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Rate of growth of polynomials with restricted zeros

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ABSTRACT. In this paper we consider for a fixed μ, the class of polynomials $P(z) = a_0 + \sum\limits_{\nu=\mu}^{n} a_\nu z_\nu$, $1 \leq \mu \leq n$, of degree at most n not vanishing in the disk $|z| < k, k > 0$. For any $\rho > \sigma \geq 1$ and $0 < r \leq R \leq k$, we investigate the dependence of $\| P(\rho z) - P(\sigma z) \|_R$ on $\| P \|_r$ and derive various refinements and generalizations of some well known results.

1. INTRODUCTION

Let P_n be the class of polynomials $P(z) = \sum\limits_{\nu=0}^{n} a_\nu z_\nu$ of degree at most n. For $P \in P_n$, we define

$$\| P \| := \max_{|z|=1}|P(z)|, \| P \|_R := \max_{|z|=R}|P(z)|,$$

$$\| P(\rho z) - P(\sigma z) \|_R := \max_{|z|=R}|P(\rho z) - P(\sigma z)|$$

$$\text{and } m := \min_{|z|=k}|P(z)|.$$

If $P \in P_n$, then concerning the estimate of the maximum of $|P'(z)|$ on the unit circle $|z| = 1$ and the estimate of the maximum of $|P(z)|$ on a larger circle $|z| = R > 1$, we have

$$\| P' \| \leq n \| P \| \tag{1.1}$$

and

$$\| P \|_R \leq R^n \| P \| . \tag{1.2}$$

Inequality (1.1) is a well-known result of S. Bernstein (for reference see [15, p-508]), whereas inequality (1.2) is a simple deduction from maximum modulus principle (see [15, p-405]).

If we restrict ourselves to the class of polynomials $P \in P_n$ with $P(z) \neq 0$ in $|z| < 1$, then Erdös conjectured and later Lax (for reference see [15, p-562]), verified that the inequality (1.1) can be replaced by

$$\| P' \| \leq \frac{n}{2} \| P \| . \tag{1.3}$$

As an extension of (1.3), it was shown by Malik (for reference see [15, p-563]), that if $P \in P_n$ and $P(z) \neq 0$ in $|z| < k,\ k \geq 1$, then

$$\| P' \| \leq \frac{n}{1+k} \| P \| . \tag{1.4}$$

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Bidkham and Dewan [3] obtained a generalization of inequality (1.4) and proved that if $P \in P_n$ and $P(z) \neq 0$ in $|z| < k,\ k \geq 1$, then

$$\| P' \|_r \leq \frac{n(r+k)^{n-1}}{(1+k)^n} \| P \|, \tag{1.5}$$

where $1 \leq r \leq k$.

In the literature, there already exist various refinements and generalizations of (1.3), (1.4) and (1.5), for example see Mir, Dewan and Singh [10]-[11], Dewan, Singh and Mir [5], Mir, Dewan, Singh and Dar [13], Mir and Dar [12], Govil and Nyuydinkong [9], Gardner, Govil and Weems [6]-[7],Gardner, Govil and Musukula[8],etc.

In this paper, we denote by $P_{n,\mu}$, $1 \leq \mu \leq n$, the linear space of all polynomials of the form $P(z) = a_0 + \sum\limits_{\nu=\mu}^{n} a_\nu z_\nu$ of degree at most n. Note that $P_{n,1} = P_n$. Aziz and Shah [2] improved as well as extended the inequalities (1.3), (1.4) and (1.5) by showing that if $P \in P_{n,\mu}$ and $P(z) \neq 0$ in $|z| < k,\ k > 0$, then for $0 < r \leq R \leq k$,

$$\| P' \|_R \leq \frac{nR^{\mu-1}(R^\mu + k^\mu)^{\frac{n}{\mu}-1}}{(k^\mu + r^\mu)^{\frac{n}{\mu}}} \Big(\| P \|_r - m \Big). \tag{1.6}$$

More recently Aziz and Aliya [1] besides proving some other results, also calculated the growth of $\| P(\rho z) - P(z) \|_R$ where $\rho > 1,\ 0 < r \leq R \leq k$ and proved the following interesting generalization of inequality (1.6).

Theorem 1.1. *If $P \in P_{n,\mu}$ and $P(z) \neq 0$ in $|z| < k,\ k > 0$, then for every $\rho > 1$ and $0 < r \leq R \leq k$,*

$$\| P(\rho z) - P(z) \|_R \leq \frac{R^\mu(\rho^n - 1)}{r^\mu + k^\mu} \Big(\frac{R^\mu + k^\mu}{r^\mu + k^\mu} \Big)^{\frac{n}{\mu}-1} \Big(\| P \|_r - m \Big). \tag{1.7}$$

Note 1: If we divide both sides of (1.7) by $\rho - 1$ and let $\rho \to 1$, we get (1.6).
As a refinement of Theorem (1.1), Mir and Dar [12] proved the following result by involving some of the coefficients of the polynomial $P(z)$.

Theorem 1.2. *If $P \in P_{n,\mu}$ and $P(z) \neq 0$ in $|z| < k,\ k > 0$, then for every $\rho > 1$, $0 < r \leq R \leq k$ and $0 \leq \lambda \leq 1$,*

$$\| P(\rho z) - P(z) \|_R \leq (\rho^n - 1) \left(\frac{\Big(\frac{\rho^\mu - 1}{\rho^n - 1}\Big) \frac{|a_\mu|}{|a_0| - \lambda m} k^{\mu+1} R^\mu + R^{\mu+1}}{R^{\mu+1} + k^{\mu+1} + \Big(\frac{\rho^\mu - 1}{\rho^n - 1}\Big) \frac{|a_\mu|}{|a_0| - \lambda m} (k^{\mu+1} R^\mu + k^{2\mu} R)} \right)$$

$$\times\, exp \left\{ n \int_r^R \frac{\frac{\mu}{n} \frac{|a_\mu|}{|a_0| - \lambda m} k^{\mu+1} \varsigma^{\mu-1} + \varsigma^\mu}{\varsigma^{\mu+1} + \frac{\mu}{n} \frac{|a_\mu|}{|a_0| - \lambda m} (k^{\mu+1} \varsigma^\mu + k^{2\mu} \varsigma) + k^{\mu+1}} d\varsigma \right\} \Big(\| P \|_r - \lambda m \Big). \tag{1.8}$$

Note 2: If we divide both sides of (1.8) by $\rho - 1$, let $\rho \to 1$ and take $\lambda = 1$, we get a result of Chanam and Dewan [4, Theorem (2.4)].

2. Main results

In this paper, we shall prove the following result which generalises and refines the bounds of Theorems (1.1) and (1.2). More precisely, we prove

Theorem 2.3. *If $P \in P_{n,\mu}$ and $P(z) \neq 0$ in $|z| < k,\ k > 0$, then for every $\rho > \sigma \geq 1,\ 0 < r \leq R \leq k$, $0 \leq \lambda \leq 1$ and $n > 2$, we have,*

$$\| P(\rho z) - P(\sigma z) \|_R \leq \left(\frac{\left(\frac{\rho^\mu - \sigma^\mu}{\rho^n - \sigma^n}\right) \frac{|a_\mu|}{|a_0| - \lambda_m} k^{\mu+1} R^\mu + R^{\mu+1}}{R^{\mu+1} + k^{\mu+1} + \left(\frac{\rho^\mu - \sigma^\mu}{\rho^n - \sigma^n}\right) \frac{|a_\mu|}{|a_0| - \lambda m} (k^{\mu+1} R^\mu + k^{2\mu} R)} \right)$$
$$\times \left[(\rho^n - \sigma^n) exp\left\{ n \int_r^R \frac{\frac{\mu}{n} \frac{|a_\mu|}{|a_0| - \lambda m} k^{\mu+1} \varsigma^{\mu-1} + \varsigma^\mu}{\varsigma^{\mu+1} + \frac{\mu}{n} \frac{|a_\mu|}{|a_0| - \lambda m} (k^{\mu+1} \varsigma^\mu + k^{2\mu} \varsigma) + k^{\mu+1}} d\varsigma \right\} \right.$$
$$\left. \times \Big(\| P \|_r - \lambda m \Big) - \Big| R|P'(0)| - R^{n-1} |Q'(0)| \Big| \left(\frac{\rho^n - \sigma^n}{n} - \frac{\rho^{n-2} - \sigma^{n-2}}{n-2} \right) \right], \tag{2.9}$$

where here and throughout $Q(z) = z^n \overline{P(1/\bar{z})}$.

Remark 2.1. To show that Theorem (2.3) is, in general, an improvement and generalisation of Theorem (1.1), we first prove that

$$\frac{\left\{ \left(\frac{\rho^\mu - \sigma^\mu}{\rho^n - \sigma^n}\right) \frac{|a_\mu|}{|a_0| - \lambda m} k^{\mu+1} R^\mu + R^{\mu+1} \right\}}{R^{\mu+1} + k^{\mu+1} + \left(\frac{\rho^\mu - \sigma^\mu}{\rho^n - \sigma^n}\right) \frac{|a_\mu|}{|a_0| - \lambda m} (k^{\mu+1} R^\mu + k^{2\mu} R)}$$
$$\times exp\left\{ n \int_r^R \frac{\frac{\mu}{n} \frac{|a_\mu|}{|a_0| - \lambda m} k^{\mu+1} \varsigma^{\mu-1} + \varsigma^\mu}{\varsigma^{\mu+1} + \frac{\mu}{n} \frac{|a_\mu|}{|a_0| - \lambda m} (k^{\mu+1} \varsigma^\mu + k^{2\mu} \varsigma) + k^{\mu+1}} d\varsigma \right\}$$
$$\leq \frac{R^\mu}{r^\mu + k^\mu} \left[\frac{R^\mu + k^\mu}{r^\mu + k^\mu} \right]^{\frac{n}{\mu} - 1}. \tag{2.10}$$

Since, we have that

$$\frac{\rho^\mu - \sigma^\mu}{\rho^n - \sigma^n} \leq \frac{\mu}{n} \tag{2.11}$$

holds for all $\rho > \sigma \geq 1$ and $1 \leq \mu \leq n$, by considering the first derivative test for the function $\phi(t) = nt^\mu - \mu t^n,$ where $t \geq 1$.

Also, it is easy to see that for $R \leq k$, the function

$$S(x) = \frac{x \frac{|a_\mu|}{|a_0| - \lambda m} k^{\mu+1} R^\mu + R^{\mu+1}}{R^{\mu+1} + k^{\mu+1} + x \frac{|a_\mu|}{|a_0| - \lambda m} (k^{\mu+1} R^\mu + k^{2\mu} R)},$$

is a non-decreasing function of x, hence by using (2.11), we get

$$\frac{\left(\frac{\rho^\mu - \sigma^\mu}{\rho^n - \sigma^n}\right) \frac{|a_\mu|}{|a_0| - \lambda m} k^{\mu+1} R^\mu + R^{\mu+1}}{R^{\mu+1} + k^{\mu+1} + \left(\frac{\rho^\mu - \sigma^\mu}{\rho^n - \sigma^n}\right) \frac{|a_\mu|}{|a_0| - \lambda m} (k^{\mu+1} R^\mu + k^{2\mu} R)}$$
$$\leq \frac{\left(\frac{\mu}{n}\right) \frac{|a_\mu|}{|a_0| - \lambda m} k^{\mu+1} R^\mu + R^{\mu+1}}{R^{\mu+1} + k^{\mu+1} + \left(\frac{\mu}{n}\right) \frac{|a_\mu|}{|a_0| - \lambda m} (k^{\mu+1} R^\mu + k^{2\mu} R)}. \tag{2.12}$$

Since $R \leq k$, if we put $\varsigma = R$ in (3.21) of Lemma (3.3), we have

$$\frac{\left(\frac{\mu}{n}\right)\frac{|a_\mu|}{|a_0|-\lambda m}k^{\mu+1}R^{\mu-1} + R^\mu}{R^{\mu+1} + k^{\mu+1} + \left(\frac{\mu}{n}\right)\frac{|a_\mu|}{|a_0|-\lambda m}(k^{\mu+1}R^\mu + k^{2\mu}R)} \leq \frac{R^{\mu-1}}{R^\mu + k^\mu}. \tag{2.13}$$

Combining (2.12) and (2.13), we get

$$\frac{\left(\frac{\rho^\mu-\sigma^\mu}{\rho^n-\sigma^n}\right)\frac{|a_\mu|}{|a_0|-\lambda m}k^{\mu+1}R^{\mu} + R^{\mu+1}}{R^{\mu+1} + k^{\mu+1} + \left(\frac{\rho^\mu-\sigma^\mu}{\rho^n-\sigma^n}\right)\frac{|a_\mu|}{|a_0|-\lambda m}(k^{\mu+1}R^\mu + k^{2\mu}R)} \leq \frac{R^{\mu}}{R^\mu + k^\mu}, \tag{2.14}$$

and Lemma (3.3) gives

$$exp\left\{n\int\limits_r^R \frac{\frac{\mu}{n}\frac{|a_\mu|}{|a_0|-\lambda m}k^{\mu+1}\varsigma^{\mu-1} + \varsigma^\mu}{\varsigma^{\mu+1} + \frac{\mu}{n}\frac{|a_\mu|}{|a_0|-\lambda m}(k^{\mu+1}\varsigma^\mu + k^{2\mu}\varsigma) + k^{\mu+1}}d\varsigma\right\} \leq \left(\frac{R^\mu + k^\mu}{r^\mu + k^\mu}\right)^{\frac{n}{\mu}}. \tag{2.15}$$

On combining inequalities (2.14) and (2.15), we get (2.10). The following generalisation and refinement of Theorem (1.1) is obtained by using (2.10) in Theorem (2.3).

Theorem 2.4. *If $P \in P_{n,\mu}$ and $P(z) \neq 0$ in $|z| < k,\ k > 0$, then for every $\rho > \sigma \geq 1,\ 0 < r \leq R \leq k,\ 0 \leq \lambda \leq 1$ and $n > 2$,*

$$\begin{aligned}\parallel P(\rho z) - P(\sigma z) \parallel_R \leq & \frac{R^\mu(\rho^n - \sigma^n)}{r^\mu + k^\mu}\left(\frac{R^\mu + k^\mu}{r^\mu + k^\mu}\right)^{\frac{n}{\mu}-1}\left\{\parallel P \parallel_r - m\right\} \\ & - \frac{R^\mu\Big|R|P'(0)| - R^{n-1}|Q'(0)|\Big|}{R^\mu + k^\mu}\left(\frac{\rho^n - \sigma^n}{n} - \frac{\rho^{n-2} - \sigma^{n-2}}{n-2}\right). \end{aligned} \tag{2.16}$$

Since for $\rho > \sigma \geq 1$, $\dfrac{\rho^x - \sigma^x}{x}$ is increasing in $x > 0$, the expression

$$\frac{R^\mu}{R^\mu + k^\mu}\Big|R|P'(0)| - R^{n-1}|Q'(0)|\Big|\left(\frac{\rho^n - \sigma^n}{n} - \frac{\rho^{n-2} - \sigma^{n-2}}{n-2}\right)$$

is non-negative. Thus for polynomials of degree $n > 2$, Theorem (2.4) generalises and sharpens the bound obtained in Theorem (1.1). It is easy to see that for $\sigma = 1$, the R.H.S. of (2.9) is less than or equal to the R.H.S. of (1.8). Hence, for $n > 2$ and $\sigma = 1$, Theorem (2.3) provides a refinement of Theorem (1.2) as well.

3. Lemmas

For the proof of Theorem (2.3) we need the following lemmas.

Lemma 3.1. *Let $P \in P_{n,\mu}$ and $P(z)$ does not vanish in $|z| < k$, where $k \geq 1$ then for every $\rho > \sigma \geq 1, 0 \leq \lambda \leq 1,\ n > 2$ and $|z| = 1$,*

$$\begin{aligned}|P(\rho z) - P(\sigma z)| \leq & \left(\frac{\rho^n - \sigma^n}{1 + \psi_1(\rho)}\right)\left\{\parallel P \parallel - \lambda m\right\} \\ & - \frac{\Big||P'(0)| - |Q'(0)|\Big|}{1 + \psi_1(\rho)}\left(\frac{\rho^n - \sigma^n}{n} - \frac{\rho^{n-2} - \sigma^{n-2}}{n-2}\right), \end{aligned} \tag{3.17}$$

where

$$\psi_1(\rho) = k^{\mu+1}\left\{\frac{\left(\dfrac{\rho^\mu-\sigma^\mu}{\rho^n-\sigma^n}\right)\dfrac{|a_\mu|k^{\mu-1}}{|a_0|-\lambda m}+1}{\left(\dfrac{\rho^\mu-\sigma^\mu}{\rho^n-\sigma^n}\right)\dfrac{|a_\mu|k^{\mu+1}}{|a_0|-\lambda m}+1}\right\}.$$

The above Lemma is due to Mir, Imtiaz and Dawood [14].

Lemma 3.2. *If* $P \in P_{n,\mu}$ *and* $P(z) \neq 0$ *in* $|z| < k,\ k \geq 1$ *, then for* $0 \leq \lambda \leq 1$,

$$\frac{|a_\mu|k^\mu}{|a_0|-\lambda m} \leq \frac{n}{\mu}. \tag{3.18}$$

The above result is due to Mir and Dar [[12], inequality (2.6)].

Lemma 3.3. *If* $P \in P_{n,\mu}$ *and* $P(z) \neq 0$ *in* $|z| < k,\ k > 0$*, then for* $0 < r \leq R \leq k$ *and* $0 \leq \lambda \leq 1$,

$$exp\left\{n\int_r^R \frac{\frac{\mu}{n}\frac{|a_\mu|}{|a_0|-\lambda m}k^{\mu+1}\varsigma^{\mu-1}+\varsigma^\mu}{\varsigma^{\mu+1}+\frac{\mu}{n}\frac{|a_\mu|}{|a_0|-\lambda m}(k^{\mu+1}\varsigma^\mu+k^{2\mu}\varsigma)+k^{\mu+1}}d\varsigma\right\} \leq \left(\frac{R^\mu+k^\mu}{r^\mu+k^\mu}\right)^{\frac{n}{\mu}}. \tag{3.19}$$

Proof. The above Lemma is due to Mir and Dar [12], however for the sake of completeness we give the brief outlines of its proof. Since $P(z) \neq 0$ in $|z| < k,\ k > 0$, the polynomial $T(z) = P(\varsigma z) \neq 0$ in $|z| < \frac{k}{\varsigma},\ \frac{k}{\varsigma} \geq 1$, where $0 < \varsigma \leq k$. Hence applying inequality (3.18) of Lemma (3.2) to $T(z)$, we get

$$\frac{|a_\mu|\varsigma^\mu}{|a_0|-\lambda m}\left(\frac{k}{\varsigma}\right)^\mu \leq \frac{n}{\mu}, \tag{3.20}$$

where $m = \min\limits_{|z|=k/\varsigma}|T(z)| = \min\limits_{|z|=k/\varsigma}|P(\varsigma z)| = \min\limits_{|z|=k}|P(z)|$.
Now inequality (3.20) becomes

$$\left(\frac{\mu}{n}\right)\frac{|a_\mu|k^\mu}{|a_0|-\lambda m} \leq 1,$$

which is equivalent to

$$\frac{\left(\frac{\mu}{n}\right)\frac{|a_\mu|}{|a_0|-\lambda m}k^{\mu+1}\varsigma^{\mu-1}+\varsigma^\mu}{\varsigma^{\mu+1}+\left(\frac{\mu}{n}\right)\frac{|a_\mu|}{|a_0|-\lambda m}(k^{\mu+1}\varsigma^\mu+k^{2\mu}\varsigma)+k^{\mu+1}} \leq \frac{\varsigma^{\mu-1}}{\varsigma^\mu+k^\mu}. \tag{3.21}$$

Integrating both sides of (3.21) with respect to ς from r to R, where $0 < r \leq R \leq k$, we get

$$n\int_r^R \frac{\left(\frac{\mu}{n}\right)\frac{|a_\mu|}{|a_0|-\lambda m}k^{\mu+1}\varsigma^{\mu-1}+\varsigma^\mu}{\varsigma^{\mu+1}+\left(\frac{\mu}{n}\right)\frac{|a_\mu|}{|a_0|-\lambda m}(k^{\mu+1}\varsigma^\mu+k^{2\mu}\varsigma)+k^{\mu+1}}d\varsigma \leq n\int_r^R \frac{\varsigma^{\mu-1}}{\varsigma^\mu+k^\mu}d\varsigma,$$

which is equivalent to

$$exp\left\{n\int_r^R \frac{\left(\frac{\mu}{n}\right)\frac{|a_\mu|}{|a_0|-\lambda m}k^{\mu+1}\varsigma^{\mu-1}+\varsigma^\mu}{\varsigma^{\mu+1}+\left(\frac{\mu}{n}\right)\frac{|a_\mu|}{|a_0|-\lambda m}(k^{\mu+1}\varsigma^\mu+k^{2\mu}\varsigma)+k^{\mu+1}}d\varsigma\right\} \leq \left(\frac{k^\mu+R^\mu}{k^\mu+r^\mu}\right)^{\frac{n}{\mu}},$$

which proves Lemma (3.3) completely. □

Lemma 3.4. *If* $P \in P_{n,\mu}$ *and* $P(z) \neq 0$ *in* $|z| < k,\ k > 0$*, then for* $0 < r \leq R \leq k$ *and* $0 \leq \lambda \leq 1$,

$$\| P \|_r \geq exp\Bigg\{ -n \int_r^R \frac{\left(\frac{\mu}{n}\right)\frac{|a_\mu|}{|a_0|-\lambda m}k^{\mu+1}t^{\mu-1} + t^\mu}{t^{\mu+1} + k^{\mu+1} + \left(\frac{\mu}{n}\right)\frac{|a_\mu|}{|a_0|-\lambda m}(k^{\mu+1}t^\mu + k^{2\mu}t)}dt\Bigg\} \| P \|_R$$
$$+ \Bigg[1 - exp\Bigg\{ -n \int_r^R \frac{\left(\frac{\mu}{n}\right)\frac{|a_\mu|}{|a_0|-\lambda m}k^{\mu+1}t^{\mu-1} + t^\mu}{t^{\mu+1} + k^{\mu+1} + \left(\frac{\mu}{n}\right)\frac{|a_\mu|}{|a_0|-\lambda m}(k^{\mu+1}t^\mu + k^{2\mu}t)}dt\Bigg\}\Bigg] m. \tag{3.22}$$

The above result is due to Mir and Dar [[12], Corollary 1].

4. Proof of the Theorem

Proof of Theorem 2.3. Since $P \in P_{n,\mu}$ and $P(z) \neq 0$ in $|z| < k, k > 0$, the polynomial $F(z) = P(Rz)$ has no zeros in $|z| < k/R, k/R \geq 1$. Now applying inequality (3.17) of Lemma 3.1 to the polynomial $F(z)$, we have for every $\rho > \sigma \geq 1$ and $n > 2$,

$$\| F(\rho z) - F(\sigma z) \| \leq \frac{1}{1 + (k/R)^{\mu+1}\left\{\frac{\left(\frac{\rho^\mu-\sigma^\mu}{\rho^n-\sigma^n}\right)\frac{|a_\mu|}{|a_0|-\lambda m}R^\mu (k/R)^{\mu-1}+1}{\left(\frac{\rho^\mu-\sigma^\mu}{\rho^n-\sigma^n}\right)\frac{|a_\mu|}{|a_0|-\lambda m}R^\mu (k/R)^{\mu+1}+1}\right\}}$$
$$\times \left[(\rho^n - \sigma^n)\Bigg(\| F \| -\lambda m\Bigg) - \Bigg||F'(0)| - |H'(0)|\Bigg|\Bigg(\frac{\rho^n - \sigma^n}{n} - \frac{\rho^{n-2} - \sigma^{n-2}}{n-2}\Bigg)\right],$$

where $m = \min\limits_{|z|=k/R}|F(z)| = \min\limits_{|z|=k/R}|P(Rz)| = \min\limits_{|z|=k}|P(z)|$ and $H(z) = z^n\overline{F(1/\bar{z})}$.
This gives

$$\| P(R\rho z) - P(R\sigma z) \| \leq \frac{\left\{\left(\frac{\rho^\mu-\sigma^\mu}{\rho^n-\sigma^n}\right)\frac{|a_\mu|}{|a_0|-\lambda m}k^{\mu+1}R^\mu + R^{\mu+1}\right\}}{R^{\mu+1} + k^{\mu+1} + \left(\frac{\rho^\mu-\sigma^\mu}{\rho^n-\sigma^n}\right)\frac{|a_\mu|}{|a_0|-\lambda m}(k^{\mu+1}R^\mu + k^{2\mu}R)}$$
$$\times \left[(\rho^n-\sigma^n)\Bigg(\| P \|_R -m\Bigg) - \Bigg|R|P'(0)| - R^{n-1}|Q'(0)|\Bigg|\Bigg(\frac{\rho^n-\sigma^n}{n} - \frac{\rho^{n-2}-\sigma^{n-2}}{n-2}\Bigg)\right], \tag{4.23}$$

for every $\rho > \sigma \geq 1$ and $0 < R \leq k$.
Now if $0 < r \leq R \leq k$, then by using (3.22) of Lemma 3.4 in (4.23), we obtain

$$\| P(\rho z) - P(\sigma z) \|_R \leq \frac{\left\{\left(\frac{\rho^\mu-\sigma^\mu}{\rho^n-\sigma^n}\right)\frac{|a_\mu|}{|a_0|-\lambda m}k^{\mu+1}R^\mu + R^{\mu+1}\right\}}{R^{\mu+1} + k^{\mu+1} + \left(\frac{\rho^\mu-\sigma^\mu}{\rho^n-\sigma^n}\right)\frac{|a_\mu|}{|a_0|-\lambda m}(k^{\mu+1}R^\mu + k^{2\mu}R)}$$
$$\times \Bigg[(\rho^n - \sigma^n)exp\Bigg\{n \int_r^R \frac{\left(\frac{\mu}{n}\right)\frac{|a_\mu|}{|a_0|-\lambda m}k^{\mu+1}\varsigma^\mu + \varsigma^{\mu+1}}{\varsigma^{\mu+1} + k^{\mu+1} + \left(\frac{\mu}{n}\right)\frac{|a_\mu|}{|a_0|-\lambda m}(k^{\mu+1}\varsigma^\mu + k^{2\mu}\varsigma)}d\varsigma\Bigg\}$$
$$\times \Bigg(\| P \|_r -m\Bigg) - \Bigg|R|P'(0)| - R^{n-1}|Q'(0)|\Bigg|\Bigg(\frac{\rho^n - \sigma^n}{n} - \frac{\rho^{n-2} - \sigma^{n-2}}{n-2}\Bigg)\Bigg],$$

which is (2.9) and this completes the proof of Theorem (2.3). □

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