A note on an Engel condition with derivations in rings

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ABSTRACT. Let $R$ be a prime ring with center $Z(R)$, $C$ the extended centroid of $R$, $d$ a derivation of $R$ and $n, k$ be two fixed positive integers. In the present paper we investigate the behavior of a prime ring $R$ satisfying any one of the properties (i) $d([x, y]_k)^n = [x, y]_k$ (ii) if $\text{char}(R) \neq 2$, $d([x, y]_k) - [x, y]_k \in Z(R)$ for all $x, y$ in some appropriate subset of $R$. Moreover, we also examine the case when $R$ is a semiprime ring.

1. Introduction, Notation and Statements of the Results

Throughout this paper, unless specifically stated, $R$ is a (semi)-prime ring, $Z(R)$ is the center of $R$, $Q$ is the Martindale quotient ring of $R$ and $U$ is the Utumi quotient ring of $R$. The center of $U$, denoted by $C$, is called the extended centroid of $R$ (we refer the reader to [3], for the definitions and related properties of these objects). For each $x, y \in R$ and each $k \geq 0$, define $[x, y]_k$ inductively by $[x, y]_0 = x$, $[x, y]_1 = xy - yx$ and $[x, y]_k = ([x, y]_{k-1}, y)$ for $k > 1$. The ring $R$ is said to satisfy an Engel condition if there exists a positive integer $k$ such that $[x, y]_k = 0$. Note that an Engel condition is a polynomial $[x, y]_k = \sum_{m=0}^{k}(-1)^m \binom{k}{m} y^m x y^{k-m}$ in non-commuting indeterminates $x, y$ and $[x + z, y]_k = [x, y]_k + [z, y]_k$. Recall that a ring $R$ is prime if for any $a, b \in R$, $aRb = \{0\}$ implies $a = 0$ or $b = 0$, and is semiprime if for any $a \in R$, $aRa = \{0\}$ implies $a = 0$. An additive mapping $d : R \rightarrow R$ is called a derivation if $d(xy) = d(x)y + xd(y)$ holds for all $x, y \in R$. In particular, $d$ is an inner derivation induced by an element $a \in R$, if $d(x) = [a, x]$ for all $x \in R$.

Many results in literature indicate that the global structure of a ring $R$ is often tightly connected to the behavior of additive mappings defined on $R$. During the past few decades, there has been an ongoing interest concerning the relationship between the commutativity of a ring and the existence of certain specific types of derivations. Derivation with certain properties investigated in various paper (see [1, 2, 4, 7, 19] and references therein). Starting from these results, many author studied derivations in the context of prime and semiprime rings. The Engel type identity with derivation appeared in the well-known paper of Posner [19], who proved that a prime ring admitting a nonzero derivation $d$ such that $[d(x), x] \in Z(R)$ for all $x \in R$, must be commutative. Since then several authors have studied this kind of identities with derivations acting on one-sided, two-sided and Lie ideals of prime and semiprime rings (see [8], for a partial bibliography).

In 1992, Daif and Bell [7, Theorem 3], showed that if in a semiprime ring $R$ there exists a nonzero ideal $I$ of $R$ and a derivation $d$ such that $d([x, y]) = [x, y]$ for all $x, y \in I$, then $I \subseteq Z(R)$. If $R$ is a prime ring, this implies that $R$ is commutative. Recently in 2011, Huang [12] generalized Daif and Bell result. More precisely he prove that, if $R$ is a prime ring, $I$ is a nonzero ideal of $R$, $m, n$ are two fixed positive integers and $d$ a derivation of $R$
satisfy \( d([x, y])^n = [x, y]_n \) for all \( x, y \in I \), then \( R \) is commutative. In 1994, Giambruno et al. [10] established that a ring must be commutative if it satisfy \( [x, y]_k^n = [x, y]_k \).

It is natural to ask what we can say about the commutativity of \( R \) satisfying any of the following conditions: (P1) \( d([x, y]_k)^n = [x, y]_k \) (P2) \( d([x, y]_k) - [x, y]_k \in Z(R) \) for all \( x, y \in I \). This result generalized a theorem of Huang [12], and for derivation Giambruno theorem [10].

2. Derivations in prime rings

We have started with the following proposition which is very crucial for developing the proof of our main result.

**Proposition 2.1.** Let \( R \) be a prime ring, \( Q \) the Martindale quotient ring of \( R \), \( I \) a nonzero ideal of \( R \) and \( n, k \) be two fixed positive integers. If \( d \) is a nonzero inner derivation on \( Q \), in the sense that there exists \( q \in Q \) such that \( d(x) = [q, x] \) for all \( x \in R \), and \( I \) satisfies \( ([q, [x, y]_k])^n = [x, y]_k \) for all \( x, y \in I \), then \( R \) is commutative.

**Proof.** Assume that \( R \) is non-commutative. We have given that \( ([q, [x, y]_k])^n = [x, y]_k \) for all \( x, y \in I \). Since \( d \neq 0 \), \( q \notin Z(R) \) and hence \( I \) satisfied generalized polynomial identity(GPI). By Chuang [5, Theorem 2], \( I \) and \( Q \) satisfy the same generalized polynomial identities, thus we have

\[
([q, [x, y]_k])^n = [x, y]_k \quad \text{for all } x, y \in Q.
\]

In case the center \( C \) of \( Q \) is infinite, we have

\[
([q, [x, y]_k])^n = [x, y]_k \quad \text{for all } x, y \in Q \otimes_C \overline{C},
\]

where \( \overline{C} \) is algebraic closure of \( C \). Since both \( Q \) and \( Q \otimes_C \overline{C} \) are prime and centrally closed [9, Theorems 2.5 and 3.5], we may replace \( R \) by \( Q \) or \( Q \otimes_C \overline{C} \) according as \( C \) is finite or infinite. Thus we may assume that \( R \) is centrally closed over \( C \) (i.e., \( RC = R \)) which is either finite or algebraically closed and \( ([q, [x, y]_k])^n = [x, y]_k \) for all \( x, y \in R \). By Martindale [17, Theorem 3], \( RC \) (and so \( R \)) is a primitive ring having nonzero socle \( H \) with \( D \) as the associated division ring.

Hence by Jacobson’s theorem [13, p.75], \( R \) is isomorphic to a dense ring of linear transforms of some vector space \( V \) over \( D \) and \( H \) consists of the finite rank linear transformations in \( R \). If \( V \) is a finite dimensional over \( D \), then the density of \( R \) on \( V \) implies that \( R \cong M_t(D) \), where \( t = \text{dim}_D V \). Assume first that \( \text{dim}_D V \geq 3 \).

**Step 1.** We want to show that, for any \( v \in V \), \( v \) and \( qv \) are linearly \( D \)-dependent. If \( v = 0 \), then \( \{v, qv\} \) is linearly \( D \)-dependent. Now let \( v \neq 0 \) and \( \{v, qv\} \) is linearly \( D \)-independent, since \( \text{dim}_D V \geq 3 \), then there exists \( w \in V \) such that \( \{v, qv, w\} \) is also linearly \( D \)-independent. By the density of \( R \), there exist \( x, y \in R \) such that:

\[
\begin{align*}
xv &= v, & xqv &= 0, & xw &= v, \\
yv &= 0, & yqv &= w, & yw &= w.
\end{align*}
\]

These imply that \((-1)^n v = ([q, [x, y]_k])^n v - ([x, y]_k)v = 0 \), a contradiction. So, we conclude that \( \{v, qv\} \) is linearly \( D \)-dependent, for all \( v \in V \).

**Step 2.** We show here that there exists \( \alpha \in D \) such that \( qv = v\alpha \), for any \( v \in V \). Now choose \( v, w \in V \) linearly independent. By Step 1, there exist \( \alpha_v, \alpha_w, \alpha_{v+w} \in D \) such that

\[
qv = v\alpha_v, \quad qw = w\alpha_w, \quad q(v+w) = (v+w)\alpha_{v+w}
\]

Moreover,

\[
v\alpha_v + w\alpha = (v+w)\alpha_{v+w}.
\]
and because \( v, w \) are linearly \( D \)-independent, we have \( \alpha_v = \alpha_w = \alpha_{v+w} \), that is, \( \alpha \) does not depend on the choice of \( v \). This completes the proof of Step 2.

Let now for \( r \in R, v \in V \). By Step 2, \( qv = v\alpha, r(qv) = r(v\alpha) \), and also \( q(rv) = (rv)\alpha \). Thus \( 0 = [q, r]v \), for any \( v \in V \), that is \( [q, r]V = 0 \). Since \( V \) is a left faithful irreducible \( R \)-module, hence \( [q, r] = 0 \), for all \( r \in R \), i.e., \( q \in Z(R) \) and \( d = 0 \), which contradicts our hypothesis.

Therefore \( \text{dim}_D V \) must be \( \leq 2 \). In this case \( R \) is a simple GPI-ring with 1, and so it is a central simple algebra finite dimensional over its center. By Lanski [15, Lemma 2], it follows that there exists a suitable filed \( \mathbb{F} \) such that \( R \subseteq M_t(\mathbb{F}) \), the ring of all \( t \times t \) matrices over \( \mathbb{F} \), and moreover, \( M_t(\mathbb{F}) \) satisfies the same generalized polynomial identity of \( R \).

If we assume \( t \geq 3 \), then by the same argument as in Steps 1 and 2, we get a contradiction. Obviously if \( t = 1 \), then \( R \) is commutative. Thus we may assume that \( t = 2 \), i.e., \( R \subseteq M_2(\mathbb{F}) \), where \( M_2(\mathbb{F}) \) satisfies \( ([q, [x, y]]_k) = [x, y]_k \). Denote by \( e_{ij} \) the usual unit matrix with 1 in \( (i, j) \)-entry and zero elsewhere. Since by choosing \( x = e_{12}, y = e_{22} \). In this case we have \( (qe_{12} - e_{12}q)^n = e_{12} \). Right multiplying by \( e_{12} \), we get \((qe_{12} - e_{12}q)^n = e_{12}e_{12} = 0 \). Now set \( q = \begin{pmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{pmatrix} \). By calculation, we find that \((-1)^n \begin{pmatrix} 0 & q_{21} \\ 0 & 0 \end{pmatrix} = 0 \), which implies that \( q_{21} = 0 \). In the same manner, we can see that \( q_{12} = 0 \). Thus we conclude that \( q \) is a diagonal matrix in \( M_2(\mathbb{F}) \). Let \( \chi \in \text{Aut}(M_2(\mathbb{F})) \). Since \( ([\chi(q), [\chi(x), \chi(y)]_k]) = [\chi(x), \chi(y)]_k \), then \( \chi(q) \) must be diagonal matrix in \( M_2(\mathbb{F}) \). In particular, let \( \chi(x) = (1 - e_{ii})x(1 + e_{ij}) \) for \( i \neq j \). Then \( \chi(q) = q + (q_{ii} - q_{jj})e_{ij} \), that is \( q_{ii} = q_{jj} \) for \( i \neq j \). This implies that \( q \) is central in \( M_2(\mathbb{F}) \), which leads to \( d = 0 \), a contradiction. Thus \( t = 1 \), that is \( R \) is commutative. This completes the proof of the proposition.

\textbf{Theorem 2.1.} Let \( R \) be a prime ring, \( I \) a nonzero ideal of \( R \) and \( n, k \) be two fixed positive integers. If \( R \) admits a derivation \( d \) such that \( d([x, y]_k)^n = [x, y]_k \) for all \( x, y \in I \), then \( R \) is commutative.

\textbf{Proof.} If \( d = 0 \), then \([x, y]_k = 0 \) which is rewritten as \([I_x(y), y]_{k-1} = 0 \) for all \( x, y \in I \). By Lanski [15, Theorem 1], either \( R \) is commutative or \( I_x = 0 \) i.e., \( I \subseteq Z(R) \) in which case \( R \) is also commutative by Mayne [18, Lemma 3]. Now we assume that \( d \neq 0 \) and \( d([x, y]_k)^n = [x, y]_k \) for all \( x, y \in I \), that is \( I \) satisfies the differential identity

\[
\left( \sum_{m=0}^{k} (-1)^m \binom{k}{m} \left( \sum_{i+j=m-1} y^i d(y) y^j \right) x y^{k-m} \right) + \sum_{m=0}^{k} (-1)^m \binom{k}{m} y^m d(x) y^{k-m} \\
+ \sum_{m=0}^{k} (-1)^m \binom{k}{m} y^m x \left( \sum_{r+s=k-m-1} y^r d(y) y^s \right) \right)^n = \sum_{m=0}^{k} (-1)^m \binom{k}{m} y^m x y^{k-m}.
\]

(2.1)

In the light of Kharchenko’s theory [14], we split the proof into two cases:
Firstly we assume that $d$ is an inner derivation induced by an element $q \in Q$ such that $d(x) = [q, x]$ for all $x \in R$. Therefore from (2.1), we have

$$
\left( \sum_{m=0}^{k} (-1)^m \binom{k}{m} \left( \sum_{i+j=m-1} y^i([q, y]) y^j \right) x y^{k-m} \right.
+ \sum_{m=0}^{k} (-1)^m \binom{k}{m} y^m ([q, x]) y^{k-m}
+ \sum_{m=0}^{k} (-1)^m \binom{k}{m} y^m x \left( \sum_{r+s=k-m-1} y^r ([q, y]) y^s \right)^n

= \sum_{m=0}^{k} (-1)^m \binom{k}{m} y^m x y^{k-m} \text{ for all } x, y \in I.
$$

It can be easily seen that

$$
\left( q \left( \sum_{m=0}^{k} (-1)^m \binom{k}{m} y^m x y^{k-m} \right) - \sum_{m=0}^{k} (-1)^m \binom{k}{m} y^m x y^{k-m} \right) q

= \sum_{m=0}^{k} (-1)^m \binom{k}{m} y^m x y^{k-m}.
$$

And hence we can write $([q, [x, y]]_k)^n = [x, y]_k$ for all $x, y \in I$. In this case we are done from Proposition 2.1.

Secondly we now assume that $d$ is an outer derivation on $Q$. Now by Kharchencko’s theorem [14], $I$ satisfy the generalized polynomial identity

$$
\left( \sum_{m=0}^{k} (-1)^m \binom{k}{m} \left( \sum_{i+j=m-1} y^i z y^j \right) x y^{k-m} \right.
+ \sum_{m=0}^{k} (-1)^m \binom{k}{m} y^m w y^{k-m}
+ \sum_{m=0}^{k} (-1)^m \binom{k}{m} y^m x \left( \sum_{r+s=k-m-1} y^r z y^s \right)^n

= \sum_{m=0}^{k} (-1)^m \binom{k}{m} y^m x y^{k-m},
$$

and in particular $I$ satisfy the polynomial identity

$$
\sum_{m=0}^{k} (-1)^m \binom{k}{m} y^m x y^{k-m} = 0 \text{ for all } x, y \in I.
$$

That is $[x, y]_k = 0$ for all $x, y \in I$, and hence $R$ is commutative by the same argument presented above. This completes the proof of the theorem. \hfill \Box

We immediately get the following corollary from the above theorem:

**Corollary 2.1.** Let $R$ be a prime ring, $I$ a nonzero ideal of $R$ and $k$ be a fixed positive integer. If $R$ admits a derivation $d$ such that $d([x, y]_k) = [x, y]_k$ for all $x, y \in I$, then $R$ is commutative.
Theorem 2.2. Let \( R \) be a prime ring of characteristic different from 2 with center \( Z(R) \), \( I \) a nonzero ideal of \( R \) and \( k \) be a fixed positive integer. If \( R \) admits a derivation \( d \) such that \( d([x, y]_k) - [x, y]_k \in Z(R) \) for all \( x, y \in I \), then \( R \) satisfies \( s_4 \), the standard identity in four variables.

Proof. If \( d = 0 \), then \([x, y]_k \in Z(R)\) for all \( x, y \in I \) and hence \( R \) satisfies the same identities. In this case the identity is a polynomial so that there exists a field \( \mathbb{F} \) such that \( R \) and \( \mathbb{F}_t \) satisfy the same identities. Thus pick \( x = e_{31}, y = e_{11} - e_{22} \), we see that \([x, y]_k = e_{31} \notin Z(R)\), a contradiction. Therefore \( t \leq 2 \) and \( R \) satisfies \( s_4 \). Now, we assume that \( d \neq 0 \).

If \( d([x, y]_k) = [x, y]_k \) for all \( x, y \in I \), then \( R \) is commutative by Corollary 2.1. Otherwise we have \( I \cap Z(R) \neq 0 \) by our assumptions. Let now \( J \) be a nonzero two-sided ideal of \( R_Z \), the ring of the central quotient of \( R \). Since \( J \cap R \) is an ideal of \( R \), then \( J \cap R \cap Z(R) \neq 0 \). That is \( J \) contains an invertible element in \( R_Z \), and so \( R_Z \) is simple with 1. By the hypothesis for any \( x, y \in I \) and \( r \in R \), thus \( I \) satisfies the differential identity \([d([x, y]_k) - [x, y]_k, r] = 0\). Which can be rewritten as, that is, \( I \) satisfy the polynomial identity

\[
f(x, y, r, d(x), d(y)) = \sum_{m=0}^{k} (-1)^{m} \binom{k}{m} \left( \sum_{i+j=m-1} y^i d(y) y^j \right) x y^{k-m} + \sum_{m=0}^{k} (-1)^{m} \binom{k}{m} y^m d(x) y^{k-m} + \sum_{m=0}^{k} (-1)^{m} \binom{k}{m} y^m x \left( \sum_{r+s=k-m-1} y^r d(y) y^s \right) - \sum_{m=0}^{k} (-1)^{m} \binom{k}{m} y^m x y^{k-m}, r \right] = 0.
\]

If \( d \) is not an inner derivation, then \( I \) satisfies the polynomial identity

\[
f(x, y, r, w, z) = \sum_{m=0}^{k} (-1)^{m} \binom{k}{m} \left( \sum_{i+j=m-1} y^i z y^j \right) x y^{k-m} + \sum_{m=0}^{k} (-1)^{m} \binom{k}{m} y^m w y^{k-m} + \sum_{m=0}^{k} (-1)^{m} \binom{k}{m} y^m x \left( \sum_{r+s=k-m-1} y^r z y^s \right) - \sum_{m=0}^{k} (-1)^{m} \binom{k}{m} y^m x y^{k-m}, r \right] = 0.
\]

By Kharchenko’s theorem [14], and setting \( z = w = 0 \) yields the identity

\[
\left[ \sum_{m=0}^{k} (-1)^{m} \binom{k}{m} y^m x y^{k-m}, r \right] = 0.
\]

In this case it is well known that there exists a field \( \mathbb{F} \) such that \( R \) and \( \mathbb{F}_t \) satisfy the same polynomial identities. Thus \( \sum_{m=0}^{k} (-1)^{m} \binom{k}{m} y^m x y^{k-m} \) is central in \( \mathbb{F}_t \). Suppose \( t \geq 3 \) and choose \( x = e_{31}, y = e_{33} \). Then \( \sum_{m=0}^{k} (-1)^{m} \binom{k}{m} y^m x y^{k-m} = (-1)^k e_{31} \notin Z(\mathbb{F}_3) \), contrary to our assumptions. This forces \( t \leq 2 \), i.e., \( R \) satisfies \( s_4 \). Notice that in this case \( t = 1 \),
then \( R \) is commutative. But if \( t \geq 2 \) and \( x = e_{12}, y = e_{22} \), we get the contradiction
\[
\sum_{m=0}^{k} (-1)^m \binom{k}{m} y^m x y^{k-m} = e_{12} \notin Z(\mathbb{F}_2).
\]

Now let \( d \) be an inner derivation induced by an element \( q \in Q \), that is, \( d(x) = [q, x] \) for all \( x \in R \). Since \( d \neq 0 \), we may assume that \( q \notin Z(R) \). By localizing \( R \) at \( Z(R) \) it is easy to see that
\[
\sum_{m=0}^{k} (-1)^m \binom{k}{m} \left( \sum_{i+j=m-1} y^i[q, y]y^j \right) x y^{k-m} + \sum_{m=0}^{k} (-1)^m \binom{k}{m} y^m [q, x] y^{k-m} + \sum_{m=0}^{k} (-1)^m \binom{k}{m} y^m x \left( \sum_{r+s=k-m-1} y^r[q, y]y^s \right) - \sum_{m=0}^{k} (-1)^m \binom{k}{m} y^m x y^{k-m} \in Z(R_Z), \quad \text{for any } x, y \in R_Z.
\]

Since \( R \) and \( R_Z \) satisfy the same polynomial identities, in order to prove that \( R \) is commutative, we may assume that \( R \) is simple with 1. In this case,
\[
\sum_{m=0}^{k} (-1)^m \binom{k}{m} \left( \sum_{i+j=m-1} y^i[q, y]y^j \right) x y^{k-m} + \sum_{m=0}^{k} (-1)^m \binom{k}{m} y^m [q, x] y^{k-m} + \sum_{m=0}^{k} (-1)^m \binom{k}{m} y^m x \left( \sum_{r+s=k-m-1} y^r[q, y]y^s \right) - \sum_{m=0}^{k} (-1)^m \binom{k}{m} y^m x y^{k-m} \in Z(R), \quad \text{for all } x, y \in R.
\]

Therefore \( R \) satisfies a generalized polynomial identity and it is simple with 1, which implies that \( Q = RC = R \) and \( R \) has a minimal right ideal. Thus \( q \in R = Q \) and \( R \) is simple artinian, that is, \( R = D_1 \), where \( D \) is a division ring finite dimensional over \( Z(R) \) by [17]. From [15, Lemma 2], it follows that there exists a suitable field \( \mathbb{F} \) such that \( R \subseteq M_t(\mathbb{F}) \), the ring of all \( t \times t \) matrices over \( \mathbb{F} \), and moreover \( M_t(\mathbb{F}) \) satisfies the generalized polynomial identity
\[
\begin{bmatrix}
\sum_{m=0}^{k} (-1)^m \binom{k}{m} \left( \sum_{i+j=m-1} y^i[q, y]y^j \right) x y^{k-m} + \sum_{m=0}^{k} (-1)^m \binom{k}{m} y^m [q, x] y^{k-m} + \sum_{m=0}^{k} (-1)^m \binom{k}{m} y^m x \left( \sum_{r+s=k-m-1} y^r[q, y]y^s \right) - \sum_{m=0}^{k} (-1)^m \binom{k}{m} y^m x y^{k-m} \\
\end{bmatrix} = 0 \text{ for all } x, y, r \in M_t(\mathbb{F}).
\]
In this case, as already see in Theorem 2.1, we have \([q, [x, y]_k] - [x, y]_k\) is central in \(M_t(\mathbb{F})\). Suppose that \(t \geq 3\) and \(M_t(\mathbb{F})\) satisfy
\[
[[q, [x, y]_k] - [x, y]_k, r] = 0 \text{ for all } x, y, r \in M_t(\mathbb{F}).
\] (2.2)

Let \(q = \sum_t a_t e_t\), with \(a_t \in \mathbb{F}\), and choose \(x = e_{i,j}, y = e_{j,i}\), and \(r = e_{i,j}\), where \(i \neq j\). Then by using the same argument presented in Theorem 2.1, we get
\[
[[q, [x, y]_k] - [x, y]_k, r] = -2e_{i,j} q e_{i,j},
\]
which has rank 1 and so it cannot be central in \(M_t(\mathbb{F})\), with \(t \geq 3\). This implies that \(t \leq 2\) and \(R\) satisfy \(s_4\). Now let \(e\) and \(f\) be any two orthogonal idempotent elements in \(M_t(\mathbb{F})\). Now, we replace \(x\) with \(e x f\), \(y\) with \(e\), and \(r\) by \(e x f\) in (2.2) and let \(Y = [q, [e x f, e]_k] - [e, e x f]_k\). Then we compute
\[
[x, y]_k = [e x f, e]_k = (-1)^k e x f
\]
\[
Y e = ([q, (-1)^k e x f] - (-1)^k e x f) e = (-1)^{(k+1)}(e x f q) e.
\]

And
\[
f Y = f ([q, (-1)^k e x f] - (-1)^k e x f) = (-1)^k (f q e x f).
\]

Hence
\[
0 = [[q, [e x f, e]_k] - [e, e x f]_k, e x f] = [Y, e x f] = (-1)^{k+1} 2(e x f q) e x f.
\]

Since \(\text{char}(R) \neq 2\), this implies that \((f q e x f)^3 = 0\) for all \(x \in M_t(\mathbb{F})\). By Levitzki’s lemma [11, Lemma 1.1], \(f q e x = 0\) for all \(x \in M_t(\mathbb{F})\) and by primeness of \(R\), we get \(f q e = 0\). Since \(f\) and \(e\) are any two orthogonal idempotent elements in \(M_t(\mathbb{F})\), we have for any idempotent \(e\) in \(M_t(\mathbb{F})\), \((1 - e) q e = 0 = eq (1 - e)\), that is, \(eq = q e = q e\). Which implies \([q, e] = 0\). Since \(q\) commutes with all idempotents in \(M_t(\mathbb{F})\), \(q \in C\) and hence \(d = 0\), a contradiction. This completes the proof. \(\square\)

The following example shows that the main results are not true in the case of arbitrary rings.

**Example 2.1.** Let \(S\) be any non-commutative ring. Consider \(R = \left\{ \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} : a, b \in S \right\}\) and \(I = \left\{ \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} : a \in S \right\}\). Clearly, \(R\) is a ring with identity under the natural operations which is not prime. Define the maps on \(R\) as follows \(d(x) = [e_{11}, x]\) for all \(x \in R\). Then, it is easy to see that \(I\) is a nonzero ideal of \(R\), \(d\) is a nonzero ideal of \(R\) and \(d\) satisfies the requirements of Theorems 2.1 and 2.2 but \(R\) is not prime.

Hence, the hypothesis of primeness is crucial.

**Example 2.2.** Let \(R = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} : a, b, c \in S \right\}\) and \(I = \left\{ \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} : a \in S \right\}\). Clearly, \(R\) is a ring with identity which is not prime and \(I\) is a nonzero ideal of \(R\). Define \(d : R \to R\) such that \(d(x) = [x, e_{11} + e_{22}]\). Then, it is easy to see that \(d\) is a nonzero derivation of \(R\). Further,
for any \( x, y \in R \) the following conditions: \( d([x, y]_k)^n = [x, y]_k \) and \( d([x, y]_k) - [x, y]_k \in Z(R) \) are satisfied, where \( n, k \) are fixed positive integer.

Hence, in Theorems 2.1 and 2.2, the hypothesis of primeness cannot be omitted.

3. Derivations in semiprime rings

From now on, \( R \) is a semiprime ring and \( U \) is the left Utumi quotient ring of \( R \). In order to prove the main results of this section we will make use of the following facts:

Fact 3.1 ([3, Proposition 2.5.1]). Any derivation of a semiprime ring \( R \) can be uniquely extended to a derivation of its left Utumi quotient ring \( U \), and so any derivation of \( R \) can be defined on the whole \( U \).

Fact 3.2 ([6, p.38]). If \( R \) is semiprime, then so is its left Utumi quotient ring. The extended centroid \( C \) of a semiprime ring coincides with the center of its left Utumi quotient ring.

Fact 3.3 ([6, p.42]). Let \( B \) be the set of all the idempotents in \( C \), the extended centroid of \( R \). Suppose that \( R \) is an orthogonally complete \( B \)-algebra. For any maximal ideal \( P \) of \( B \), \( PR \) forms a minimal prime ideal of \( R \), which is invariant under any derivation of \( R \).

Now we are ready to prove the following:

Theorem 3.3. Let \( R \) be a semiprime ring, \( U \) the left Utumi quotient ring of \( R \) and \( k \) be a fixed positive integer. If \( R \) admits a nonzero derivation \( d \) such that \( d([x, y]_k)^n = [x, y]_k \) for all \( x, y \in R \), then there exists a central idempotent element \( e \) in \( U \) such that on the direct sum decomposition \( U = eU \oplus (1 - e)U \), \( d \) vanishes identically on \( eU \) and the ring \( (1 - e)U \) is commutative.

Proof. Since \( R \) is semiprime and \( d \) is a derivation of \( R \), we have given that \( d([x, y]_k)^n = [x, y]_k \) for all \( x, y \in R \). By Fact 3.2, \( Z(U) = C \), the extended centroid of \( R \), and, by Fact 3.1, the derivation \( d \) can be uniquely extended on \( U \). As we know that \( R \) and \( U \) satisfy the same differential identities [16], therefore \( R \) satisfies \( d([x, y]_k)^n = [x, y]_k \). Let \( B \) be the complete Boolean algebra of idempotents in \( C \) and \( M \) be any maximal ideal of \( B \). Since \( U \) is an orthogonally complete \( B \)-algebra [6, p.42], thus by Fact 3.3, \( MU \) is a prime ideal of \( U \), which is \( d \)-invariant. Denote \( U = U/MU \) and \( \overline{d} \) the derivation induced by \( d \) on \( U \), i.e., \( \overline{d}(u) = d(u) \) for all \( u \in U \). For any \( x, y \in U \), \( \overline{d}([x, y]_k)^n = [x, y]_k \). It is obvious that \( U \) is prime. Therefore, by Theorem 2.1, we have either \( U \) is commutative or \( \overline{d} = 0 \) in \( U \). This implies that, for any maximal ideal \( M \) of \( B \), \( d(U) \subseteq MU \) or \( [U, U] \subseteq MU \), where \( MU \) runs over all minimal prime ideals of \( U \). In any case \( d(U)[U, U] \subseteq MU = 0 \), for all \( M \). Therefore \( d(U)[U, U] \subseteq \bigcap_M MU = 0 \).

By using the theory of orthogonal completion for semiprime rings [3, Chapter 3], it is clear that there exists a central idempotent element \( e \) in \( U \) such that on the direct sum decomposition \( U = eU \oplus (1 - e)U \), \( d \) vanishes identically on \( eU \) and the ring \( (1 - e)U \) is commutative. With this completes the proof. \( \square \)

We come now to our last result of this section:

Theorem 3.4. Let \( R \) be a semiprime ring of characteristic different from 2 with center \( Z(R) \), \( U \) the left Utumi quotient ring of \( R \) and \( k \) be a fixed positive integer. If \( R \) admits a nonzero derivation \( d \) such that \( d([x, y]_k) - [x, y]_k \in Z(R) \) for all \( x, y \in R \), then there exists a central idempotent element \( e \) in \( U \) such that on the direct sum decomposition \( U = eU \oplus (1 - e)U \), \( d \) vanishes identically on \( eU \) and the ring \( (1 - e)U \) satisfies \( s_4 \), the standard identity in four variables.

Proof. By Fact 3.2, \( Z(U) = C \), the extended centroid of \( R \), and by Fact 3.1, the derivation \( d \) can be uniquely extended on \( U \). Since \( R \) and \( U \) satisfy the same differential identities, then \( d([x, y]_k)^n - [x, y]_k \in C \) for all \( x, y \in U \). Let \( B \) be the complete Boolean algebra of
idempotents in $C$ and $M$ be any maximal ideal of $B$. As already pointed out in the proof of Theorem 3.3, $U$ is an orthogonally complete $B$-algebra, and by Fact 3.3, $MU$ is a prime ideal of $U$, which is $d$-invariant. Let $\overline{d}$ be the derivation induced by $d$ on $\overline{U} = U/MU$. Since $Z(\overline{U}) = (C + MU)/MU = C/MU$, then $d([x,y]_{k})^{n} - [x,y]_{k} \in (C + MU)/MU$, for all $x, y \in \overline{U}$. Moreover $\overline{U}$ is prime, hence we may conclude, by Theorem 2.2, either $\overline{d} = 0$ in $\overline{U}$ or $\overline{U}$ satisfies $s_{4}$. This implies that, for any maximal ideal $M$ of $B$, either $d(U) \subseteq MU$ or $s_{4}(x_{1}, x_{2}, x_{3}, x_{4}) \subseteq MU$, for all $x_{1}, x_{2}, x_{3}, x_{4} \in U$. In any case $d(U)s_{4}(x_{1}, x_{2}, x_{3}, x_{4}) \subseteq \bigcap_{M}MU = 0$. From [3, Chapter 3], there exists a central idempotent element $e$ of $U$, the left Utumi quotient ring of $R$ such that on the direct sum decomposition $U = eU \oplus (1 - e)U$, $d(eU) = 0$ and the ring $(1 - e)U$ satisfies $s_{4}$. This completes the proof of the theorem. □

According to Theorem 2.1 and Theorem 2.2, we conclude with the following conjecture.

**Conjecture 3.1.** Let $R$ be a prime or semiprime ring with suitable torsion free restriction, $Z(R)$ be the center of $R$, $I$ be a nonzero ideal of $R$, and $n, k$ be the fixed positive integers. If $R$ admits a derivation $d$ such that $d([x,y]_{k})^{n} - [x,y]_{k} \in Z(R)$ for all $x, y \in I$, then $R$ is commutative (or satisfies $s_{4}$).

**Acknowledgement.** The authors wish to thank the referee for his/her suggestions which improve the quality of the paper.

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