CREAT. MATH. INFORM.
Volume **25** (2016), No. 2,
Pages 165 - 173

Online version at https://creative-mathematics.cunbm.utcluj.ro/
Print Edition: ISSN 1584 - 286X; Online Edition: ISSN 1843 - 441X
DOI: https://doi.org/10.37193/CMI.2016.02.07

An improvement of Batir's asymptotic formula and some estimates related to the gamma function

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ABSTRACT. In this article, we improve Batir's asymptotic formula related to the gamma function. We prove the monotonicity of some functions related to the gamma function.

1. INTRODUCTION

The Euler Gamma function $\Gamma(x)$ is defined for real numbers $x > 0$ by

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt.$$

Batir has established in [3] the following asymptotic formula

$$\Gamma(n+1) \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n\left(1+\frac{1}{3n}+\frac{1}{18n^2}-\frac{2}{405n^3}-\frac{31}{9720n^4}\right)^{\frac{1}{4}}, \tag{1.1}$$

as $n \to \infty$. Batir has showed the following double inequality in [2]

$$\sqrt{(2x+1)}x^x \exp\left(-\left[x+\frac{1}{6\left(x+\frac{3}{8}\right)}-\frac{4}{9}\right]\right) < \Gamma(x+1)$$

$$< \sqrt{(2x+1)}x^x \exp\left(-\left[x+\frac{1}{6\left(x+\frac{3}{8}\right)}\right]\right). \tag{1.2}$$

Qi and Li have refined in [6], [7] the left hand side inequality (1.2) to

$$\left(\frac{x^2+1}{x+1}\right)^\alpha < \Gamma(x+1) < \left(\frac{x^2+1}{x+1}\right)^\beta,$$

for $x \in (0,1)$, $\alpha \geq 2(1-\gamma)$, $\beta \leq \gamma$ and have showed that the function h_1 is completely monotonic on $(0,\infty)$, where $h_\tau : (0,\infty) \to \mathbb{R}$ is given by

$$h_\tau(x) = \begin{cases} \frac{\ln x}{\ln(x^2+\tau)-\ln(x+\tau)}, & \text{if } x \neq 1 \\ (1+\tau), & \text{if } x = 1 \end{cases}$$

and $\gamma = 0,577215...$ is Euler-Mascheroni's constant. Kupán and Szász have studied in [5] the problem of the largest $\tau \in [0,6)$ such that the function

$$f_\tau(x) = \begin{cases} \frac{\ln\Gamma(x+1)}{\ln(x^2+\tau)-\ln(x+\tau)}, & \text{if } x \neq 1 \\ (1+\tau)(1-\gamma), & \text{if } x = 1 \end{cases}$$

Received: 27.11.2015. In revised form: 23.02.2016. Accepted: 29.02.2016
2010 *Mathematics Subject Classification.* 26D15, 41A25, 34E05.
Key words and phrases. *Gamma function, asymptotic series, inequalities.*
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is monotonic. More precisely, it is increasing on $(0,1)$ for $\tau < \frac{6\gamma}{\pi^2-12\gamma} = 1.176..$ and decreasing on $(0,1)$ for $\tau > \frac{\pi^2-6\gamma}{18-12\gamma-\pi^2} = 5.321...$Guo and Qi have proved in [4] that f_1 is strictly increasing on $(0,\infty)$.

2. Improvement of Batir's asymptotic formula

We improve Batir's asymptotic formula (1.1). Let us consider the following asymptotic formula

$$\Gamma(n+1) \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \left(1+\frac{1}{3n}+\frac{1}{18n^2}-\frac{2}{425n^3}-\frac{31}{9720n^4}\right)^{\frac{1}{4}},$$

as $n\to\infty$.We are going to find those real numbers a, b and c for which we obtain the best estimate of the form

$$\Gamma(n+1) \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \left(1+\frac{1}{3n}+\frac{1}{18n^2}-\frac{2}{425n^3}-\frac{31}{9720n^4}\right)^{a+\frac{b}{n^2}+\frac{c}{n^3}}. \tag{2.3}$$

Let us consider the relative error sequence w_n given by

$$\Gamma(n+1) = \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \left(1+\frac{1}{3n}+\frac{1}{18n^2}-\frac{2}{425n^3}-\frac{31}{9720n^4}\right)^{a+\frac{b}{n^2}+\frac{c}{n^3}} \exp w_n. \tag{2.4}$$

By using the logarithm function, we obtain:

$$w_n = \ln n! - \frac{1}{2}\ln(2n) - \frac{1}{2}\ln\pi - n\ln n + n$$
$$-\left(a+\frac{b}{n^2}+\frac{c}{n^3}\right)\ln\left(1+\frac{1}{3n}+\frac{1}{18n^2}-\frac{2}{425n^3}-\frac{31}{9720n^4}\right).$$

By calculating the difference $w_n - w_{n+1}$, we get

$$w_n - w_{n+1} = -\ln(n+1) - \frac{1}{2}\ln\frac{n}{n+1} - n\ln n + (n+1)\ln(n+1) - 1$$
$$-\left(a+\frac{b}{n^2}+\frac{c}{n^3}\right)\ln\left(1+\frac{1}{3n}+\frac{1}{18n^2}-\frac{2}{425n^3}-\frac{31}{9720n^4}\right)$$
$$+\left(a+\frac{b}{(n+1)^2}+\frac{c}{(n+1)^3}\right)$$
$$\cdot\ln\left(1+\frac{1}{3(n+1)}+\frac{1}{18(n+1)^2}-\frac{2}{425(n+1)^3}-\frac{31}{9720(n+1)^4}\right),$$

or

$$w_n - w_{n+1} = \left(-\frac{1}{3}a+\frac{1}{12}\right)\frac{1}{n^2}+\left(\frac{1}{3}a-\frac{1}{12}\right)\frac{1}{n^3}$$
$$+\left(-\frac{6901}{22\,950}a-b+\frac{3}{40}\right)\frac{1}{n^4}+\left(\frac{27\,716}{103\,275}a+2b-\frac{4}{3}c-\frac{1}{15}\right)\frac{1}{n^5}$$
$$+\left(-\frac{113\,173}{495\,720}a-\frac{45\,151}{13\,770}b+\frac{10}{3}c+\frac{5}{84}\right)\frac{1}{n^6}+O\left(\frac{1}{n^7}\right). \tag{2.5}$$

In order to get the best approximation of the form (2.3), we need as much as possible null coefficients of $\frac{1}{n^2}, \frac{1}{n^3}, \frac{1}{n^4}, \ldots$, such that the difference (2.5) $w_n - w_{n+1}$ would have the highest possible convergence speed. This leads us to $a=\frac{1}{4}$, $b=-\frac{2}{11\,475}$ and $c=\frac{2}{34\,425}$. Now, we introduce two terms d and g in the approximation (2.4) to obtain :

$$\Gamma(n+1) = \left(1+\frac{1}{3n}+\frac{1}{18n^2}-\frac{2}{425n^3}-\frac{31}{9720n^4}+\frac{d}{n^5}+\frac{g}{n^6}\right)^{\frac{1}{4}-\frac{2}{11\,475n^2}+\frac{2}{34\,425n^3}}$$

$$\cdot\sqrt{2\pi n}\left(\frac{n}{e}\right)^{n}\exp w_{n}. \tag{2.6}$$

We have:

$$w_n = \ln n! - \frac{1}{2}\ln(2n) - \frac{1}{2}\ln\pi - n\ln n + n$$
$$-\left(\frac{1}{4}-\frac{2}{11\,475n^2}+\frac{2}{34\,425n^3}\right)\ln\left(1+\frac{1}{3n}+\frac{1}{18n^2}-\frac{2}{425n^3}-\frac{31}{9720n^4}+\frac{d}{n^5}+\frac{g}{n^6}\right),$$

then

$$w_n - w_{n+1} = -\ln(n+1) - \frac{1}{2}\ln\frac{n}{n+1} - n\ln n + (n+1)\ln(n+1) - 1$$
$$-\left(\frac{1}{4}-\frac{2}{11\,475n^2}+\frac{2}{34\,425n^3}\right)\ln\left(1+\frac{1}{3n}+\frac{1}{18n^2}-\frac{2}{425n^3}-\frac{31}{9720n^4}+\frac{d}{n^5}+\frac{g}{n^6}\right)$$
$$+\left(\frac{1}{4}-\frac{2}{11\,475(n+1)^2}+\frac{2}{34\,425(n+1)^3}\right)$$
$$\cdot\ln\left(1+\frac{1}{3(n+1)}+\frac{1}{18(n+1)^2}-\frac{2}{425(n+1)^3}-\frac{31}{9720(n+1)^4}+\frac{d}{(n+1)^5}+\frac{g}{(n+1)^6}\right).$$

Then, we obtain

$$w_n - w_{n+1} = \left(-\frac{5}{4}d+\frac{56\,876\,999}{17\,697\,204\,000}\right)\frac{1}{n^6}+\left(\frac{17}{4}d-\frac{3}{2}g-\frac{413\,919\,781}{44\,243\,010\,000}\right)\frac{1}{n^7}$$
$$+\left(-\frac{972\,713}{91\,800}d+\frac{35}{6}g+\frac{3937\,201\,397}{227\,535\,480\,000}\right)\frac{1}{n^8}+O\left(\frac{1}{n^9}\right). \tag{2.7}$$

By the same procedure, we obtain $d = \frac{56\,876\,999}{22\,121\,505\,000}$, $g = \frac{139\,069\,421}{132\,729\,030\,000}$. Thus, we have obtained the following approximation for the Euler Gamma function:

$$\Gamma(x+1) \sim \sqrt{2\pi x}\left(\frac{x}{e}\right)^{x}(1+\frac{1}{3x}+\frac{1}{18x^2}-\frac{2}{425x^3}-\frac{31}{9720x^4}$$
$$+\frac{56\,876\,999}{22\,121\,505\,000x^5}+\frac{139\,069\,421}{132\,729\,030\,000x^6})^{\frac{1}{4}-\frac{2}{11\,475x^2}+\frac{2}{34\,425x^3}}.$$

We are now able to state and prove the main results of this paper.

Theorem 2.1. For all integers $n \geq 1$ the following inequality holds:

$$\Gamma(n+1) < \sqrt{2\pi n}\left(\frac{n}{e}\right)^{n}\left(1+\frac{1}{3n}+\frac{1}{18n^2}-\frac{2}{425n^3}-\frac{31}{9720n^4}\right)^{\frac{1}{4}+\frac{2}{34\,425n^3}}. \tag{2.8}$$

Proof. To show the inequality (2.8) we consider the sequence $(a_n)_{n\geq 1}$ given by

$$a_n = \ln\Gamma(n+1) - \frac{1}{2}\ln 2\pi - \frac{1}{2}\ln n - n\ln n + n - \left(\frac{1}{4}+\frac{2}{34\,425n^3}\right)$$
$$\cdot\ln\left(1+\frac{1}{3n}+\frac{1}{18n^2}-\frac{2}{425n^3}-\frac{31}{9720n^4}\right)$$

that converges to zero and we prove that $a_n < 0$ for all integers $n \geq 1$.It suffices that a_n is strictly increasing.We have

$$f(n) = a_{n+1} - a_n$$

where

$$f(x) = \ln(x+1) - \frac{1}{2}\ln\frac{x+1}{x} + 1 - (x+1)\ln(x+1) + x\ln x - \left(\frac{1}{4}+\frac{2}{34\,425(x+1)^3}\right)$$

$$\cdot \ln\left(1+\frac{1}{3\left(n+1\right)}+\frac{1}{18\left(n+1\right)^2}-\frac{2}{425\left(n+1\right)^3}-\frac{31}{9720\left(n+1\right)^4}\right)$$
$$+\left(\frac{1}{4}+\frac{2}{34\,425n^3}\right)\ln\left(1+\frac{1}{3n}+\frac{1}{18n^2}-\frac{2}{425n^3}-\frac{31}{9720n^4}\right).$$

We have

$$f'(x)=\ln x-\ln\left(x+1\right)+\frac{1}{x+1}+\frac{1}{2x+2x^2}$$
$$-\frac{2}{11\,475x^4}\ln\left(\frac{1}{3x}+\frac{1}{18x^2}-\frac{2}{425x^3}-\frac{31}{9720x^4}+1\right)$$
$$+\frac{2}{11\,475\left(4x+6x^2+4x^3+x^4+1\right)}$$
$$\cdot\ln(\frac{1}{3x+3}+\frac{1}{36x+18x^2+18}-\frac{2}{1275x+1275x^2+425x^3+425}$$
$$-\frac{31}{38\,880x+58\,320x^2+38\,880x^3+9720x^4+9720}+1)$$
$$+\frac{1}{1580\,107\,500x^6-133\,844\,400x^5-90\,709\,875x^4+9480\,645\,000x^7+28\,441\,935\,000x^8}$$
$$\cdot(23\,328x-183\,600x^2+90\,159\,075x^3+100\,383\,300x^4-790\,053\,750x^5$$
$$-2370\,161\,250x^6+21\,080)+(17\,499\,569\,697x+42\,580\,362\,825x^2$$
$$+54\,812\,070\,225x^3+39\,402\,304\,200x^4+15\,011\,021\,250x^5+2370\,161\,250x^6$$
$$+2969\,811\,817)(302\,348\,578\,500x+1017\,286\,634\,250x^2+1954\,471\,801\,500x^3$$
$$+2345\,699\,705\,625x^4+1801\,188\,705\,600x^5+864\,318\,802\,500x^6$$
$$+237\,016\,125\,000x^7+28\,441\,935\,000x^8+39\,278\,133\,225)^{-1} \tag{2.9}$$

and

$$f''(x)=h_1(x)+\frac{8}{11\,475x^5}\ln\left(\frac{1}{3x}+\frac{1}{18x^2}-\frac{2}{425x^3}-\frac{31}{9720x^4}+1\right)$$
$$-\frac{8}{11\,475\left(5x+10x^2+10x^3+5x^4+x^5+1\right)}\ln(\frac{1}{3x+3}+\frac{1}{36x+18x^2+18}$$
$$-\frac{2}{1275x+1275x^2+425x^3+425}$$
$$-\frac{31}{38\,880x+58\,320x^2+38\,880x^3+9720x^4+9720}+1)$$

where the function $h_1:[1,\infty)\to\mathbb{R}$ is given by

$$h_1(x)=\frac{1}{x}-\frac{1}{x+1}-\frac{1}{x^2+x^3}-\frac{1}{2x+x^2+1}+\frac{1}{2x^2+2x^3}-\frac{1}{2x+4x^2+2x^3}+...$$

We have:

$$f_1(x)<x, \tag{2.10}$$
$$f_1(x)>x-\frac{x^2}{2} \tag{2.11}$$

for all real numbers x with $|x|<1, x\neq -1$. We apply the inequality (2.10) for the function $f_2:[1,\infty)\to\mathbb{R}$ given by

$$f_2(x)=\ln(\frac{1}{3x+3}+\frac{1}{36x+18x^2+18}-\frac{2}{1275x+1275x^2+425x^3+425}$$
$$-\frac{31}{38\,880x+58\,320x^2+38\,880x^3+9720x^4+9720}+1)$$

and we get

$$\ln(\frac{1}{3x+3}+\frac{1}{36x+18x^2+18}-\frac{2}{1275x+1275x^2+425x^3+425}$$
$$-\frac{31}{38\,880x+58\,320x^2+38\,880x^3+9720x^4+9720}+1)<(\frac{1}{3x+3}+\frac{1}{36x+18x^2+18}$$
$$-\frac{2}{1275x+1275x^2+425x^3+425}-\frac{31}{38\,880x+58\,320x^2+38\,880x^3+9720x^4+9720}). \tag{2.12}$$

We apply the inequality (2.11) for the function $f_3:[1,\infty)\to\mathbb{R}$ given by

$$f_3(x)=\ln\left(1+\frac{1}{3x}+\frac{1}{18x^2}-\frac{2}{425x^3}-\frac{31}{9720x^4}\right)$$

and we obtain

$$\ln\left(1+\frac{1}{3x}+\frac{1}{18x^2}-\frac{2}{425x^3}-\frac{31}{9720x^4}\right)$$
$$>\left(\left(\frac{1}{3x}+\frac{1}{18x^2}-\frac{2}{425x^3}-\frac{31}{9720x^4}\right)-\frac{\left(\frac{1}{3x}+\frac{1}{18x^2}-\frac{2}{425x^3}-\frac{31}{9720x^4}\right)^2}{2}\right). \tag{2.13}$$

Using (2.9),(2.12) and (2.13) we get that

$$f''(x)>h_1(x)+\frac{8}{11\,475x^5}$$
$$\cdot\left(\left(\frac{1}{3x}+\frac{1}{18x^2}-\frac{2}{425x^3}-\frac{31}{9720x^4}\right)-\frac{\left(\frac{1}{3x}+\frac{1}{18x^2}-\frac{2}{425x^3}-\frac{31}{9720x^4}\right)^2}{2}\right)$$
$$-\frac{8}{11\,475\,(5x+10x^2+10x^3+5x^4+x^5+1)}(\frac{1}{3x+3}+\frac{1}{36x+18x^2+18}$$
$$-\frac{2}{1275x+1275x^2+425x^3+425}$$
$$-\frac{31}{38\,880x+58\,320x^2+38\,880x^3+9720x^4+9720}+1).$$

We consider the function $h:[1,\infty)\to\mathbb{R}$ given by

$$h(x)=h_1(x)+\frac{8}{11\,475x^5}$$
$$\cdot\left(\left(\frac{1}{3x}+\frac{1}{18x^2}-\frac{2}{425x^3}-\frac{31}{9720x^4}\right)-\frac{\left(\frac{1}{3x}+\frac{1}{18x^2}-\frac{2}{425x^3}-\frac{31}{9720x^4}\right)^2}{2}\right)$$
$$-\frac{8}{11\,475\,(5x+10x^2+10x^3+5x^4+x^5+1)}(\frac{1}{3x+3}+\frac{1}{36x+18x^2+18}$$
$$-\frac{2}{1275x+1275x^2+425x^3+425}$$
$$-\frac{31}{38\,880x+58\,320x^2+38\,880x^3+9720x^4+9720})$$

that can be represented as

$$h(x)=\frac{R(x-6)}{1958\,227\,224\,750\,000x^{13}}$$
$$\cdot(x+1)^{-9}\left(45\,900x^2-3888x+275\,400x^3+826\,200x^4-2635\right)^{-2}$$
$$\cdot\left(4218\,912x+5829\,300x^2+3580\,200x^3+826\,200x^4+1140\,977\right)^{-2} \tag{2.14}$$

where $R(x)$ is a polynomial with positive coefficients for all real numbers $x \geq 6$ .Then from (2.14) we get

$$f''(x) > 0$$

for all real numbers $x \geq 1$ that is equivalent to f' is strictly increasing and using that $\lim_{x\to\infty} f'(x) = 0$, it follows that $f'(x) < 0$ for all real numbers $x \geq 1$. Then we get that f is strictly decreasing and using that $\lim_{x\to\infty} f(x) = 0$, it follows that $f(x) > 0$ for all real numbers $x \geq 1$ or a_n is strictly increasing.This concludes the proof for the inequality (2.8). □

Theorem 2.2. We have the following inequality for all real numbers $x \geq 2$:

$$\sqrt{2\pi x}\left(\frac{x}{e}\right)^x \cdot (1 + \frac{1}{3x} - \frac{2}{425x^3} - \frac{31}{9720x^4} + \frac{56\,876\,999}{22\,121\,505\,000x^5} + \frac{139\,069\,421}{132\,729\,030\,000x^6})^{\frac{1}{4}} < \Gamma(x+1). \tag{2.15}$$

Proof. To show the inequality (2.15) we consider the function $H : [1, \infty) \to \mathbb{R}$ given by

$$H(x) = \ln\Gamma(x+1) - \frac{1}{2}\ln 2\pi - \frac{1}{2}\ln x - x\ln x + x - \frac{1}{4}\ln\left(1 + \frac{1}{3x} - \frac{2}{425x^3} - \frac{31}{9720x^4} + \frac{56\,876\,999}{22\,121\,505\,000x^5} + \frac{139\,069\,421}{132\,729\,030\,000x^6}\right)$$

and we prove that $H(x) > 0$ for all real numbers $x \geq 1$.We have

$$H'(x) = \psi(x) + \frac{1}{2x} - \ln x + (853\,154\,985x - 846\,625\,500x^2 - 936\,910\,800x^3 + 7373\,835\,000x^4 + \left(853\,154\,985x - 846\,625\,500x^2 - 936\,910\,800x^3 + 22\,121\,505\,000x^5 + 417\,208\,263\right) \cdot (278\,138\,842x + 682\,523\,988x^2 - 846\,625\,500x^3 - 1249\,214\,400x^4 + 88\,486\,020\,000x^6 + 265\,458\,060\,000x^7)^{-1}$$

and

$$H''(x) = \psi'(x) - \frac{1}{x} - \frac{1}{2x^2} + h_2(x), \tag{2.16}$$

where the function $h_2 : [1, \infty) \to \mathbb{R}$ is given by

$$h_2(x) = (853\,154\,985x - 846\,625\,500x^2 - ....$$

Knowing that [1, p. 260, Rel. 6.4.12]

$$\psi'(x) - \frac{1}{x} - \frac{1}{2x^2} > \frac{1}{6x^3} - \frac{1}{30x^5}$$

and using (2.16), we get

$$H''(x) > \frac{1}{6x^3} - \frac{1}{30x^5} + h_2(x).$$

The function $f_4 : [1, \infty) \to \mathbb{R}$ given by

$$f_4(x) = \frac{1}{6x^3} - \frac{1}{30x^5} + h_2(x)$$

can be represented as

$$f_4(x) = \frac{T_1(x-2)}{90\,x^5\,(T_2(x))^2} > 0, \tag{2.17}$$

where $T_1(x)$ is a nonnegative polynomial for all real numbers $x \geq 2$ and $T_2(x)$ is a polynomial for all real numbers $x \geq 1$ .Using (2.17), it follows that

$$H''(x) > 0$$

for all real numbers $x \geq 2$ that is equivalent to H' is strictly increasing and using that $\lim_{x\to\infty} H(x) = 0$, it follows that $H'(x) < 0$ for all real numbers $x \geq 2$.Then we obtain that H is strictly decreasing and using that $\lim_{x\to\infty} H(x) = 0$, it follows that $H(x) > 0$ for all real numbers $x \geq 2$.This concludes the proof for the inequality (2.15). □

3. Comparison tests

We demonstrate the following proposition:

Proposition 3.1. For all real numbers $x \geq 6$ we have the following inequality:

$$\sqrt{2\pi x}\left(\frac{x}{e}\right)^x (1+\frac{1}{3x}+\frac{1}{18x^2}-\frac{2}{425x^3}-\frac{31}{9720x^4}$$
$$+\frac{56\,876\,999}{22\,121\,505\,000x^5}+\frac{139\,069\,421}{132\,729\,030\,000x^6})^{\frac{1}{4}-\frac{2}{11\,475x^2}+\frac{2}{34\,425x^3}}$$
$$<\sqrt{2\pi x}\left(\frac{x}{e}\right)^x\left(1+\frac{1}{3x}+\frac{1}{18x^2}-\frac{2}{425x^3}-\frac{31}{9720x^4}\right)^{\frac{1}{4}}. \tag{3.18}$$

Proof. The inequality (3.18) is equivalent to $p < 0$, where

$$p(x) = \frac{1}{2}\ln 2\pi + \frac{1}{2}\ln x + x\ln x - x$$
$$+\left(\frac{1}{4}-\frac{2}{11\,475x^2}+\frac{2}{34\,425x^3}\right)\ln(1+\frac{1}{3x}-\frac{2}{425x^3}-\frac{31}{9720x^4}+\frac{56\,876\,999}{22\,121\,505\,000x^5}$$
$$+\frac{139\,069\,421}{132\,729\,030\,000x^6})-\frac{1}{2}\ln 2\pi-\frac{1}{2}\ln x - x\ln x + x$$
$$-\frac{1}{4}\ln\left(1+\frac{1}{3x}+\frac{1}{18x^2}-\frac{2}{425x^3}-\frac{31}{9720x^4}\right).$$

We have

$$p'(x) = \frac{22\,950x^2-2916x+68\,850x^3-2635}{45\,900x^3-3888x^2-2635x+275\,400x^4+826\,200x^5}$$
$$+\frac{2\,(4x-2)}{11\,475x^4}\ln(1+\frac{1}{3x}+\frac{1}{18x^2}-\frac{2}{425x^3}-\frac{31}{9720x^4}+\frac{56\,876\,999}{22\,121\,505\,000x^5}$$
$$+\frac{139\,069\,421}{132\,729\,030\,000x^6})+1062\,586\,144x+9082\,907\,880x^2-4791\,616\,013\,925x^3$$
$$-9797\,448\,739\,275x^4+9656\,036\,932\,500x^5+10\,928\,023\,470\,000x^6$$
$$-253\,844\,269\,875\,000x^8-1112\,555\,368$$
$$\cdot(3191\,643\,211\,950x^4+7831\,962\,762\,300x^5-9715\,027\,612\,500x^6-14\,334\,735\,240\,000x^7$$
$$+1015\,377\,079\,500\,000x^9+3046\,131\,238\,500\,000x^{10})^{-1}. \tag{3.19}$$

We apply the inequality (2.11) for the function $f_5 : [1,\infty) \to \mathbb{R}$ given by

$$f_5(x) = \ln\left(1+\frac{1}{3x}+\frac{1}{18x^2}-\frac{2}{425x^3}-\frac{31}{9720x^4}+\frac{56\,876\,999}{22\,121\,505\,000x^5}+\frac{139\,069\,421}{132\,729\,030\,000x^6}\right)$$

and we obtain

$$\ln\left(1+\frac{1}{3x}+\frac{1}{18x^2}-\frac{2}{425x^3}-\frac{31}{9720x^4}+\frac{56\,876\,999}{22\,121\,505\,000x^5}+\frac{139\,069\,421}{132\,729\,030\,000x^6}\right)$$
$$>\left(\frac{1}{3x}+\frac{1}{18x^2}-\frac{2}{425x^3}-\frac{31}{9720x^4}+\frac{56\,876\,999}{22\,121\,505\,000x^5}+\frac{139\,069\,421}{132\,729\,030\,000x^6}\right)$$
$$-\frac{\left(\frac{1}{3x}+\frac{1}{18x^2}-\frac{2}{425x^3}-\frac{31}{9720x^4}+\frac{56\,876\,999}{22\,121\,505\,000x^5}+\frac{139\,069\,421}{132\,729\,030\,000x^6}\right)^2}{2}. \tag{3.20}$$

Using (3.20) and (3.19), we get

$$p'(x) > f_6(x)$$

where the function $f_6 : [1, \infty) \to \mathbb{R}$ given by

$$f_6(x) = \frac{22\,950x^2 - 2916x + 68\,850x^3 - 2635}{45\,900x^3 - 3888x^2 - 2635x + 275\,400x^4 + 826\,200x^5}$$

$$+\frac{2\,(4x-2)}{11\,475x^4}((\frac{1}{3x} + \frac{1}{18x^2} - \frac{2}{425x^3} - \frac{31}{9720x^4} + \frac{56\,876\,999}{22\,121\,505\,000x^5}$$

$$+\frac{139\,069\,421}{132\,729\,030\,000x^6}) - \frac{1}{2}(\frac{1}{3x} + \frac{1}{18x^2} - \frac{2}{425x^3} - \frac{31}{9720x^4} + \frac{56\,876\,999}{22\,121\,505\,000x^5}$$

$$+\frac{139\,069\,421}{132\,729\,030\,000x^6})^2) + 1062\,586\,144x + 9082\,907\,880x^2 - 4791\,739\,393\,125x^3$$

$$-9797\,448\,739\,275x^4 + 9656\,036\,932\,500x^5 + 10\,928\,023\,470\,000x^6$$

$$-253\,844\,269\,875\,000x^8 - 1112\,555\,368)$$

$$\cdot(3191\,643\,211\,950x^4 + 7831\,962\,762\,300x^5 - 9715\,027\,612\,500x^6 - 14\,334\,735\,240\,000x^7$$

$$+1015\,377\,079\,500\,000x^9 + 3046\,131\,238\,500\,000x^{10})^{-1},$$

can be represented as

$$f_6(x) = \frac{P(x-6)}{202\,155\,022\,269\,401\,827\,500\,000\,000x^{16}Q(x-1)Q_1(x-1)} > 0, \tag{3.21}$$

where $P(x)$, $Q(x)$ and $Q_1(x)$ are polynomials with positive coefficients, for all real numbers $x \geq 6, x \geq 1$ and $x \geq 1$ respectively. By using (3.21) we get

$$p'(x) > 0$$

for all real numbers $x \geq 6$.Then we obtain that p is strictly increasing and using that $\lim_{x\to\infty} p(x) = 0$, it follows that $p(x) < 0$ for all real numbers $x \geq 6$.This concludes the proof for the inequality (3.18). □

We remark that our asymptotic formula

$$\Gamma(n+1) \approx \eta_n = \sqrt{2\pi n}\left(\frac{n}{e}\right)^n (1 + \frac{1}{3n} + \frac{1}{18n^2} - \frac{2}{425n^3} - \frac{31}{9720n^4}$$

$$+\frac{56\,876\,999}{22\,121\,505\,000n^5} + \frac{139\,069\,421}{132\,729\,030\,000n^6})^{\frac{1}{4} - \frac{2}{11\,475n^2} + \frac{2}{34\,425n^3}}$$

estimates better the following asymptotic formula

$$\Gamma(n+1) \approx \beta_n = \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \left(1 + \frac{1}{3n} + \frac{1}{18n^2} - \frac{2}{425n^3} - \frac{31}{9720n^4}\right)^{\frac{1}{4}}$$

or Batir's asymptotic formula (1.1)

$$\Gamma(n+1) \approx \delta_n = \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \left(1 + \frac{1}{3n} + \frac{1}{18n^2} - \frac{2}{405n^3} - \frac{31}{9720n^4}\right)^{\frac{1}{4}}$$

as it can be seem from the below table, in the case of some particular values of n:

n	$\beta_n - \Gamma(n+1)$	$\eta_n - \Gamma(n+1)$	$\lvert\delta_n - \Gamma(n+1)\rvert$
50	$1.397\,88 \times 10^{55}$	$2.129\,03 \times 10^{49}$	$6.312\,61 \times 10^{52}$
100	$5.397\,91 \times 10^{147}$	$5.104\,47 \times 10^{140}$	$6.050\,49 \times 10^{144}$
1000	$2.336\,96 \times 10^{2554}$	$2.242\,63 \times 10^{2543}$	$2.607\,27 \times 10^{2549}$
5000	$1.965\,22 \times 10^{16310}$	$2.513\,06 \times 10^{16300}$	$8.764\,56 \times 10^{16303}$
10000	$1.653\,55 \times 10^{35643}$	$3.494\,23 \times 10^{35634}$	$1.808\,88 \times 10^{35636}$
40000	$1.898\,76 \times 10^{166695}$	$1.257\,64 \times 10^{166689}$	$1.268\,04 \times 10^{166689}$

Acknowledgements. The authors would like to thank Prof. Cristinel Mortici for suggesting the problem and for his guidance through out the progress of this work.

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