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An elementary proof for $\sin x = x - \frac{x^3}{6} + o(x^3)$

RADU GOLOGAN

ABSTRACT. Using only elementary trigonometrical calculations we prove the power series development for the sin and cos functions up to the terms of power three and four respectively.

The history of power series expantions is strongly connected with the developpement of mathematical analysis. In his PhD thesis in 1949, J. E. Westberry ([2]) says: "The development of functions in series first attracted the attention of mathematicians in the seventeenth century. Sir Isaac Newton (1669), James Gregory (1671), and Nicolaus Mercator (1667) were among the first to make such developments. Brook Taylor published his formula for the expansion of a function into a power series in the early eighteenth century, and a short time later Colin Maclaurin published a similar formula for the expansion of a function about the origin".

Well-known facts. By elementary geometry on the unit circle one gets $\sin x < x < \tan x$. These imply the following:

Remark 0.1. *For* $x \in (0, \frac{\pi}{2})$ *we have* $\cos x < \dfrac{\sin x}{x} < 1$.

This immediately imply that we have $\sin x = x + o(x)$ and $1 - \cos x = 2\sin^2 \dfrac{x}{2} = 2\left(\dfrac{x}{2} + o(x)\right)^2 = \dfrac{x^2}{2} + o(x^2)$ that is $\cos x = 1 - \dfrac{x^2}{2} + o(x^2)$.

Using the simple geometric idea that an arc is well approximated by the sum of the chords of its dyadic division in equal parts, we shall give a completely elementary proof of the following weak version of the power series development of the sinus and cosinus functions.

Proposition 0.1. *For sufficiently small* x *we have* $\sin x = x - \dfrac{x^3}{6} + o(x^3)$ *and* $\cos x = \dfrac{x^2}{2} - \dfrac{x^4}{24} + o(x^4)$.

Proof. We shall prove the following equivalent form of the result

$$\lim_{x\to 0} \frac{x - \sin x}{x^3} = \frac{1}{6}.$$

It is easy to see that it is sufficient to consider the case when x is positive. Fix an arbitrary positive integer n. We can write

$$x - \sin x = x - 2^n \sin \frac{x}{2^n} \cos \frac{x}{2} \cos \frac{x}{2^2} \cdots \cos \frac{x}{2^n} =$$

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$$= x - x\cos\frac{x}{2}\cos\frac{x}{2^2}\cdots\cos\frac{x}{2^n} + (x - 2^n\sin\frac{x}{2^n})\cos\frac{x}{2}\cos\frac{x}{2^2}\cdots\cos\frac{x}{2^n}.$$

Using $\cos x = 1 - \frac{x^2}{2} + o(x^2)$ the first difference in the last term is, modulo an $0(x^3)$,

$$x\left(1-\left(1-\frac{x^2}{4\cdot 2}\right)\left(1-\frac{x^2}{4^2\cdot 2}\right)\cdots\left(1-\frac{x^2}{4^n\cdot 2}\right)\right) =$$

$$= x^3\left(\frac{1}{4\cdot 2}+\frac{1}{4^2\cdot 2}+\cdots+\frac{1}{4^n\cdot 2}\right) = \frac{x^3}{6}\left(1-\frac{1}{4^n}\right).$$

Thus

$$\lim_{x\to 0}\frac{x - x\cos\frac{x}{2}\cos\frac{x}{2^2}\cdots\cos\frac{x}{2^n}}{x^3} = \frac{1}{6}\left(1-\frac{1}{4^n}\right).$$

In what concerns the second term, due to the Remark, we have that for sufficiently small x,

$$0 \le \left(x - 2^n\sin\frac{x}{2^n}\right)\cos\frac{x}{2}\cos\frac{x}{2^2}\cdots\cos\frac{x}{2^n} \le x\left(1-\cos\frac{x}{2^n}\right)$$

$$= 2x\sin^2\frac{x}{2^{n+1}} \le \frac{x^3}{4^{n+1}}.$$

We thus get

$$0 \le \limsup_{x\to 0}\frac{(x-2^n\sin\frac{x}{2^n})\cos\frac{x}{2}\cos\frac{x}{2^2}\cdots\cos\frac{x}{2^n}}{x^3} \le \frac{1}{4^{n+1}}.$$

Collecting all, we obtain

$$\liminf_{x\to 0}\frac{x-\sin x}{x^3} \ge \frac{1}{6}\left(1-\frac{1}{4^n}\right)$$

and

$$\limsup_{x\to 0}\frac{x-\sin x}{x^3} \le \frac{1}{6}\left(1-\frac{1}{4^n}\right) + \frac{1}{4^{n+1}}.$$

As n was an arbitrary positive integer, the last two inequilies give the result.

As for the development for the cosinus, we write $1 - \cos x = 2\sin^2\frac{x}{2} = = 2\left(\frac{x}{2} - \frac{x^3}{48} + o(x^3)\right) = \frac{x^2}{2} - \frac{x^4}{24} + o(x^4).$ □

Remark 0.2. It should be interesting to find elementary arguments for the proof for higher order terms in the power series of the trigonometrical functions.

References

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Institute of Mathematics Bucharest
Romanian Academy
P.O. Box 1-764, 014700, Bucharest Romania

University Politehnica Bucharest
Splaiul Independentei 313
Bucharest, Romania
Email address: radu.gologan@imar.ro; rgolog@yahoo.com