CREAT. MATH. INFORM.
Volume **25** (2016), No. 2,
Pages 183 - 196

Online version at `https://creative-mathematics.cunbm.utcluj.ro/`
Print Edition: ISSN 1584 - 286X; Online Edition: ISSN 1843 - 441X
DOI: https://doi.org/10.37193/CMI.2016.02.10

Iterative methods for a fixed point of hemicontractive-type mapping and a solution of a variational inequality problem

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ABSTRACT. In this paper, we introduce and study an iterative process for finding a common point of the fixed point set of a Lipschitz hemicontractive-type multi-valued mapping and the solution set of a variational inequality problem for a monotone mapping. Our results improve and extend most of the results that have been proved for this class of nonlinear mappings.

1. INTRODUCTION

Let H be a real Hilbert space with inner product $\langle .,. \rangle$ and norm $\|.\|$. Let C be a nonempty subset of H. A mapping $T : C \to H$ is called *Lipschitzian* if there exists $L \geq 0$ such that $||Tx - Ty|| \leq L||x - y||\ \forall x, y \in C$. If $L = 1$ then T is called *nonexpansive* and if $L \in [0, 1)$ then T is called a *contraction*. A mapping $T : C \longrightarrow H$ is called pseudocontractive if

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + \|x - Tx - (y - Ty)\|^2 \text{ for all } x, y \in C. \tag{1.1}$$

We know that the class of pseudocontractive mappings is a more general class of mappings in the sense that it includes the class of nonexpansive and hence the class of contraction mappings (see, [2]).

Let $CB(C)$ denotes the family of nonempty closed bounded subsets of C. The Pompeiu-Hausdorff metric ([1]) on $CB(C)$ is defined by

$$D(A, B) = max\{\sup_{x\in A} d(x, B), \sup_{y\in B} d(y, A)\},$$

for all $A, B \in CB(C)$, where $d(x, B) = \inf\{\|x - b\| : b \in B\}$.
A multi-valued mapping $T : C \longrightarrow CB(C)$ is said to be nonexpansive if

$$D(Tx, Ty) \leq \|x - y\| \quad \forall x, y \in C.$$

T is called $k-$strictly pseudocontractive if there exists $k \in [0, 1)$ such that

$$D^2(Tx, Ty) \leq \|x - y\|^2 + k\|(x - u) - (y - v)\|^2, \tag{1.2}$$

for all $x, y \in C$ and $u \in Tx, v \in Ty$. If in (1.2), $k = 1$, then T is called pseudocontractive.

An element $x \in C$ is called a fixed point of $T : C \longrightarrow C$ (resp., $T : C \longrightarrow CB(C)$) if $x = Tx$ (resp., $x \in Tx$). The set of fixed points of T is denoted by $F(T)$. We write $x_n \rightharpoonup x$ to indicate that the sequence $\{x_n\}$ converges weakly to x and $x_n \to x$ to indicate that the sequence $\{x_n\}$ converges strongly to x.

A multi-valued mapping $T : C \longrightarrow CB(C)$ is said to be hemicontractive-type if $F(T) \neq \emptyset$ and for all $p \in F(T)$, $x \in C$,

$$D^2(Tx, Tp) \leq \|x - p\|^2 + \|x - u\|^2,\ \forall u \in Tx. \tag{1.3}$$

Received: 08.03.2016. In revised form: 26.05.2016. Accepted: 03.06.2016
2010 *Mathematics Subject Classification.* 47H09, 47H10, 49J40, 49J53.
Key words and phrases. *Fixed points of mappings, metric projection, monotone mapping, Hemicontractive multi-valued mapping, strong convergence .*
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T is called demicontractive if $F(T) \neq \emptyset$, and there exists $k \in [0,1)$ such that for all $p \in F(T)$, $x \in C$

$$D^2(Tx, Tp) \leq \|x-p\|^2 + k\|x-u\|^2, \ \forall u \in Tx. \tag{1.4}$$

And T is called strongly demicontractive if $F(T) \neq \emptyset$ and there exist $\alpha, k \in [0,1)$ such that for all $p \in F(T)$, $x \in C$,

$$D^2(Tx, Tp) \leq \alpha\|x-p\|^2 + k\|x-u\|^2, \ \forall u \in Tx.$$

If in (1.4), $k = 0$, then T is said to be quasi-nonexpansive.

We note that the class of multi-valued hemicontractive-type mappings is a more general class of mappings in the sense that it includes the class of multi-valued pseudocontractive mappings T with $F(T) \neq \emptyset$ and $T(p) = \{p\}, \forall p \in F(T)$ and the class of multi-valued demicontractive mappings and hence the class of multi-valued quasi-nonexpansive (see, [8, 19]).

Let $T : C \longrightarrow CB(C)$ be a multi-valued mapping, $I - T$ (where I is the identity mapping on C) is said to be demiclosed at zero if $\{x_n\} \subset C$ such that $x_n \rightharpoonup x$ and $\lim_{n\to\infty} d(x_n, Tx_n) = 0$ implies $x \in Tx$.

Many authors have extended the existence and approximation of fixed points for single-valued mappings to multi-valued mappings; see, for example, [6, 10, 14, 15] and the references therein. Recently, Woldeamanual et al. [19] considered the problem of finding a common point of fixed points of a finite family of hemicontractive-type multi-valued mappings and they introduced the following iterative algorithm:

$$\begin{cases} z_n = (1-\gamma_n)x_n + \gamma_n w_n, & w_n \in T_n y_n, \\ y_n = (1-\beta_n)x_n + \beta_n u_n, & u_n \in T_n x_n, \\ x_{n+1} = \alpha_n w + (1-\alpha_n) z_n, & \forall n \geq 1, \end{cases}$$

where $T_n := T_{n(modN)+1}$ and $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \subset (0,1)$ satisfying certain conditions. Then, they proved that the sequence $\{x_n\}$ converges strongly to some point p in $\cap_{i=1}^N F(T_i)$ nearest to w.

A mapping $A : C \longrightarrow H$ is called η- strongly monotone if there exists a positive real number η such that

$$\langle Ax - Ay, x - y\rangle \geq \eta\|x-y\|^2, \quad \forall x, y \in C.$$

A is called α-inverse strongly monotone if there exists a positive real number α such that

$$\langle Ax - Ay, x - y\rangle \geq \alpha\|Ax - Ay\|^2, \quad \forall x, y \in C.$$

Note that every α-inverse strongly monotone mapping is $\frac{1}{\alpha}$ -Lipschitz mapping. However, the converse may not hold.

A mapping $A : C \longrightarrow H$ is called monotone if

$$\langle Ax - Ay, x - y\rangle \geq 0, \quad \forall x, y \in C.$$

Clearly, the class of monotone mappings includes the class of α-inverse strongly monotone and the class of $\eta-$strongly monotone mappings and the inclusion is proper.

Let C be a nonempty, closed and convex subset of a real Hilbert space H and let $A : C \longrightarrow H$ be a nonlinear mapping. The classical variational inequality problem is the problem of finding $u \in C$ such that

$$\langle v - u, Au\rangle \geq 0 \text{ for all } v \in C. \tag{1.5}$$

The set of solutions of the variational inequality problem (1.5) is denoted by $VI(C, A)$.

Variational inequality theory, which was first introduced by Stampacchia [13] in 1964, emerged as an interesting and fascinating branch of applicable mathematics with a wide range of applications in economics, industry, network analysis, optimizations, pure and applied sciences etc. In recent years, much attention has been given to developing efficient iterative methods for treating solution problems of variational inequalities [3, 21, 25] and the references therein. The classical variational inequality is equivalent to a fixed point problem. This alternative equivalent formulation has played a major role in finding solutions of varational inequalities using iterative algorithms.

Recently, finding a common element of the fixed point set of nonexpansive mapping and the solution set of variational inequality problem has been considered by many authors; see, for example [4, 5, 16, 11, 24] and the references therein. We describe some of them as follows:

In 2004, Iiduka et al. [3] considered the following iterative algorithm:

$$\begin{cases} x_0 = x \in C, \\ x_{n+1} = \alpha_n x + (1-\alpha_n) T P_C(x_n - \lambda_n A x_n), \quad \forall n \geq 0. \end{cases}$$

where $T : C \longrightarrow C$ is a nonexpansive mapping, $A : C \longrightarrow H$ is a $\alpha-$inverse strongly monotone mapping, $\{\alpha_n\}$ is a sequence in $(0,1)$, and $\{\lambda_n\}$ is a sequence in $(0, 2\alpha)$. Then, they proved that the sequence $\{x_n\}$ strongly converges to some point $z \in F(T) \cap VI(C, A)$.

In 2006, Nadezhkina and Takahashi [9] introduced the following iterative algorithm for finding an element of $F(T) \cap VI(C, A)$ under the assumptions that C is a nonempty, closed convex subset of a real Hilbert space H, A is a monotone and $L-$Lipschitz mapping of C into H and T is a nonexpansive mapping of C into itself:

$$\begin{cases} x_0 = x \in C, \\ y_n = P_C(x_n - \lambda_n A x_n), \\ x_{n+1} = \alpha_n x_n + (1-\alpha_n) T P_C(x_n - \lambda_n A y_n), \quad \forall n \geq 0. \end{cases} \tag{1.6}$$

where $\{\lambda_n\} \subset (0, \frac{1}{L})$ and $\{\alpha_n\} \subset [c, d]$ for some $c, d \in (0,1)$. They proved that the sequence $\{x_n\}, \{y_n\}$ generated by (1.6) converge weakly to the same point $z \in F(T) \cap VI(C, A)$, where $z = \lim_{n\to\infty} P_{F(T)\cap VI(C,A)} x_n$.

In this paper, inspired by the papers surveyed above, we introduce an iterative algorithm for finding a common element of the solution set of a variational inequality problem (1.5) and the fixed point set of a hemicontractive-type mapping. Strong convergence theorem is established in the framework of Hilbert spaces. The results presented in this paper improve and extend the corresponding results announced by Iiduka et al. [3], Nadezhkina and Takahashi [9], Zegeye and Shahzad [25] and some other results in this direction.

2. Preliminaries

Let C be a nonempty, closed and convex subset of a real Hilbert space H. Let A be a monotone mapping from C into H. Then, in the context of variational inequality problem, it is easy to see that

$$u \in VI(C, A) \text{ if and only if } u = P_C(u - \lambda A u), \quad \forall \lambda > 0.$$

So, to find a solution of a classical variational inequality problem, we shall use projection mappings. Now, we describe some properties of projection mappings: For every point $x \in H$, there exists a unique nearest point in C, denoted by $P_C x$, such that

$$\|x - P_C x\| = \inf\{\|x - y\| : y \in C\}.$$

P_C is called the metric projection of H onto C. The metric projection P_C satisfies

$$\langle x - y, P_C x - P_C y \rangle \geq \|P_C x - P_C y\|^2, \quad \forall x, y \in H,$$

which implies that P_C is nonexpansive mapping. It is also characterized by the following properties (see, e.g., [17]):

$$z = P_C x \in C \text{ if and only if } \langle x - z, z - y\rangle \geq 0, \text{ for all } x \in H, y \in C, \tag{2.7}$$

and

$$\|y - P_C x\|^2 \leq \|x - y\|^2 - \|x - P_C x\|^2, \text{for all } x \in H, y \in C. \tag{2.8}$$

Let C be a bounded, closed and convex subset of a real Hilbert space H and let A be continuous monotone mapping of C into H. Then, $VI(C, A)$ is nonempty (see, [18, 22]).

A multi-valued mapping $A : H \longrightarrow 2^H$ is called monotone if for all $x, y \in H$, $\langle x - y, u - v\rangle \geq 0$ for all $u \in Ax$ and $v \in Ay$ holds. A is called maximal if its graph $G(A)$ is not properly contained in the graph of any other monotone mapping. Equivalently, a monotone mapping A is maximal if and only if, for $(x, u) \in H \times H$, $\langle x - y, u - v\rangle \geq 0$, for every $(y, v) \in G(A)$ implies $u \in Ax$. The normal cone to C at $x \in C$, denoted by $N_C x$, is given by

$$N_C x = \{z \in H : \langle x - y, z\rangle \geq 0, \quad \forall y \in C\}.$$

Let A be continuous monotone mapping of C into H. Then, the mapping $B : H \longrightarrow 2^H$ define by

$$Bv = \begin{cases} Av + N_C v, & \textit{if} \quad v \in C, \\ \emptyset, & \textit{if} \quad v \notin C \end{cases}$$

is maximal monotone and $0 \in Bv$ if and only if $v \in VI(C, A)$ (see, *e.g.*, [12]).

We need the following lemmas for the proof of our main result.

Lemma 2.1. [23] *Let H be a real Hilbert space. Then for all $x_i \in H$ and $\alpha_i \in [0, 1]$ for $i = 1, 2, \cdots, n$ such that $\alpha_1 + \alpha_2 + \cdots + \alpha_n = 1$ the following equality holds:*

$$\|\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n\|^2 = \sum_{i=1}^{n} \alpha_i \|x_i\|^2 - \sum_{1 \leq i,j \leq n} \alpha_i \alpha_j \|x_i - x_j\|^2.$$

Lemma 2.2. *Let H be a real Hilbert space. Then, for any given $x, y \in H$, we have the following inequality:*

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y\rangle.$$

Lemma 2.3. [10] *Let (X, d) be a metric space. Let $A, B \in CB(X)$ and $a \in A$. Then, for $\varepsilon > 0$, there exists a point $b \in B$ such that $d(a, b) \leq D(A, B) + \varepsilon$.*

Lemma 2.4. [20] *Let $\{b_n\}$ be a sequence of nonnegative real numbers satisfying the following relation:*

$$b_{n+1} \leq (1 - \alpha_n) b_n + \alpha_n \delta_n, \textit{for } n \geq n_0,$$

where $\{\alpha_n\} \subset (0, 1)$ and $\delta_n \subset \mathbb{R}$ satisfying the following conditions:

$$\lim_{n\to\infty} \alpha_n = 0, \ \sum_{n=1}^{\infty} \alpha_n = \infty, \ \textit{and} \ \limsup_{n\to\infty} \delta_n \leq 0.$$

Then, $\lim_{n\to\infty} b_n = 0$.

Lemma 2.5. [7] *Let $\{a_n\}$ be a sequence of real numbers such that there exist a subsequence $\{n_i\}$ of $\{n\}$ such that $a_{n_i} < a_{n_i+1}$, for all $i \in \mathbb{N}$. Then, there exists a nondecreasing sequence $\{m_k\} \subset \mathbb{N}$ such that $m_k \to \infty$ and the following properties are satisfied by all (sufficiently large) numbers $k \in \mathbb{N}$:*

$$a_{m_k} \leq a_{m_k+1} \quad \textit{and} \quad a_k \leq a_{m_k+1}.$$

In fact, $m_k = max\{j \le k : a_j \le a_{j+1}\}$.

3. Main result

Theorem 3.1. *Let C be a nonempty, closed and convex subset of a real Hilbert space H. Let $T : C \longrightarrow CB(C)$ be a Lipschitz hemicontractive-type mapping with Lipschitz constant L. Let $A : C \longrightarrow H$ be a d-Lipschitz monotone mapping. Assume that $\mathcal{F} = F(T) \cap VI(C, A)$ is nonempty, closed and convex, $I - T$ is demiclosed at zero and $Tp = \{p\}$ for all $p \in \mathcal{F}$. Let $\{x_n\}$ be a sequence generated from an arbitrary $x_0, u \in C$ by*

$$\begin{cases} z_n = P_C(x_n - \gamma_n Ax_n), \\ u_n = P_C(x_n - \gamma_n Az_n), \\ y_n = (1 - \lambda_n)x_n + \lambda_n v_n, \\ x_{n+1} = \alpha_n u + (1 - \alpha_n)(a_n x_n + b_n w_n + c_n u_n), \end{cases} \tag{3.9}$$

for all $n \ge 0$, where $v_n \in Tx_n, w_n \in Ty_n$ such that $\|v_n - w_n\| \le 2D(Tx_n, Ty_n)$ and P_C is a metric projection from H onto C and $\gamma_n \subset [a, b]$ for some $a, b \in (0, \frac{1}{d})$, $\{a_n\}, \{b_n\}, \{c_n\} \subset [e, f]$, and $\{\alpha_n\} \subset (0, c)$ for some $c, e, f \in (0, 1)$, satisfying the following conditions: (i) $a_n + b_n + c_n = 1$; (ii) $\lim_{n\to\infty} \alpha_n = 0$, $\sum \alpha_n = \infty$; (iii) $b_n + c_n \le \lambda_n \le \lambda < \frac{1}{\sqrt{1+4L^2}+1}$. Then, the sequence $\{x_n\}$ is bounded.

Proof. Let $p \in \mathcal{F}$. Then, from (2.8) and (3.9), we have

$$\begin{aligned} \|u_n - p\|^2 &= \|P_C(x_n - \gamma_n Az_n) - p\|^2 \\ &\le \|x_n - \gamma_n Az_n - p\|^2 - \|x_n - \gamma_n Az_n - u_n\|^2 \\ &= \Big\langle x_n - \gamma_n Az_n - p, x_n - \gamma_n Az_n - p\Big\rangle \\ &\quad - \Big\langle x_n - \gamma_n Az_n - u_n, x_n - \gamma_n Az_n - u_n\Big\rangle \\ &= \|x_n - p\|^2 - \|x_n - u_n\|^2 + 2\gamma_n\langle Az_n, p - u_n\rangle \\ &= \|x_n - p\|^2 - \|x_n - u_n\|^2 \\ &\quad + 2\gamma_n\Big(\langle Az_n - Ap, p - z_n\rangle + \langle Ap, p - z_n\rangle + \langle Az_n, z_n - u_n\rangle\Big) \\ &\le \|x_n - p\|^2 - \langle x_n - u_n, x_n - u_n\rangle + 2\gamma_n\langle Az_n, z_n - u_n\rangle, \end{aligned} \tag{3.10}$$

which gives

$$\begin{aligned} \|u_n - p\|^2 &\le \|x_n - p\|^2 - \|x_n - z_n\|^2 - 2\langle x_n - z_n, z_n - u_n\rangle \\ &\quad - \|z_n - u_n\|^2 + 2\gamma_n\langle Az_n, z_n - u_n\rangle \\ &= \|x_n - p\|^2 - \|x_n - z_n\|^2 - \|z_n - u_n\|^2 \\ &\quad + 2\langle x_n - \gamma_n Az_n - z_n, u_n - z_n\rangle. \end{aligned} \tag{3.11}$$

And from (2.7), we get that

$$\begin{aligned} \langle x_n - \gamma_n Az_n - z_n, u_n - z_n\rangle &= \langle x_n - \gamma_n Ax_n - z_n, u_n - z_n\rangle \\ &\quad + \langle \gamma_n Ax_n - \gamma_n Az_n, u_n - z_n\rangle \\ &\le \langle \gamma_n Ax_n - \gamma_n Az_n, u_n - z_n\rangle \\ &\le \gamma_n d\|x_n - z_n\| \times \|u_n - z_n\|. \end{aligned} \tag{3.12}$$

Thus, from (3.11) and (3.12), we obtain that

$$\begin{aligned}
\|u_n - p\|^2 &\leq \|x_n - p\|^2 - \|x_n - z_n\|^2 - \|z_n - u_n\|^2 \\
&\quad + 2\gamma_n d\|x_n - z_n\| \times \|u_n - z_n\| \\
&\leq \|x_n - p\|^2 - \|x_n - z_n\|^2 - \|z_n - u_n\|^2 \\
&\quad + \gamma_n^2 d^2\|x_n - z_n\|^2 + \|z_n - u_n\|^2 \\
&= \|x_n - p\|^2 + (\gamma_n^2 d^2 - 1)\|x_n - z_n\|^2. \qquad (3.13)
\end{aligned}$$

In addition, from (3.9), Lemma 2.1 and definition of hemicontractive-type mapping, we have that

$$\begin{aligned}
\|y_n - p\|^2 &= \|(1-\lambda_n)(x_n - p) + \lambda_n(v_n - p)\|^2 \\
&= (1-\lambda_n)\|x_n - p\|^2 + \lambda_n\|v_n - p\|^2 \\
&\quad - \lambda_n(1-\lambda_n)\|x_n - v_n\|^2 \\
&= (1-\lambda_n)\|x_n - p\|^2 + \lambda_n\Big(d(v_n, Tp)\Big)^2 \\
&\quad - \lambda_n(1-\lambda_n)\|x_n - v_n\|^2 \\
&\leq (1-\lambda_n)\|x_n - p\|^2 + \lambda_n D^2(Tx_n, Tp) \\
&\quad - \lambda_n(1-\lambda_n)\|x_n - v_n\|^2 \\
&\leq (1-\lambda_n)\|x_n - p\|^2 + \lambda_n\Big(\|x_n - p\|^2 + \|x_n - v_n\|^2\Big) \\
&\quad - \lambda_n(1-\lambda_n)\|x_n - v_n\|^2 \\
&= (1-\lambda_n)\|x_n - p\|^2 + \lambda_n\|x_n - p\|^2 + \lambda_n\|x_n - v_n\|^2 \\
&\quad - \lambda_n(1-\lambda_n)\|x_n - v_n\|^2 \\
&= \|x_n - p\|^2 + \lambda_n^2\|x_n - v_n\|^2. \qquad (3.14)
\end{aligned}$$

Thus, from (3.9), Lemma 2.1, (3.13), (3.14) and definition of hemicontractive-type mapping, we have the following:

$$\begin{aligned}
\|x_{n+1} - p\|^2 &= \|\alpha_n u + (1-\alpha_n)(a_n x_n + b_n w_n + c_n u_n) - p\|^2 \\
&\leq \alpha_n\|u - p\|^2 + (1-\alpha_n) \\
&\quad \times \|a_n(x_n - p) + b_n(w_n - p) + c_n(u_n - p)\|^2 \\
&\leq \alpha_n\|u - p\|^2 + (1-\alpha_n)\Big(a_n\|x_n - p\|^2 + b_n\|w_n - p\|^2 \\
&\quad + c_n\|u_n - p\|^2\Big) - (1-\alpha_n)a_n b_n\|w_n - x_n\|^2 \\
&\leq \alpha_n\|u - p\|^2 + (1-\alpha_n)\Big(a_n\|x_n - p\|^2 + b_n D^2(Ty_n, Tp) \\
&\quad + c_n\|u_n - p\|^2\Big) - (1-\alpha_n)a_n b_n\|w_n - x_n\|^2 \\
&\leq \alpha_n\|u - p\|^2 + (1-\alpha_n)a_n\|x_n - p\|^2 + (1-\alpha_n)b_n \\
&\quad \times \Big(\|y_n - p\|^2 + \|y_n - w_n\|^2\Big) + (1-\alpha_n)c_n \\
&\quad \times \Big(\|x_n - p\|^2 + (\gamma_n^2 d^2 - 1)\|x_n - z_n\|^2\Big) \\
&\quad - (1-\alpha_n)a_n b_n\|w_n - x_n\|^2,
\end{aligned}$$

which gives

$$\begin{aligned}
\|x_{n+1}-p\|^2 &\le \alpha_n\|u-p\|^2+(1-\alpha_n)(a_n+c_n)\|x_n-p\|^2\\
&\quad +(1-\alpha_n)b_n\Big(\|x_n-p\|^2+\lambda_n^2\|x_n-v_n\|^2\Big)\\
&\quad +(1-\alpha_n)b_n\|y_n-w_n\|^2+(1-\alpha_n)c_n\\
&\quad \times(\gamma_n^2d^2-1)\|x_n-z_n\|^2-(1-\alpha_n)a_nb_n\|w_n-x_n\|^2\\
&= \alpha_n\|u-p\|^2+(1-\alpha_n)\|x_n-p\|^2+(1-\alpha_n)b_n\lambda_n^2\\
&\quad \times\|x_n-v_n\|^2+(1-\alpha_n)b_n\|y_n-w_n\|^2\\
&\quad +(1-\alpha_n)c_n(\gamma_n^2d^2-1)\|x_n-z_n\|^2\\
&\quad -(1-\alpha_n)a_nb_n\|w_n-x_n\|^2. \qquad (3.15)
\end{aligned}$$

Furthermore, from (3.9) and Lemma 2.1, we get that

$$\begin{aligned}
\|y_n-w_n\|^2 &= \|(1-\lambda_n)(x_n-w_n)+\lambda_n(v_n-w_n)\|^2\\
&= (1-\lambda_n)\|x_n-w_n\|^2+\lambda_n\|v_n-w_n\|^2\\
&\quad -\lambda_n(1-\lambda_n)\|x_n-v_n\|^2\\
&\le (1-\lambda_n)\|x_n-w_n\|^2+4\lambda_nD^2(Tx_n,Ty_n)\\
&\quad -\lambda_n(1-\lambda_n)\|x_n-v_n\|^2\\
&\le (1-\lambda_n)\|x_n-w_n\|^2+4\lambda_nL^2\|x_n-y_n\|^2\\
&\quad -\lambda_n(1-\lambda_n)\|x_n-v_n\|^2\\
&= (1-\lambda_n)\|x_n-w_n\|^2+4\lambda_n^3L^2\|x_n-v_n\|^2\\
&\quad -\lambda_n(1-\lambda_n)\|x_n-v_n\|^2\\
&= (1-\lambda_n)\|x_n-w_n\|^2+\lambda_n(4L^2\lambda_n^2+\lambda_n-1)\|x_n-v_n\|^2. \qquad (3.16)
\end{aligned}$$

Now, substituting (3.16) into (3.15), we obtain that

$$\begin{aligned}
\|x_{n+1}-p\|^2 &\le \alpha_n\|u-p\|^2+(1-\alpha_n)\|x_n-p\|^2+(1-\alpha_n)b_n\lambda_n^2\|x_n-v_n\|^2\\
&\quad +(1-\alpha_n)b_n\Big((1-\lambda_n)\|x_n-w_n\|^2-\lambda_n(1-4L^2\lambda_n^2-\lambda_n)\|x_n-v_n\|^2\Big)\\
&\quad -(1-\alpha_n)a_nb_n\|w_n-x_n\|^2+(1-\alpha_n)c_n(\gamma_n^2d^2-1)\|x_n-z_n\|^2\\
&= \alpha_n\|u-p\|^2+(1-\alpha_n)\|x_n-p\|^2\\
&\quad +\Big((1-\alpha_n)b_n\lambda_n^2-(1-\alpha_n)b_n\lambda_n(1-4L^2\lambda_n^2-\lambda_n)\Big)\|x_n-v_n\|^2\\
&\quad +\Big((1-\alpha_n)b_n(1-\lambda_n)-(1-\alpha_n)a_nb_n\Big)\|x_n-w_n\|^2\\
&\quad +(1-\alpha_n)c_n(\gamma_n^2d^2-1)\|x_n-z_n\|^2, \qquad (3.17)
\end{aligned}$$

which gives

$$\begin{aligned}
\|x_{n+1}-p\|^2 &\leq \alpha_n\|u-p\|^2+(1-\alpha_n)\|x_n-p\|^2 \\
&\quad -(1-\alpha_n)b_n\lambda_n\Big(1-4L^2\lambda_n^2-2\lambda_n\Big)\|x_n-v_n\|^2+(1-\alpha_n)b_n \\
&\quad \times\Big(1-a_n-\lambda_n\Big)\|x_n-w_n\|^2+(1-\alpha_n)c_n(\gamma_n^2d^2-1)\|x_n-z_n\|^2 \\
&= \alpha_n\|u-p\|^2+(1-\alpha_n)\|x_n-p\|^2 \\
&\quad -(1-\alpha_n)b_n\lambda_n\Big(1-4L^2\lambda_n^2-2\lambda_n\Big)\|x_n-v_n\|^2 \\
&\quad +(1-\alpha_n)b_n\Big(b_n+c_n-\lambda_n\Big)\|x_n-w_n\|^2 \\
&\quad +(1-\alpha_n)c_n(\gamma_n^2d^2-1)\|x_n-z_n\|^2. \qquad (3.18)
\end{aligned}$$

Now, since from the hypothesis, we have $\gamma_n < \frac{1}{d}$ and

$$1-4L^2\lambda_n^2-2\lambda_n \geq 1-4L^2\lambda^2-2\lambda > 0 \quad and \quad (b_n+c_n)-\lambda_n \leq 0, \qquad (3.19)$$

for all $n \geq 0$, then inequality (3.18) implies that

$$\begin{aligned}
\|x_{n+1}-p\|^2 &\leq \alpha_n\|u-p\|^2+(1-\alpha_n)\|x_n-p\|^2 \qquad (3.20) \\
&\leq \alpha_n\max\{\|u-p\|^2,\|x_n-p\|^2\} \\
&\quad +(1-\alpha_n)\max\{\|u-p\|^2,\|x_n-p\|^2\} \\
&= \max\{\|u-p\|^2,\|x_n-p\|^2\}.
\end{aligned}$$

Thus, by induction, we have that

$$\|x_{n+1}-p\|^2 \leq max\{\|u-p\|^2,\|x_0-p\|^2\}, \quad \forall n \geq 0,$$

which implies that $\{x_n\}$ is bounded. □

Theorem 3.2. *Let C be a nonempty, closed and convex subset of a real Hilbert space H. Let $T: C \longrightarrow CB(C)$ be a Lipschitz hemicontractive-type mapping with Lipschitz constant L. Let $A: C \longrightarrow H$ be a d-Lipschitz monotone mapping. Assume that $\mathcal{F} = F(T) \cap VI(C,A)$ is nonempty, closed and convex, $I-T$ is demiclosed at zero and $Tp = \{p\}$ for all $p \in \mathcal{F}$. Let $\{x_n\}$ be a sequence generated from an arbitrary $x_0, u \in C$ by*

$$\begin{cases} z_n = P_C(x_n-\gamma_nAx_n), \\ u_n = P_C(x_n-\gamma_nAz_n), \\ y_n = (1-\lambda_n)x_n+\lambda_nv_n, \quad v_n \in T_nx_n, \\ x_{n+1} = \alpha_nu+(1-\alpha_n)(a_nx_n+b_nw_n+c_nu_n), \end{cases} \qquad (3.21)$$

for all $n \geq 0$, where $v_n \in Tx_n, w_n \in Ty_n$ such that $\|v_n-w_n\| \leq 2D(Tx_n,Ty_n)$ and P_C is a metric projection from H onto C and $\gamma_n \subset [a,b]$ for some $a,b \in (0,\frac{1}{d})$, $\{a_n\},\{b_n\},\{c_n\} \subset [e,f]$, and $\{\alpha_n\} \subset (0,c)$ for some $c,e,f \in (0,1)$, satisfying the following conditions: (i) $a_n+b_n+c_n = 1$; (ii) $\lim_{n\to\infty}\alpha_n = 0$, $\sum\alpha_n = \infty$; (iii) $b_n+c_n \leq \lambda_n \leq \lambda < \frac{1}{\sqrt{1+4L^2}+1}$. Then, the sequence $\{x_n\}$ converges strongly to the point $x^* = P_{\mathcal{F}}(u)$.

Proof. Clearly, from Theorem 3.1 the sequence $\{x_n\}$ and hence $\{y_n\},\{z_n\}$ are bounded. Let $x^* = P_{\mathcal{F}}(u)$. Then, using (3.21), Lemma 2.2, and following the methods used to get

(3.18) we obtain that

$$\begin{aligned}
\|x_{n+1}-x^*\|^2 &= \|\alpha_n u+(1-\alpha_n)\Big(a_n x_n+b_n w_n+c_n u_n\Big)-x^*\|^2\\
&\leq (1-\alpha_n)\|a_n x_n+b_n w_n+c_n u_n-x^*\|^2+2\alpha_n\langle u-x^*,x_{n+1}-x^*\rangle\\
&\leq (1-\alpha_n)a_n\|x_n-x^*\|^2+(1-\alpha_n)b_n\|w_n-x^*\|^2\\
&\quad+(1-\alpha_n)c_n\|u_n-x^*\|^2-(1-\alpha_n)a_n b_n\|w_n-x_n\|^2\\
&\quad-(1-\alpha_n)a_n c_n\|u_n-x_n\|^2+2\alpha_n\langle u-x^*,x_{n+1}-x^*\rangle\\
&\leq (1-\alpha_n)\|x_n-x^*\|^2-(1-\alpha_n)b_n\lambda_n\Big(1-4L^2\lambda_n^2-2\lambda_n\Big)\|x_n-v_n\|^2\\
&\quad+(1-\alpha_n)b_n\Big(b_n+c_n-\lambda_n\Big)\|x_n-w_n\|^2-(1-\alpha_n)a_n c_n\|u_n-x_n\|^2\\
&\quad+(1-\alpha_n)c_n(\gamma_n^2 d^2-1)\|x_n-z_n\|^2+2\alpha_n\langle u-x^*,x_{n+1}-x^*\rangle. \qquad (3.22)
\end{aligned}$$

This together with (3.19) implies that

$$\|x_{n+1}-x^*\|^2 \leq (1-\alpha_n)\|x_n-x^*\|^2+2\alpha_n\langle u-x^*,x_{n+1}-x^*\rangle. \qquad (3.23)$$

Now, we consider two cases:

Case 1. Suppose that there exists $n_0\in\mathbb{N}$ such that $\{\|x_n-x^*\|\}$ is decreasing for all $n\geq n_0$. Then, we get that, $\{\|x_n-x^*\|\}$ is convergent. Thus, from (3.22) and (3.19), we have that

$$\begin{aligned}
(1-\alpha_n)b_n\lambda_n\Big(1-4L^2\lambda_n^2-2\lambda_n\Big)\|x_n-v_n\|^2 &\leq (1-\alpha_n)\|x_n-x^*\|^2-\|x_{n+1}-x^*\|^2\\
&\quad+2\alpha_n\langle u-x^*,x_{n+1}-x^*\rangle.
\end{aligned}$$

Hence, from (3.19) and the fact that $\alpha_n\to 0$ as $n\to\infty$, we have that

$$x_n-v_n\to 0 \text{ as } n\to\infty, \qquad (3.24)$$

and from (3.22), the fact that $\alpha_n\to 0$ as $n\to\infty$ and $\gamma_n^2 d^2-1<0$, we also have that

$$u_n-x_n\to 0,\quad z_n-x_n\to 0 \text{ as } n\to\infty. \qquad (3.25)$$

Moreover, from (3.21) and (3.24), we obtain that

$$\|y_n-x_n\|=\|(1-\lambda_n)x_n+\lambda_n v_n-x_n\|=\lambda_n\|x_n-v_n\|\to 0 \text{ as } n\to\infty, \qquad (3.26)$$

and hence Lipschitz continuity of T_n, (3.24) and (3.26) imply that

$$\begin{aligned}
\|w_n-x_n\| &\leq \|w_n-v_n\|+\|v_n-x_n\|\\
&\leq 2L\|y_n-x_n\|+\|v_n-x_n\|\to 0 \text{ as } n\to\infty. \qquad (3.27)
\end{aligned}$$

In addition, from the fact that $\alpha_n\to 0$ as $n\to\infty$, (3.25) and (3.27) we have that

$$\begin{aligned}
\|x_{n+1}-x_n\| &= \|\alpha_n(u-x_n)+(1-\alpha_n)\Big(b_n(w_n-x_n)+c_n(u_n-x_n)\Big)\|\\
&\leq \alpha_n\|u-x_n\|+(1-\alpha_n)b_n\|w_n-x_n\|\\
&\quad+(1-\alpha_n)c_n\|u_n-x_n\|\to 0 \text{ as } n\to\infty, \qquad (3.28)
\end{aligned}$$

and from (3.24) we get

$$d(x_n,Tx_n)\leq ||x_n-v_n||\to 0 \text{ as } n\to\infty. \qquad (3.29)$$

Furthermore, from Theorem 3.1 we have that $\{x_{n+1}\}$ is a bounded sequence in H. Thus, since Hilbert space is reflexive, we can choose a subsequence $\{x_{n_j+1}\}$ of $\{x_{n+1}\}$ such that $x_{n_j+1}\rightharpoonup z$ as $j\to\infty$ and

$$\limsup_{n\to\infty}\langle u-x^*,x_{n+1}-x^*\rangle=\lim_{j\to\infty}\langle u-x^*,x_{n_j+1}-x^*\rangle.$$

Then, from (3.28) we have $x_{n_j} \rightharpoonup z$ as $j \to \infty$. Thus, from (3.29) and demiclosedness of $I - T$ at zero, we obtain that

$$z \in F(T).$$

Next, we show that $z \in VI(C, A)$. But, since A is Lipschitz continuous and $z_n - u_n = z_n - x_n + x_n - u_n \to 0$ as $n \to \infty$, we have $Az_n - Au_n \to 0$ as $n \to \infty$. Thus, from (3.25), we get that $u_{n_j} \rightharpoonup z$ and $z_{n_j} \rightharpoonup z$ as $j \to \infty$. Let

$$Bv = \begin{cases} Av + N_C v, & \textit{if} \quad v \in C, \\ \emptyset, & \textit{if} \quad v \notin C. \end{cases} \tag{3.30}$$

Then, B is maximal monotone and $0 \in Bv$ if and only if $v \in VI(C, A)$. Let $(v, w) \in G(B)$, where $G(B)$ denotes the graph of B, then we have $w \in Bv = Av + N_C v$ and hence $w - Av \in N_C v$. Thus, from the definition of $N_C v$, we get that

$$\langle v - u, w - Av\rangle \geq 0, \quad \forall u \in C.$$

On the other hand, since $u_n = P_C(x_n - \gamma_n Az_n)$ and $v \in C$, we have that

$$\langle x_n - \gamma_n Az_n - u_n, u_n - v\rangle \geq 0,$$

and hence

$$\left\langle v - u_n, \frac{(u_n - x_n)}{\gamma_n} + Az_n \right\rangle \geq 0.$$

Therefore, from $w - Av \in N_C v$ and $u_{n_j} \in C$, we obtain the following:

$$\begin{aligned}
\langle v - u_{n_j}, w\rangle &\geq \langle v - u_{n_j}, Av\rangle \\
&\geq \langle v - u_{n_j}, Av\rangle - \left\langle v - u_{n_j}, \frac{(u_{n_j} - x_{n_j})}{\gamma_{n_j}} + Az_{n_j} \right\rangle \\
&= \left\langle v - u_{n_j}, Av + Au_{n_j} - Au_{n_j} \right\rangle - \left\langle v - u_{n_j}, \frac{(u_{n_j} - x_{n_j})}{\gamma_{n_j}} \right\rangle \\
&\quad - \langle v - u_{n_j}, Az_{n_j}\rangle \\
&= \langle v - u_{n_j}, Av - Au_{n_j}\rangle - \left\langle v - u_{n_j}, \frac{(u_{n_j} - x_{n_j})}{\gamma_{n_j}} \right\rangle \\
&\quad + \langle v - u_{n_j}, Au_{n_j} - Az_{n_j}\rangle \\
&\geq \langle v - u_{n_j}, Au_{n_j} - Az_{n_j}\rangle - \left\langle v - u_{n_j}, \frac{(u_{n_j} - x_{n_j})}{\gamma_{n_j}} \right\rangle.
\end{aligned}$$

This implies that $\langle v - z, w\rangle \geq 0$, as $j \to \infty$. Then, maximality of B gives that $0 \in Bz$ and hence $z \in VI(C, A)$. Therefore,

$$z \in F(T) \cap VI(C, A).$$

Thus, since $x^* = P_{\mathcal{F}}(u)$, using (2.7) we obtain that

$$\begin{aligned}
\limsup_{n\to\infty} \langle u - x^*, x_{n+1} - x^*\rangle &= \lim_{j\to\infty} \langle u - x^*, x_{n_j+1} - x^*\rangle. \\
&= \langle u - x^*, z - x^*\rangle \leq 0.
\end{aligned} \tag{3.31}$$

Hence, it follows from (3.23), (3.31), assumptions of $\{\alpha_n\}$ and Lemma 2.4 that

$$\|x_n - x^*\| \to 0 \text{ as } n \to \infty .$$

Consequently, $x_n \to x^* = P_{\mathcal{F}}(u)$.

Case 2. Suppose that there exists a subsequence $\{n_j\}$ of $\{n\}$ such that

$$\|x_{n_j} - x^*\| < \|x_{n_j+1} - x^*\|,$$

for all $j \in \mathbb{N}$. Then, by Lemma 2.5, there exist a nondecreasing sequence $\{m_k\} \subset \mathbb{N}$ such that $m_k \to \infty$, and

$$\|x_{m_k} - x^*\| \leq \|x_{m_k+1} - x^*\| \text{ and } \|x_k - x^*\| \leq \|x_{m_k+1} - x^*\|, \tag{3.32}$$

for all $k \in \mathbb{N}$. Thus, from (3.22), (3.19) and the fact that $\alpha_n \to 0$, we get that

$$x_{m_k} - v_{m_k}, u_{m_k} - x_{m_k}, z_{m_k} - x_{m_k} \to 0 \text{ as } k \to \infty \ .$$

Hence, following the method in Case 1, we obtain that

$$\limsup_{k\to\infty} \langle u - x^*, x_{m_k+1} - x^*\rangle \leq 0. \tag{3.33}$$

Now, from (3.23), we have that

$$\|x_{m_k+1} - x^*\|^2 \leq (1-\alpha_{m_k})\|x_{m_k} - x^*\|^2 + 2\alpha_{m_k}\langle u - x^*, x_{m_k+1} - x^*\rangle, \tag{3.34}$$

and hence (3.32) and (3.34) imply that

$$\begin{aligned}
\alpha_{m_k}\|x_{m_k} - x^*\|^2 &\leq \|x_{m_k} - x^*\|^2 - \|x_{m_k+1} - x^*\|^2 + 2\alpha_{m_k}\langle u - x^*, x_{m_k+1} - x^*\rangle \\
&\leq 2\alpha_{m_k}\langle u - x^*, x_{m_k+1} - x^*\rangle.
\end{aligned}$$

Hence, the fact that $\alpha_{m_k} > 0$ imply that

$$\|x_{m_k} - x^*\|^2 \leq 2\langle u - x^*, x_{m_k+1} - x^*\rangle.$$

Thus, using (3.33) we get that $\|x_{m_k} - x^*\| \to 0$ as $k \to \infty$. This together with (3.34) implies that $\|x_{m_k+1} - x^*\| \to 0$ as $k \to \infty$. Since $\|x_k - x^*\| \leq \|x_{m_k+1} - x^*\|$ for all $k \in \mathbb{N}$, we get that $x_k \to x^*$. Therefore, from the above two cases, we can conclude that $\{x_n\}$ converges strongly to $x^* = P_{\mathcal{F}}(u)$. □

If, in Theorem 3.2, we assume that T with $F(T) \neq \emptyset$ and $T(p) = \{p\}, \forall p \in F(T)$ is pseudo-contractive multi-valued mapping. Then, we have that T is hemicontractive-type and hence we get the following theorem.

Theorem 3.3. *Let C be a nonempty, closed and convex subset of a real Hilbert space H. Let $T : C \longrightarrow CB(C)$ be a Lipschitz pseudocontractive mapping with Lipschitz constant L. Let $A : C \longrightarrow H$ be a $d-$Lipschitz monotone mapping. Assume that $\mathcal{F} = F(T) \cap VI(C, A)$ is nonempty, closed and convex, $I - T$ is demiclosed at zero and $Tp = \{p\}$ for all $p \in \mathcal{F}$. Let $\{x_n\}$ be a sequence generated from an arbitrary $x_0, u \in C$ by*

$$\begin{cases}
z_n = P_C(x_n - \gamma_n A x_n), \\
u_n = P_C(x_n - \gamma_n A z_n), \\
y_n = (1-\lambda_n)x_n + \lambda_n v_n, \\
x_{n+1} = \alpha_n u + (1-\alpha_n)(a_n x_n + b_n w_n + c_n u_n),
\end{cases}$$

for all $n \geq 0$, where $v_n \in Tx_n, w_n \in Ty_n$ such that $\|v_n - w_n\| \leq 2D(Tx_n, Ty_n)$ and P_C is a metric projection from H onto C and $\gamma_n \subset [a, b]$ for some $a, b \in (0, \frac{1}{d}), \{a_n\}, \{b_n\}, \{c_n\} \subset [e, f]$, and $\{\alpha_n\} \subset (0, c)$ for some $c, e, f \in (0, 1)$, satisfying the following conditions: (i) $a_n + b_n + c_n = 1$; (ii) $\lim_{n\to\infty} \alpha_n = 0$, $\sum \alpha_n = \infty$; (iii) $b_n + c_n \leq \lambda_n \leq \lambda < \frac{1}{\sqrt{1+4L^2}+1}$. Then, the sequence $\{x_n\}$ converges strongly to the point $x^* = P_{\mathcal{F}}(u)$.

If, in Theorem 3.2 we assume that $A = 0$, then we get the following corollary which is the main result of Woldeamanual et al. [19].

Corollary 3.1. *Let C be a nonempty, closed and convex subset of a real Hilbert space H. Let $T : C \longrightarrow CB(C)$ be a Lipschitz hemicontractive-type mapping with Lipschitz constant L.*

Assume that $F(T)$ is nonempty, closed and convex, $I-T$ is demiclosed at zero and $Tp=\{p\}$ for all $p \in F(T)$. Let $\{x_n\}$ be a sequence generated from an arbitrary $x_0, u \in C$ by

$$\begin{cases} y_n = (1-\lambda_n)x_n + \lambda_n v_n, \\ x_{n+1} = \alpha_n u + (1-\alpha_n)((1-a_n)x_n + a_n w_n), \end{cases}$$

for all $n \geq 0$, where $v_n \in Tx_n, w_n \in Ty_n$ such that $\|v_n - w_n\| \leq 2D(Tx_n, Ty_n)$ and $\{a_n\} \subset [e,f]$, and $\{\alpha_n\} \subset (0,c)$ for some $c, e, f \in (0,1)$, satisfying the following conditions:(i) $\lim_{n\to\infty} \alpha_n = 0$, $\sum \alpha_n = \infty$; (ii) $a_n \leq \lambda_n \leq \lambda < \frac{1}{\sqrt{1+4L^2}+1}$. Then, the sequence $\{x_n\}$ converges strongly to the point $x^* = P_{F(T)}(u)$.

If, in Theorem 3.2 we assume that $T = I$, where I is the identity mapping on C, then we obtain the following corollary.

Corollary 3.2. *Let C be a nonempty, closed and convex subset of a real Hilbert space H. Let $A : C \longrightarrow H$ be a $d-$Lipschitz monotone mapping. Assume that $VI(C,A)$ is nonempty. Let $\{x_n\}$ be a sequence generated from an arbitrary $x_0, u \in C$ by*

$$\begin{cases} z_n = P_C(x_n - \gamma_n A x_n), \\ u_n = P_C(x_n - \gamma_n A z_n), \\ x_{n+1} = \alpha_n u + (1-\alpha_n)((1-a_n)x_n + a_n u_n), \end{cases}$$

for all $n \geq 0$, where P_C is a metric projection from H onto C and $\gamma_n \subset [a,b]$ for some $a, b \in (0, \frac{1}{d}), \{a_n\} \subset [e,f]$, and $\{\alpha_n\} \subset (0,c)$ for some $c, e, f \in (0,1)$, satisfying the following conditions: (i) $\lim_{n\to\infty} \alpha_n = 0$, $\sum \alpha_n = \infty$; (ii) $a_n \leq \lambda_n \leq \lambda < \frac{1}{\sqrt{5}+1}$. Then, the sequence $\{x_n\}$ converges strongly to the point $x^* = P_{VI(C,A)}(u)$.

If, in Theorem 3.2 we assume that $C = H$, then we have $VI(C,A) = A^{-1}(0)$ and $P_H = I$, identity mapping on H. Hence, we have the following corollary.

Corollary 3.3. *Let H be a real Hilbert space. Let $T \colon H \longrightarrow CB(H)$ be a Lipschitz hemicontractive-type mapping with Lipschitz constant L. Let $A : H \longrightarrow H$ be a $d-$Lipschitz monotone mapping. Assume that $\mathcal{F} = F(T) \cap A^{-1}(0)$ is nonempty, closed and convex, $I-T$ is demiclosed at zero and $Tp = \{p\}$ for all $p \in \mathcal{F}$. Let $\{x_n\}$ be a sequence generated from an arbitrary $x_0, u \in C$ by*

$$\begin{cases} z_n = x_n - \gamma_n A x_n, \\ u_n = x_n - \gamma_n A z_n, \\ y_n = (1-\lambda_n)x_n + \lambda_n v_n, \\ x_{n+1} = \alpha_n u + (1-\alpha_n)(a_n x_n + b_n w_n + c_n u_n), \end{cases}$$

for all $n \geq 0$, where $v_n \in Tx_n, w_n \in Ty_n$ such that $\|v_n - w_n\| \leq 2D(Tx_n, Ty_n)$ and $\gamma_n \subset [a,b]$ for some $a, b \in (0, \frac{1}{d}), \{a_n\}, \{b_n\}, \{c_n\} \subset [e,f]$, and $\{\alpha_n\} \subset (0,c)$ for some $c, e, f \in (0,1)$, satisfying the following conditions:(i) $a_n + b_n + c_n = 1$; (ii) $\lim_{n\to\infty} \alpha_n = 0$, $\sum \alpha_n = \infty$; (iii) $b_n + c_n \leq \lambda_n \leq \lambda < \frac{1}{\sqrt{1+4L^2}+1}$. Then, the sequence $\{x_n\}$ converges strongly to the point $x^* = P_{\mathcal{F}}(u)$.

If, in Corollary 3.3 we assume that $T = I$, identity mapping on H, we obtain the following corollary.

Corollary 3.4. *Let H be a real Hilbert space. Let $A : H \longrightarrow H$ be a $d-$Lipschitz monotone mapping. Assume that $A^{-1}(0)$ is nonempty. Let $\{x_n\}$ be a sequence generated from an arbitrary $x_0, u \in C$ by*

$$\begin{cases} z_n = x_n - \gamma_n A x_n, \\ u_n = x_n - \gamma_n A z_n, \\ x_{n+1} = \alpha_n u + (1-\alpha_n)((1-a_n)x_n + a_n u_n), \end{cases}$$

for all $n \geq 0$*, where* $\gamma_n \subset [a,b]$ *for some* $a, b \in (0, \frac{1}{d}), \{a_n\} \subset [e,f]$*, and* $\{\alpha_n\} \subset (0,c)$ *for some* $c, e, f \in (0,1)$*, satisfying the following conditions:(i)* $\lim_{n\to\infty} \alpha_n = 0$, $\sum \alpha_n = \infty$*; (ii)* $a_n \leq \lambda_n \leq \lambda < \frac{1}{\sqrt{5}+1}$*. Then, the sequence* $\{x_n\}$ *converges strongly to the point* $x^* = P_{A^{-1}(0)}(u)$*.*

Remark 3.1. Theorem 3.1, 3.2 and 3.3 extends the results of Iiduka et al. [3], Nadezhkina and Takahashi [9], Zegeye and Shahzad [25] in the sense that our scheme provides strong convergence to a common point of solution set of a variational inequality problem for monotone mapping and the fixed point set of a Lipschitz hemicontractive-type multi-valued mapping.

Acknowledgements. This work was completed with the partial support of Simmons Foundation based in Botswana University

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