y_0, y_1, \ldots, y_{q-1} & ; f \end{bmatrix} \right)$$

and, if $p \geq 2$, then

$$\begin{bmatrix} x_0, x_1, \ldots, x_p & \\ y_0, y_1, \ldots, y_q & ; f \end{bmatrix} \tag{1.12}$$

$$= \frac{1}{x_p - x_{p-1}} \left(\begin{bmatrix} x_0, x_1, \ldots, x_{p-2}, x_p & \\ y_0, y_1, \ldots, y_q & ; f \end{bmatrix} - \begin{bmatrix} x_0, x_1, \ldots, x_{p-1} & \\ y_0, y_1, \ldots, y_q & ; f \end{bmatrix} \right).$$

Proof. By using the method of parametric extensions (see [8]) and (1.10), we have that

$$\begin{bmatrix} x_0, x_1, \ldots, x_p & \\ y_0, y_1, \ldots, y_q & ; f \end{bmatrix} = \begin{bmatrix} x_0, x_1, \ldots, x_p & \\ y_{q-1}, y_0, y_1, \ldots, y_{q-2}, y_q & ; f \end{bmatrix}$$

$$[x_0; [y_{q-1}, y_0, y_1, \ldots, y_{q-2}, y_q; f]_y]_x$$

$$= \left[x_0, x_1, \ldots, x_p \frac{[y_0, y_1, \ldots, y_{q-2}, y_q; f]_y - [y_{q-1}, y_0, y_1, \ldots, y_{q-2}; f]_y}{y_q - y_{q-1}} \right]_x$$

$$= \frac{1}{y_q - y_{q-1}} \big([x_0, x_1, \ldots, x_p; [y_0, y_1, \ldots, y_{q-2}, y_q; f]_y]_x$$

$$- [x_0, x_1, \ldots, x_p; [y_{q-1}, y_0, y_1, \ldots, y_{q-2}; f]_y]_x \big)$$

$$= \frac{1}{y_q - y_{q-1}} \left(\begin{bmatrix} x_0, x_1, \ldots, x_p & \\ y_0, y_1, \ldots, y_{q-2}, y_q & ; f \end{bmatrix} - \begin{bmatrix} x_0, x_1, x_2, \ldots, x_p & \\ y_{q-1}, y_0, y_1, \ldots, y_{q-2} & ; f \end{bmatrix} \right),$$

which yields the identity (1.11).
In a similar way, one proves (1.12). □

From Theorem 1.1, the following identities

$$\begin{bmatrix} x_0, x_1, x_2 & \\ y_0, y_1 & ; f \end{bmatrix} = \frac{1}{x_2 - x_1} \left(\begin{bmatrix} x_0, x_2 & \\ y_0, y_1 & ; f \end{bmatrix} - \begin{bmatrix} x_0, x_1 & \\ y_0, y_1 & ; f \end{bmatrix} \right), \tag{1.13}$$

$$\begin{bmatrix} x_0, x_1 & \\ y_0, y_1, y_2 & ; f \end{bmatrix} = \frac{1}{y_2 - y_1} \left(\begin{bmatrix} x_0, x_1 & \\ y_0, y_2 & ; f \end{bmatrix} - \begin{bmatrix} x_0, x_1 & \\ y_0, y_1 & ; f \end{bmatrix} \right), \tag{1.14}$$

hold and from (1.9) we have

$$\begin{bmatrix} x_0, x_1, x_2 & \\ y_0, y_1, y_2 & ; f \end{bmatrix} = \frac{1}{(x_2 - x_1)(y_2 - y_1)} \left(\begin{bmatrix} x_0, x_2 & \\ y_0, y_2 & ; f \end{bmatrix} - \begin{bmatrix} x_0, x_1 & \\ y_0, y_2 & ; f \end{bmatrix} \right. \tag{1.15}$$

$$\left. - \begin{bmatrix} x_0, x_2 & \\ y_0, y_1 & ; f \end{bmatrix} + \begin{bmatrix} x_0, x_1 & \\ y_0, y_1 & ; f \end{bmatrix} \right).$$

We shall use these identities in the proof of Theorem 2.3. Now, we recall the following theorem from [9].

Theorem 1.2. *Let $p, q \in \mathbb{N}$, $a \leq x_0 < x_1 < \cdots < x_p \leq b$, $c \leq y_0 < y_1 < \cdots < y_q \leq d$ and $f : [a,b] \times [c,d] \to \mathbb{R}$. If $f \in C^{(p-1,q-1)}([a,b] \times [c,d])$ and exists $\dfrac{\partial^{p+q} f}{\partial x^p \partial y^q}$ on $]a,b[\times]c,d[$ then, there exists $(\xi, \eta) \in]a,b[\times]c,d[$ such that*

$$\begin{bmatrix} x_0, x_1, \ldots, x_p & \\ y_0, y_1, \ldots, y_q & ; f \end{bmatrix} = \frac{1}{p!q!} \frac{\partial^{p+q} f}{\partial x^p \partial y^q} (\xi, \eta). \tag{1.16}$$

2. Main results

Let $m, n \in \mathbb{N}$, $p, q \in \mathbb{N}_0$ and $f : [0, 1 + p] \times [0, 1 + q] \to \mathbb{R}$ be a given function. In what follows, first we shall prove:

Theorem 2.3. *The remainder term of the approximation formula* (1.6) *can be represented under the form:*

$$\left(\widetilde{R}^{(\alpha,\beta,\gamma,\delta)}_{m,p,n,q} f\right)(x, y) \tag{2.17}$$
$$= xy(1-x)(1-y)\frac{(m+p)(n+q)}{(m+\beta)^2(n+\delta)^2}\sum_{k=0}^{m+p-1}\sum_{j=0}^{n+q-1}\widetilde{p}_{m-1,k}(x)\widetilde{p}_{n-1,j}(y)$$
$$\times\left[\begin{array}{l} x, \frac{k+\alpha}{m+\beta}, \frac{k+\alpha+1}{m+\beta} \\ y, \frac{j+\gamma}{n+\delta}, \frac{j+\gamma+1}{n+\delta}\end{array}; f\right] - \frac{m+p}{(m+\beta)^2(n+\delta)} x(1-x)\{(\delta-q)y-\gamma\}\sum_{k=0}^{m+p-1}\sum_{j=0}^{n+q}$$
$$\times\widetilde{p}_{m-1,k}(x)\widetilde{p}_{n,j}(y)\left[\begin{array}{c} x, \frac{k+\alpha}{m+\beta}, \frac{k+\alpha+1}{m+\beta} \\ y, \frac{j+\gamma}{n+\delta}\end{array}; f\right] - \frac{n+q}{(m+\beta)(n+\delta)^2} y(1-y)\{(\beta-p)x-\alpha\}$$
$$\times\sum_{k=0}^{m+p}\sum_{j=0}^{n+q-1}\widetilde{p}_{m,k}(x)\widetilde{p}_{n-1,j}(y)\left[\begin{array}{c} x, \frac{k+\alpha}{m+\beta} \\ y, \frac{j+\gamma}{n+\delta}, \frac{j+\gamma+1}{n+\delta}\end{array}; f\right] + \frac{1}{(m+\beta)(n+\delta)}$$
$$\times\{(\beta-p)x-\alpha\}\{(\delta-q)y-\gamma\}\sum_{k=0}^{m+p}\sum_{j=0}^{n+q}\widetilde{p}_{m,k}(x)\widetilde{p}_{n,j}(y)\left[\begin{array}{c} x, \frac{k+\alpha}{m+\beta} \\ y, \frac{j+\gamma}{n+\delta}\end{array}; f\right],$$

where $x \in [0, 1 + p]\setminus\left\{\frac{k+\alpha}{m+\beta}, k \in \{0, 1, \ldots, m+p\}\right\}$ *and* $y \in [0, 1 + q]\setminus\left\{\frac{j+\gamma}{n+\delta}, j \in \{0, 1, \ldots, n+q\}\right\}$.

Proof. From (1.6) and (1.5) we get,

$$\left(\widetilde{R}^{(\alpha,\beta,\gamma,\delta)}_{m,p,n,q} f\right)(x, y)$$
$$= \sum_{k=0}^{m+p}\sum_{j=0}^{n+q}\widetilde{p}_{m,k}(x)\widetilde{p}_{n,j}(y)\left(x - \frac{k+\alpha}{m+\beta}\right)\left(y - \frac{j+\gamma}{n+\delta}\right)\left[\begin{array}{c} x, \frac{k+\alpha}{m+\beta} \\ y, \frac{j+\gamma}{n+\delta}\end{array}; f\right]$$
$$= \frac{1}{(m+\beta)(n+\delta)}\sum_{k=0}^{m+p}\sum_{j=0}^{n+q}\widetilde{p}_{m,k}(x)\widetilde{p}_{n,j}(y)$$
$$\times\{(m+\beta)x-(k+\alpha)\}\{(n+\delta)y-(j+\gamma)\}\left[\begin{array}{c} x, \frac{k+\alpha}{m+\beta} \\ y, \frac{j+\gamma}{n+\delta}\end{array}; f\right].$$

Next, by using the following simple identities

$$(m+\beta)x - (k+\alpha) = (m+p-k)x + (\beta-p)x - \alpha - k(1-x),$$

$$(n+\delta)y - (j+\gamma) = (n+q-j)y + (\delta-q)y - \gamma - j(1-y),$$

after some elementary computations, by means of the relations (1.13)-(1.15), one obtains

$$\left(\widetilde{R}^{(\alpha,\beta,\gamma,\delta)}_{m,p,n,q}f\right)(x,y)$$

$$= xy(1-x)(1-y)\frac{(m+p)(n+q)}{(m+\beta)^2(n+\delta)^2}\sum_{k=0}^{m+p-1}\sum_{j=0}^{n+q-1}\widetilde{p}_{m-1,k}(x)\widetilde{p}_{n-1,j}(y)$$

$$\times\begin{bmatrix} x, \frac{k+\alpha}{m+\beta}, \frac{k+\alpha+1}{m+\beta} & \\ & ; f \\ y, \frac{j+\gamma}{n+\delta}, \frac{j+\delta+1}{n+\delta} & \end{bmatrix} - \frac{m+p}{(m+\beta)^2(n+\delta)}x(1-x)\{(\delta-q)y-\gamma\}$$

$$\times\sum_{k=0}^{m+p-1}\sum_{j=0}^{n+q}\widetilde{p}_{m-1,k}(x)\widetilde{p}_{n,j}(y)\begin{bmatrix} x, \frac{k+\alpha}{m+\beta}, \frac{k+\alpha+1}{m+\beta} & \\ & ; f \\ y, \frac{j+\gamma}{n+\delta} & \end{bmatrix} - \frac{n+q}{(m+\beta)(n+\delta)^2}$$

$$\times y(1-y)\{(\beta-p)x-\alpha\}\sum_{k=0}^{m+p}\sum_{j=0}^{n+q-1}\widetilde{p}_{m,k}(x)\widetilde{p}_{n-1,j}(y)\begin{bmatrix} x, \frac{k+\alpha}{m+\beta} & \\ & ; f \\ y, \frac{j+\gamma}{n+\delta}, \frac{j+\gamma+1}{n+\delta} & \end{bmatrix}$$

$$+\frac{1}{(m+\beta)(n+\delta)}\{(\beta-p)x-\alpha\}\{(\delta-q)y-\gamma\}$$

$$\times\sum_{k=0}^{m+p}\sum_{j=0}^{n+q}\widetilde{p}_{m,k}(x)\widetilde{p}_{n,j}(y)\begin{bmatrix} x, \frac{k+\alpha}{m+\beta} & \\ & ; f \\ y, \frac{j+\gamma}{n+\delta} & \end{bmatrix},$$

which is the desired equality (2.17). □

Remark 2.1. For $\alpha = \beta = \gamma = \delta = 0$, the GBS Schurer-Stancu approximation formula (1.6) becomes the following Schurer GBS approximation formula:

$$f = \widetilde{U}_{m,p,n,q}f + \widetilde{R}_{m,p,n,q}f. \tag{2.18}$$

By applying Theorem 2.3, the following corollary follows.

Corollary 2.1. *The remainder term of the GBS Schurer approximation formula can be represented under the form:*

$$\left(\widetilde{R}_{m,p,n,q}f\right)(x,y) \tag{2.19}$$

$$= xy(1-x)(1-y)\frac{(m+p)(n+q)}{m^2n^2}\sum_{k=0}^{m+p-1}\sum_{j=0}^{n+q-1}\widetilde{p}_{m-1,k}(x)\widetilde{p}_{n-1,j}(y)\begin{bmatrix} x, \frac{k}{m}, \frac{k+1}{m} & \\ & ; f \\ y, \frac{j}{n}, \frac{j+1}{n} & \end{bmatrix}$$

$$+\frac{m+p}{m^2n}x(1-x)qy\sum_{k=0}^{m+p-1}\sum_{j=0}^{n+q}\widetilde{p}_{m-1,k}\widetilde{p}_{n,j}(y)\begin{bmatrix} x, \frac{k}{m}, \frac{k+1}{m} & \\ & ; f \\ y, \frac{j}{n} & \end{bmatrix}$$

$$+\frac{n+q}{mn^2}y(1-y)px\sum_{k=0}^{m+p}\sum_{j=0}^{n+q-1}\widetilde{p}_{m,k}(x)\widetilde{p}_{n-1,j}(y)\begin{bmatrix} x, \frac{k}{m} & \\ & ; f \\ y, \frac{j}{n}, \frac{j+1}{n} & \end{bmatrix}$$

$$+\frac{pq}{mn}xy\sum_{k=0}^{m+p}\sum_{j=0}^{n+q}\widetilde{p}_{m,k}(x)\widetilde{p}_{n,j}(y)\begin{bmatrix} x, \frac{k}{m} & \\ & ; f \\ y, \frac{j}{n} & \end{bmatrix},$$

where $x \in [0, 1+p]\backslash\left\{\frac{k}{m}, k \in \{0, 1, \ldots, m+p\}\right\}$ *and*

$$y \in [0, 1+q]\backslash\left\{\frac{j}{n}, j \in \{0, 1, \ldots, n+q\}\right\}.$$

Remark 2.2. For $p = q = 0$, the GBS Schurer-Stancu approximation formula (1.6) reduces to the following Stancu GBS approximation formula:

$$f = \widetilde{U}_{m,n}^{(\alpha,\beta,\gamma,\delta)} f + \widetilde{R}_{m,n}^{(\alpha,\beta,\gamma,\delta)} f. \tag{2.20}$$

From Theorem 2.3, we also obtain

Corollary 2.2. *The remainder term of the GBS Stancu approximation formula* (2.20) *can be represented under the form:*

$$\begin{aligned}
&\left(\widetilde{R}_{m,n}^{(\alpha,\beta,\gamma,\delta)} f\right)(x,y) \qquad (2.21)\\
&= xy(1-x)(1-y)\frac{mn}{(m+\beta)^2(n+\delta)^2}\sum_{k=0}^{m-1}\sum_{j=0}^{n-1} p_{m-1,k}(x)p_{n-1,j}(y)\begin{bmatrix} x, \frac{k+\alpha}{m+\beta}, \frac{k+\alpha+1}{m+\beta} & ; f \\ y, \frac{j+\gamma}{n+\delta}, \frac{j+\gamma+1}{n+\delta} & \end{bmatrix}\\
&- \frac{m}{(m+\beta)^2(n+\delta)} x(1-x)(\delta y-\gamma)\sum_{k=0}^{m-1}\sum_{j=0}^{n} p_{m-1,k}(x)p_{n,j}(y)\begin{bmatrix} x, \frac{k+\alpha}{m+\beta}, \frac{k+\alpha+1}{m+\beta} & ; f \\ y, \frac{j+\gamma}{n+\delta} & \end{bmatrix}\\
&- \frac{n}{(m+\beta)(n+\delta)^2} y(1-y)(\beta x-\alpha)\sum_{k=0}^{m}\sum_{j=0}^{n-1} p_{m,k}(x)p_{n-1,j}(y)\begin{bmatrix} x, \frac{k+\alpha}{m+\beta} & ; f \\ y, \frac{j+\gamma}{n+\delta}, \frac{j+\gamma+1}{n+\delta} & \end{bmatrix}\\
&+ \frac{1}{(m+\beta)(n+\delta)} (\beta x-\alpha)(\delta y-\gamma)\sum_{k=0}^{m}\sum_{j=0}^{n} p_{m,k}(x)p_{n,j}(y)\begin{bmatrix} x, \frac{k+\alpha}{m+\beta} & ; f \\ y, \frac{j+\gamma}{n+\delta} & \end{bmatrix},
\end{aligned}$$

where:

$$p_{m,k}(x) = \binom{m}{k} x^k(1-x)^{m-k},\ p_{n,j}(y) = \binom{n}{j} y^j(1-y)^{n-j},$$

are the fundamental Bernstein polynomials and
$x \in [0,1]\setminus\left\{\frac{k+\alpha}{m+\beta}, k \in \{0,1,\dots,m\}\right\}$, $y \in [0,1]\setminus\left\{\frac{j+\gamma}{n+\delta}, j \in \{0,1,\dots,n\}\right\}$.

Remark 2.3. For $p = q = 0$ and $\alpha = \beta = \gamma = \delta = 0$, the GBS approximation formula (1.6) is the classical GBS Bernstein approximation formula:

$$f = U_{m,n} f + R_{m,n} f, \tag{2.22}$$

first considered by E. Dobrescu and I. Matei [7]. For other related results see [1], [10], [11], [12], [13], [14], [15], [16], [17], [21], [23], [26], [27], [28], [29], [32] and [39].

By applying Theorem 2.3, it follows:

Corollary 2.3. *The remainder term of the GBS Bernstein approximation formula* (2.22) *can be represented under the form:*

$$\begin{aligned}
&(R_{m,n} f)(x,y) \qquad (2.23)\\
&= \frac{xy(1-x)(1-y)}{mn}\sum_{k=0}^{m-1}\sum_{j=0}^{n-1} p_{m-1,k}(x)p_{n-1,j}(y)\begin{bmatrix} x, \frac{k}{m}, \frac{k+1}{m} & ; f \\ y, \frac{j}{n}, \frac{j+1}{n} & \end{bmatrix},
\end{aligned}$$

where $x \in [0,1]\setminus\left\{\frac{k}{m}, k \in \{0,1,...,m\}\right\}$ *and* $y \in [0,1]\setminus\left\{\frac{j}{n}, j \in \{0,1,...,n\}\right\}$.

Theorem 2.4. *Let $f : [0, 1+p] \times [0, 1+q] \to \mathbb{R}$ be a function with the property that $f \in C^{(2,2)}([0, 1+p] \times [0, 1+q])$. Then, for any $(x, y) \in [0, 1] \times [0, 1]$ and any $m, n \in \mathbb{N}$, we have*

$$\begin{aligned}
\left|\left(\widetilde{R}^{(\alpha,\beta,\gamma,\delta)}_{m,p,n,q} f\right)(x, y)\right| &\leq \frac{xy(1-x)(1-y)}{4} \, \frac{(m+p)(n+q)}{(m+\beta)^2(n+\delta)^2} \, M_1(f) \\
&+ \frac{x(1-x)|(\delta-q)y-\gamma|}{2} \, \frac{m+p}{(m+\beta)^2(n+\delta)} \, M_2(f) \\
&+ \frac{y(1-y)|(\beta-p)x-\alpha|}{2} \, \frac{n+q}{(m+\beta)(n+\delta)^2} \, M_3(f) \\
&+ |(\beta-p)x-\alpha| \, |(\delta-q)y-\gamma| \frac{1}{(m+\beta)(n+\delta)} \, M_4(f) \\
&\leq \frac{(m+p)(n+q)}{64m^2n^2} \, M_1(f) + \frac{m+p}{8m^2n} \, m_2 M_2(f) \\
&+ \frac{n+q}{8mn^2} \, m_1 M_3(f) + \frac{1}{mn} \, m_1 m_2 M_4(f)
\end{aligned} \tag{2.24}$$

and

$$\left|\left(\widetilde{R}_{m,p,n,q} f\right)(x, y)\right| \leq \frac{(9m+p)(9n+q)}{64m^2n^2} \, M(f), \tag{2.25}$$

where

$$M_1(f) = \sup_{(x,y)\in[0,1+p]\times[0,1+q]} \left| \frac{\partial^4 f}{\partial x^2 \partial y^2}(x, y) \right|,$$

$$M_2(f) = \sup_{(x,y)\in[0,1+p]\times[0,1+q]} \left| \frac{\partial^3 f}{\partial x^2 \partial y}(x, y) \right|,$$

$$M_3(f) = \sup_{(x,y)\in[0,1+p]\times[0,1+q]} \left| \frac{\partial^3 f}{\partial x \partial y^2}(x, y) \right|,$$

$$M_4(f) = \sup_{(x,y)\in[0,1+p]\times[0,1+q]} \left| \frac{\partial^2 f}{\partial x \partial y}(x, y) \right|,$$

$m_1 = \max\{\alpha, |\beta - p - \alpha|\}$, $m_2 = \max\{\gamma, |\delta - q - \gamma|\}$ *and*

$$M(f) = \max\{M_1(f), m_2 M_2(f), m_1 M_3(f), m_1 m_2 M_4(f)\},$$

where $x \in [0, 1+p] \backslash \left\{ \dfrac{k+\alpha}{m+\beta}, k \in \{0, 1, \dots, m+p\} \right\}$ *and*
$y \in [0, 1+q] \backslash \left\{ \dfrac{j+\gamma}{n+\delta}, j \in \{0, 1, \dots, n+q\} \right\}$.

Proof. In (2.17) we apply Theorem 1.2 and we have that
$(\xi_1(k, j), \eta_1(k, j)), (\xi_2(k, j), \eta_2(k, j)), (\xi_3(k, j), \eta_3(k, j)), (\xi_4(k, j), \eta_4(k, j))$

$\in]0, 1+p[\times]0, 1+q[$ exist so that

$$\left(\widetilde{R}_{m,p,n,q}^{(\alpha,\beta,\gamma,\delta)} f\right)(x,y) = xy(1-x)(1-y)\frac{(m+p)(n+q)}{(m+\beta)^2(n+\delta)^2}$$
$$\times \sum_{k=0}^{m+p-1}\sum_{j=0}^{n+q-1} \widetilde{p}_{m-1,k}(x)\widetilde{p}_{n-1,j}(y)\frac{1}{2!2!}\frac{\partial^4 f}{\partial x^2 \partial y^2}(\xi_1(k,j),\eta_1(k,j))$$
$$- \frac{m+p}{(m+\beta)^2(n+\delta)} x(1-x)\{(\delta-q)y-\gamma\}$$
$$\times \sum_{k=0}^{m+p-1}\sum_{j=0}^{n+q} \widetilde{p}_{m-1,k}(x)\widetilde{p}_{n,j}(y)\frac{1}{2!1!}\frac{\partial^3 f}{\partial x^2 \partial y}(\xi_2(k,j),\eta_2(k,j))$$
$$- \frac{n+q}{(m+\beta)(n+\delta)^2} y(1-y)\{(\beta-p)x-\alpha\}$$
$$\times \sum_{k=0}^{m+p}\sum_{j=0}^{n+q-1} \widetilde{p}_{m,k}(x)\widetilde{p}_{n-1,j}(y)\frac{1}{1!2!}\frac{\partial^3 f}{\partial x \partial y^2}(\xi_3(k,j),\eta_3(k,j))$$
$$+ \frac{1}{(m+\beta)(n+\delta)}\{(\beta-p)x-\alpha\}\{(\delta-q)y-\gamma\}$$
$$\times \sum_{k=0}^{m+p}\sum_{j=0}^{n+q} \widetilde{p}_{m,k}(x)\widetilde{p}_{n,j}(y)\frac{\partial^2 f}{\partial x \partial y}(\xi_4(k,j),\eta_4(k,j)).$$

Since $x(1-x) \le \dfrac{1}{4}$, $y(1-y) \le \dfrac{1}{4}$, we obtain

$$\sum_{k=0}^{m+p}\sum_{j=0}^{n+q} \widetilde{p}_{m,k}(x)\widetilde{p}_{n,j}(y) = \sum_{k=0}^{m+p-1}\sum_{j=0}^{n+q} \widetilde{p}_{m-1,k}(x)\widetilde{p}_{n,j}(y)$$
$$= \sum_{k=0}^{m+p}\sum_{j=0}^{n+q-1} \widetilde{p}_{m,k}(x)\widetilde{p}_{n-1,j}(y) = \sum_{k=0}^{m+p-1}\sum_{j=0}^{n+q-1} \widetilde{p}_{m-1,k}(x)\widetilde{p}_{n-1,j}(y) = 1,$$

By taking the absolute value in the relation above and by using the fact that the partial derivatives of f are bounded on $[0,1+p]\times[0,1+q]$, $|(\beta-p)x-\alpha| \le \max\{\alpha, |\beta-p-\alpha|\}$ for any $x\in[0,1]$ and $|(\delta-q)y-\gamma| \le \max\{\gamma, |\delta-q-\gamma|\}$ for any $y\in[0,1]$, we get the inequalities from (2.24). From (2.24), it follows (2.25). □

Remark 2.4. From (1.6) and (2.25) it follows that in the conditions of Theorem 2.2, the sequence $\left(\widetilde{U}_{m,p,n,q} f\right)_{m,n\ge 1}$ converges uniformly to function f on $[0,1]\times[0,1]$.

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