CREAT. MATH. INFORM.
Volume **26** (2017), No. 1,
Pages 19 - 27

Online version at https://creative-mathematics.cunbm.utcluj.ro/
Print Edition: ISSN 1584 - 286X; Online Edition: ISSN 1843 - 441X
DOI: https://doi.org/10.37193/CMI.2017.01.03

A note on an Engel condition with derivations in rings

MOHD ARIF RAZA and NADEEM UR REHMAN

ABSTRACT. Let R be a prime ring with center $Z(R)$, C the extended centroid of R, d a derivation of R and n, k be two fixed positive integers. In the present paper we investigate the behavior of a prime ring R satisfying any one of the properties (i) $d([x,y]_k)^n = [x,y]_k$ (ii) if $char(R) \neq 2$, $d([x,y]_k) - [x,y]_k \in Z(R)$ for all x, y in some appropriate subset of R. Moreover, we also examine the case when R is a semiprime ring.

1. INTRODUCTION, NOTATION AND STATEMENTS OF THE RESULTS

Throughout this paper, unless specifically stated, R is a (semi)-prime ring, $Z(R)$ is the center of R, Q is the Martindale quotient ring of R and U is the Utumi quotient ring of R. The center of U, denoted by C, is called the extended centroid of R (we refer the reader to [3], for the definitions and related properties of these objects). For each $x, y \in R$ and each $k \geq 0$, define $[x,y]_k$ inductively by $[x,y]_0 = x$, $[x,y]_1 = xy - yx$ and $[x,y]_k = [[x,y]_{k-1}, y]$ for $k > 1$. The ring R is said to satisfy an Engel condition if there exists a positive integer k such that $[x,y]_k = 0$. Note that an Engel condition is a polynomial $[x,y]_k = \sum_{m=0}^{k}(-1)^m\binom{k}{m}y^m x y^{k-m}$ in non-commuting indeterminates x, y and $[x+z,y]_k = [x,y]_k + [z,y]_k$. Recall that a ring R is prime if for any $a, b \in R$, $aRb = \{0\}$ implies $a = 0$ or $b = 0$, and is semiprime if for any $a \in R$, $aRa = \{0\}$ implies $a = 0$. An additive mapping $d : R \to R$ is called a derivation if $d(xy) = d(x)y + xd(y)$ holds for all $x, y \in R$. In particular, d is an inner derivation induced by an element $a \in R$, if $d(x) = [a,x]$ for all $x \in R$.

Many results in literature indicate that the global structure of a ring R is often tightly connected to the behavior of additive mappings defined on R. During the past few decades, there has been an ongoing interest concerning the relationship between the commutativity of a ring and the existence of certain specific types of derivations. Derivation with certain properties investigated in various paper (see [1, 2, 4, 7, 19] and references therein). Starting from these results, many author studied derivations in the context of prime and semiprime rings. The Engel type identity with derivation appeared in the well-known paper of Posner [19], who proved that a prime ring admitting a nonzero derivation d such that $[d(x), x] \in Z(R)$ for all $x \in R$, must be commutative. Since then several authors have studied this kind of identities with derivations acting on one-sided, two-sided and Lie ideals of prime and semiprime rings (see [8], for a partial bibliography).

In 1992, Daif and Bell [7, Theorem 3], showed that if in a semiprime ring R there exists a nonzero ideal I of R and a derivation d such that $d([x,y]) = [x,y]$ for all $x, y \in I$, then $I \subseteq Z(R)$. If R is a prime ring, this implies that R is commutative. Recently in 2011, Huang [12] generalized Daif and Bell result. More precisely he prove that, if R is a prime ring, I is a nonzero ideal of R, m, n are two fixed positive integers and d a derivation of R

Received: 24.03.2016. In revised form: 01.07.2016. Accepted: 15.07.2016
2010 *Mathematics Subject Classification.* 16N60, 16U80, 16W25.
Key words and phrases. *Prime and semiprime rings, derivation, maximal right ring of quotient, generalized polynomial identity (GPI), ideal.*
Corresponding author: Mohd Arif Raza; arifraza03@gmail.com

satisfy $d([x,y])^m = [x,y]_n$ for all $x,y \in I$, then R is commutative. In 1994, Giambruno et. al. [10] established that a ring must be commutative if it satisfy $[x,y]_k^n = [x,y]_k$.

It is natural to ask what we can say about the commutativity of R satisfying any of the following conditions: (P_1) $d([x,y]_k)^n = [x,y]_k$ (P_2) $d([x,y]_k) - [x,y]_k \in Z(R)$ for all $x,y \in I$. This result generalized a theorem of Huang [12], and for derivation Giambruno theorem [10].

2. Derivations in prime rings

We have started with the following proposition which is very crucial for developing the proof of our main result.

Proposition 2.1. *Let R be a prime ring, Q the Martindale quotient ring of R, I a nonzero ideal of R and n,k be two fixed positive integers. If d is a nonzero inner derivation on Q, in the sense that there exists $q \in Q$ such that $d(x) = [q,x]$ for all $x \in R$, and I satisfies $([q,[x,y]_k])^n = [x,y]_k$ for all $x,y \in I$, then R is commutative.*

Proof. Assume that R is non-commutative. We have given that $([q,[x,y]_k])^n = [x,y]_k$ for all $x,y \in I$. Since $d \neq 0$, $q \notin Z(R)$ and hence I satisfied generalized polynomial identity(GPI). By Chuang [5, Theorem 2], I and Q satisfy the same generalized polynomial identities, thus we have

$$([q,[x,y]_k])^n = [x,y]_k \text{ for all } x,y \in Q.$$

In case the center C of Q is infinite, we have

$$([q,[x,y]_k])^n = [x,y]_k \text{ for all } x,y \in Q \otimes_C \overline{C},$$

where $\overline{C}$ is algebraic closure of C. Since both Q and $Q \otimes_C \overline{C}$ are prime and centrally closed [9, Theorems 2.5 and 3.5], we may replace R by Q or $Q \otimes_C \overline{C}$ according as C is finite or infinite. Thus we may assume that R is centrally closed over C $(i.e., RC = R)$ which is either finite or algebraically closed and $([q,[x,y]_k])^n = [x,y]_k$ for all $x,y \in R$. By Martindale [17, Theorem 3], RC (and so R) is a primitive ring having nonzero socle H with $\mathcal{D}$ as the associated division ring.

Hence by Jacobson's theorem [13, p.75], R is isomorphic to a dense ring of linear transformations of some vector space $\mathcal{V}$ over $\mathcal{D}$ and H consists of the finite rank linear transformations in R. If $\mathcal{V}$ is a finite dimensional over $\mathcal{D}$, then the density of R on $\mathcal{V}$ implies that $R \cong M_t(\mathcal{D})$, where $t = dim_{\mathcal{D}}\mathcal{V}$. Assume first that $dim_{\mathcal{D}}\mathcal{V} \geq 3$.

Step 1. We want to show that, for any $v \in \mathcal{V}$, v and qv are linearly $\mathcal{D}$-dependent. If $v = 0$, then $\{v, qv\}$ is linearly $\mathcal{D}$-dependent. Now let $v \neq 0$ and $\{v, qv\}$ is linearly $\mathcal{D}$-independent, since $dim_{\mathcal{D}}\mathcal{V} \geq 3$, then there exists $w \in \mathcal{V}$ such that $\{v, qv, w\}$ is also linearly $\mathcal{D}$-independent. By the density of R, there exist $x,y \in R$ such that:

$$\begin{array}{lll} xv = v, & xqv = 0, & xw = v \\ yv = 0, & yqv = w, & yw = w. \end{array}$$

These imply that $(-1)^n v = ([q,[x,y]_k])^n v - ([x,y]_k)v = 0$, a contradiction. So, we conclude that $\{v, qv\}$ is linearly $\mathcal{D}$-dependent, for all $v \in \mathcal{V}$.

Step 2. We show here that there exists $\alpha \in \mathcal{D}$ such that $qv = v\alpha$, for any $v \in \mathcal{V}$. Now choose $v,w \in \mathcal{V}$ linearly independent. By Step 1, there exist $\alpha_v, \alpha_w, \alpha_{v+w} \in \mathcal{D}$ such that

$$qv = v\alpha_v,\ qw = w\alpha_w,\ q(v+w) = (v+w)\alpha_{v+w}$$

Moreover,

$$v\alpha_v + w\alpha = (v+w)\alpha_{v+w}.$$

Hence

$$v(\alpha_v - \alpha_{v+w}) + w(\alpha_w - \alpha_{v+w}) = 0,$$

and because v, w are linearly $\mathcal{D}$-independent, we have $\alpha_v = \alpha_w = \alpha_{v+w}$, that is, α does not depend on the choice of v. This completes the proof of Step 2.

Let now for $r \in R, v \in \mathcal{V}$. By Step 2, $qv = v\alpha$, $r(qv) = r(v\alpha)$, and also $q(rv) = (rv)\alpha$. Thus $0 = [q, r]v$, for any $v \in \mathcal{V}$, that is $[q, r]\mathcal{V} = 0$. Since $\mathcal{V}$ is a left faithful irreducible R-module, hence $[q, r] = 0$, for all $r \in R$, i.e., $q \in Z(R)$ and $d = 0$, which contradicts our hypothesis.

Therefore $dim_{\mathcal{D}}\mathcal{V}$ must be ≤ 2. In this case R is a simple GPI-ring with 1, and so it is a central simple algebra finite dimensional over its center. By Lanski [15, Lemma 2], it follows that there exists a suitable filed $\mathbb{F}$ such that $R \subseteq M_t(\mathbb{F})$, the ring of all $t \times t$ matrices over $\mathbb{F}$, and moreover, $M_t(\mathbb{F})$ satisfies the same generalized polynomial identity of R.

If we assume $t \geq 3$, then by the same argument as in Steps 1 and 2, we get a contradiction. Obviously if $t = 1$, then R is commutative. Thus we may assume that $t = 2$, i.e., $R \subseteq M_2(\mathbb{F})$, where $M_2(\mathbb{F})$ satisfies $([q, [x, y]_k])^n = [x, y]_k$. Denote by e_{ij} the usual unit matrix with 1 in (i, j)-entry and zero elsewhere. Since by choosing $x = e_{12}, y = e_{22}$. In this case we have $(qe_{12} - e_{12}q)^n = e_{12}$. Right multiplying by e_{12}, we get $(-1)^n(e_{12}q)^n e_{12} = (qe_{12} - e_{12}q)^n = e_{12}e_{12} = 0$. Now set $q = \begin{pmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{pmatrix}$. By calculation, we find that $(-1)^n \begin{pmatrix} 0 & q_{21}^n \\ 0 & 0 \end{pmatrix} = 0$, which implies that $q_{21} = 0$. In the same manner, we can see that $q_{12} = 0$. Thus we conclude that q is a diagonal matrix in $M_2(\mathbb{F})$. Let $\chi \in Aut(M_2(\mathbb{F}))$. Since $([\chi(q), [\chi(x), \chi(y)]_k])^n = [\chi(x), \chi(y)]_k$, then $\chi(q)$ must be diagonal matrix in $M_2(\mathbb{F})$. In particular, let $\chi(x) = (1 - e_{ij})x(1 + e_{ij})$ for $i \neq j$. Then $\chi(q) = q + (q_{ii} - q_{jj})e_{ij}$, that is $q_{ii} = q_{jj}$ for $i \neq j$. This implies that q is central in $M_2(\mathbb{F})$, which leads to $d = 0$, a contradiction. Thus $t = 1$, that is R is commutative. This completes the proof of the proposition. □

Theorem 2.1. *Let R be a prime ring, I a nonzero ideal of R and n, k be two fixed positive integers. If R admits a derivation d such that $d([x, y]_k)^n = [x, y]_k$ for all $x, y \in I$, then R is commutative.*

Proof. If $d = 0$, then $[x, y]_k = 0$ which is rewritten as $[I_x(y), y]_{k-1} = 0$ for all $x, y \in I$. By Lanski [15, Theorem 1], either R is commutative or $I_x = 0$ i.e., $I \subseteq Z(R)$ in which case R is also commutative by Mayne [18, Lemma 3]. Now we assume that $d \neq 0$ and $d([x, y]_k)^n = [x, y]_k$ for all $x, y \in I$, that is I satisfies the differential identity

$$\begin{aligned}\Bigg(\sum_{m=0}^{k}(-1)^m\binom{k}{m}\Big(\sum_{i+j=m-1} & y^i d(y) y^j)xy^{k-m} \\ &+ \sum_{m=0}^{k}(-1)^m\binom{k}{m}y^m d(x) y^{k-m} \\ &+ \sum_{m=0}^{k}(-1)^m\binom{k}{m}y^m x\Big(\sum_{r+s=k-m-1} y^r d(y) y^s\Big)\Bigg)^n \\ &= \sum_{m=0}^{k}(-1)^m\binom{k}{m}y^m x y^{k-m}.\end{aligned} \tag{2.1}$$

In the light of Kharchenko's theory [14], we split the proof into two cases:

Firstly we assume that d is an inner derivation induced by an element $q \in Q$ such that $d(x) = [q, x]$ for all $x \in R$. Therefore from (2.1), we have

$$\begin{aligned}\Big(\sum_{m=0}^{k}(-1)^m\binom{k}{m}&(\sum_{i+j=m-1} y^i([q,y])y^j)xy^{k-m}\\ &+\sum_{m=0}^{k}(-1)^m\binom{k}{m}y^m([q,x])y^{k-m}\\ &+\sum_{m=0}^{k}(-1)^m\binom{k}{m}y^m x(\sum_{r+s=k-m-1} y^r([q,y])y^s)\Big)^n\\ &=\sum_{m=0}^{k}(-1)^m\binom{k}{m}y^m xy^{k-m} \text{ for all } x,y\in I.\end{aligned}$$

It can be easily seen that

$$\begin{aligned}\Big(q(\sum_{m=0}^{k}(-1)^m\binom{k}{m}y^m xy^{k-m})-(\sum_{m=0}^{k}(-1)^m\binom{k}{m}y^m xy^{k-m})q\Big)^n\\ =\sum_{m=0}^{k}(-1)^m\binom{k}{m}y^m xy^{k-m}.\end{aligned}$$

And hence we can write $([q, [x, y]_k])^n = [x, y]_k$ for all $x, y \in I$. In this case we are done from Proposition 2.1.

Secondly we now assume that d is an outer derivation on Q. Now by Kharchencko's theorem [14], I satisfy the generalized polynomial identity

$$\begin{aligned}\Big(\sum_{m=0}^{k}(-1)^m\binom{k}{m}&(\sum_{i+j=m-1} y^i zy^j)xy^{k-m}\\ &+\sum_{m=0}^{k}(-1)^m\binom{k}{m}y^m wy^{k-m}\\ &+\sum_{m=0}^{k}(-1)^m\binom{k}{m}y^m x(\sum_{r+s=k-m-1} y^r zy^s)\Big)^n\\ &=\sum_{m=0}^{k}(-1)^m\binom{k}{m}y^m xy^{k-m},\end{aligned}$$

and in particular I satisfy the polynomial identity

$$\sum_{m=0}^{k}(-1)^m\binom{k}{m}y^m xy^{k-m}=0 \text{ for all } x,y\in I.$$

That is $[x, y]_k = 0$ for all $x, y \in I$, and hence R is commutative by the same argument presented above. This completes the proof of the theorem. □

We immediately get the following corollary from the above theorem:

Corollary 2.1. *Let R be a prime ring, I a nonzero ideal of R and k be a fixed positive integer. If R admits a derivation d such that $d([x, y]_k) = [x, y]_k$ for all $x, y \in I$, then R is commutative.*

Theorem 2.2. *Let R be a prime ring of characteristic different from 2 with center $Z(R)$, I a nonzero ideal of R and k be a fixed positive integer. If R admits a derivation d such that $d([x,y]_k)-[x,y]_k \in Z(R)$ for all $x,y \in I$, then R satisfies s_4, the standard identity in four variables.*

Proof. If $d=0$, then $[x,y]_k \in Z(R)$ for all $x,y \in I$ and hence R satisfies the same identities. In this case the identity is a polynomial so that there exists a field $\mathbb{F}$ such that R and $\mathbb{F}_t$ satisfy the same identities. Thus pick $x=e_{31}, y=e_{11}-e_{22}$, we see that $[x,y]_k = e_{31} \notin Z(R)$, a contradiction. Therefore $t \leq 2$ and R satisfies s_4. Now, we assume that $d \neq 0$.

If $d([x,y]_k)=[x,y]_k$ for all $x,y \in I$, then R is commutative by Corollary 2.1. Otherwise we have $I \cap Z(R) \neq 0$ by our assumptions. Let now J be a nonzero two-sided ideal of R_Z, the ring of the central quotient of R. Since $J \cap R$ is an ideal of R, then $J \cap R \cap Z(R) \neq 0$. That is J contains an invertible element in R_Z, and so R_Z is simple with 1. By the hypothesis for any $x,y \in I$ and $r \in R$, thus I satisfies the differential identity $[d([x,y]_k)-[x,y]_k, r]=0$. Which can be rewritten as, that is, I satisfy the polynomial identity

$$\begin{aligned} f(x,y,r,d(x),d(y)) = \Bigg[& \sum_{m=0}^{k}(-1)^m \binom{k}{m} \Big(\sum_{i+j=m-1} y^i d(y) y^j \Big) x y^{k-m} \\ & + \sum_{m=0}^{k}(-1)^m \binom{k}{m} y^m d(x) y^{k-m} \\ & + \sum_{m=0}^{k}(-1)^m \binom{k}{m} y^m x \Big(\sum_{r+s=k-m-1} y^r d(y) y^s \Big) \\ & - \sum_{m=0}^{k}(-1)^m \binom{k}{m} y^m x y^{k-m}, r \Bigg] = 0. \end{aligned}$$

If d is not an inner derivation, then I satisfies the polynomial identity

$$\begin{aligned} f(x,y,r,w,z) = \Bigg[& \sum_{m=0}^{k}(-1)^m \binom{k}{m} \Big(\sum_{i+j=m-1} y^i z y^j \Big) x y^{k-m} \\ & + \sum_{m=0}^{k}(-1)^m \binom{k}{m} y^m w y^{k-m} \\ & + \sum_{m=0}^{k}(-1)^m \binom{k}{m} y^m x \Big(\sum_{r+s=k-m-1} y^r z y^s \Big) \\ & - \sum_{m=0}^{k}(-1)^m \binom{k}{m} y^m x y^{k-m}, r \Bigg] = 0. \end{aligned}$$

By Kharchenko's theorem [14], and setting $z=w=0$ yields the identity

$$\Bigg[\sum_{m=0}^{k}(-1)^m \binom{k}{m} y^m x y^{k-m}, r \Bigg] = 0.$$

In this case it is well known that there exists a field $\mathbb{F}$ such that R and $\mathbb{F}_t$ satisfy the same polynomial identities. Thus $\sum_{m=0}^{k}(-1)^m \binom{k}{m} y^m x y^{k-m}$ is central in $\mathbb{F}_t$. Suppose $t \geq 3$ and choose $x=e_{31}, y=e_{33}$. Then $\sum_{m=0}^{k}(-1)^m \binom{k}{m} y^m x y^{k-m} = (-1)^k e_{31} \notin Z(\mathbb{F}_3)$, contrary to our assumptions. This forces $t \leq 2$, i.e., R satisfies s_4. Notice that in this case $t=1$, then R is commutative. But if $t \geq 2$ and $x = e_{12}$, $y = e_{22}$, we get the contradiction $\sum\limits_{m=0}^{k}(-1)^m\binom{k}{m}y^m x y^{k-m} = e_{12} \notin Z(\mathbb{F}_2)$.

Now let d be an inner derivation induced by an element $q \in Q$, that is , $d(x) = [q, x]$ for all $x \in R$. Since $d \neq 0$, we may assume that $q \notin Z(R)$. By localizing R at $Z(R)$ it is easy to see that

$$\sum_{m=0}^{k}(-1)^m\binom{k}{m}\Big(\sum_{i+j=m-1} y^i[q,y]y^j\Big)xy^{k-m} + \sum_{m=0}^{k}(-1)^m\binom{k}{m}y^m[q,x]y^{k-m}$$
$$+\sum_{m=0}^{k}(-1)^m\binom{k}{m}y^m x\Big(\sum_{r+s=k-m-1} y^r[q,y]y^s\Big)$$
$$-\sum_{m=0}^{k}(-1)^m\binom{k}{m}y^m x y^{k-m} \in Z(R_Z), \qquad \text{for any } x, y \in R_Z.$$

Since R and R_Z satisfy the same polynomial identities, in order to prove that R is commutative, we may assume that R is simple with 1. In this case,

$$\sum_{m=0}^{k}(-1)^m\binom{k}{m}(\sum_{i+j=m-1} y^i[q,y]y^j)xy^{k-m}$$
$$+\sum_{m=0}^{k}(-1)^m\binom{k}{m}y^m[q,x]y^{k-m}$$
$$+\sum_{m=0}^{k}(-1)^m\binom{k}{m}y^m x(\sum_{r+s=k-m-1} y^r[q,y]y^s)$$
$$-\sum_{m=0}^{k}(-1)^m\binom{k}{m}y^m x y^{k-m} \in Z(R), \text{ for all } x, y \in R.$$

Therefore R satisfies a generalized polynomial identity and it is simple with 1, which implies that $Q = RC = R$ and R has a minimal right ideal. Thus $q \in R = Q$ and R is simple artinian, that is, $R = \mathcal{D}_t$, where $\mathcal{D}$ is a division ring finite dimensional over $Z(R)$ by [17]. From [15, Lemma 2], it follows that there exists a suitable field $\mathbb{F}$ such that $R \subseteq M_t(\mathbb{F})$, the ring of all $t \times t$ matrices over $\mathbb{F}$, and moreover $M_t(\mathbb{F})$ satisfies the generalized polynomial identity

$$\Big[\sum_{m=0}^{k}(-1)^m\binom{k}{m}(\sum_{i+j=m-1} y^i[q,y]y^j)xy^{k-m}$$
$$+\sum_{m=0}^{k}(-1)^m\binom{k}{m}y^m[q,x]y^{k-m}$$
$$+\sum_{m=0}^{k}(-1)^m\binom{k}{m}y^m x(\sum_{r+s=k-m-1} y^r[q,y]y^s)$$
$$-\sum_{m=0}^{k}(-1)^m\binom{k}{m}y^m x y^{k-m}, r\Big] = 0 \text{ for all } x, y, r \in M_t(\mathbb{F}).$$

In this case, as already see in Theorem 2.1, we have $[q,[x,y]_k]-[x,y]_k$ is central in $M_t(\mathbb{F})$. Suppose that $t\geq 3$ and $M_t(\mathbb{F})$ satisfy

$$[[q,[x,y]_k]-[x,y]_k,r]=0 \text{ for all } x,y,r\in M_t(\mathbb{F}). \tag{2.2}$$

Let $q=\sum_t a_{tt}e_{tt}$, with $a_t\in\mathbb{F}$, and choose $x=e_{ij}$, $y=e_{jj}$, and $r=e_{ij}$, where $i\neq j$. Then by using the same argument presented in Theorem 2.1, we get

$$[[q,[x,y]_k]-[x,y]_k,r]=-2e_{ij}qe_{ij},$$

which has rank 1 and so it cannot be central in $M_t(\mathbb{F})$, with $t\geq 3$. This implies that $t\leq 2$ and R satisfy s_4. Now let e and f be any two orthogonal idempotent elements in $M_t(\mathbb{F})$. Now, we replace x with exf, y with e, and r by exf in (2.2) and let $Y=[q,[exf,e]_k]-[e,exf]_k$. Then we compute

$$[x,y]_k=[exf,e]_k=(-1)^k exf$$

$$\begin{aligned}Ye&=\big([q,(-1)^k exf]-(-1)^k exf\big)e\\&=(-1)^{(k+1)}(exfq)e.\end{aligned}$$

And

$$\begin{aligned}fY&=f\big([q,(-1)^k exf]-(-1)^k exf\big)\\&=(-1)^k(fqex)f.\end{aligned}$$

Hence

$$\begin{aligned}0&=[[q,[exf,e]_k]-[e,exf]_k,exf]\\&=[Y,exf]\\&=(-1)^{k+1}2(exfq)exf.\end{aligned}$$

Since $char(R)\neq 2$, this implies that $(fqex)^3=0$ for all $x\in M_t(\mathbb{F})$. By Levitzki's lemma [11, Lemma 1.1], $fqex=0$ for all $x\in M_t(\mathbb{F})$ and by primeness of R, we get $fqe=0$. Since f and e are any two orthogonal idempotent elements in $M_t(\mathbb{F})$, we have for any idempotent e in $M_t(\mathbb{F})$, $(1-e)qe=0=eq(1-e)$, that is, $eq=eqe=qe$. Which implies $[q,e]=0$. Since q commutes with all idempotents in $M_t(\mathbb{F})$, $q\in C$ and hence $d=0$, a contradiction. This completes the proof. $\square$

The following example shows that the main results are not true in the case of arbitrary rings.

Example 2.1. Let S be any non-commutative ring. Consider $R=\left\{\begin{pmatrix}a&b\\0&0\end{pmatrix}:a,b\in S\right\}$ and $I=\left\{\begin{pmatrix}0&a\\0&0\end{pmatrix}:a\in S\right\}$. Clearly, R is a ring with identity under the natural operations which is not prime. Define the maps on R as follows $d(x)=[e_{11},x]$, for all $x\in R$. Then, it is easy to see that I is a nonzero ideal of R, d is a nonzero ideal of R and d satisfies the requirements of Theorems 2.1 and 2.2 but R is not prime.

Hence, the hypothesis of primeness is crucial.

Example 2.2. Let $R=\left\{\begin{pmatrix}a&b\\0&c\end{pmatrix}:a,b,c\in S\right\}$ and $I=\left\{\begin{pmatrix}0&a\\0&0\end{pmatrix}:a\in S\right\}$. Clearly, R is a ring with identity which is not prime and I is a nonzero ideal of R. Define $d:R\to R$ such that $d(x)=[x,e_{11}+e_{22}]$. Then, it is easy to see that d is a nonzero derivation of R. Further, for any $x, y \in R$ the following conditions: $d([x,y]_k)^n = [x,y]_k$ and $d([x,y]_k) - [x,y]_k \in Z(R)$ are satisfied, where n, k are fixed positive integer.

Hence, in Theorems 2.1 and 2.2, the hypothesis of primeness cannot be omitted.

3. Derivations in semiprime rings

From now on, R is a semiprime ring and U is the left Utumi quotient ring of R. In order to prove the main results of this section we will make use of the following facts:

Fact 3.1 ([3, Proposition 2.5.1]). *Any derivation of a semiprime ring R can be uniquely extended to a derivation of its left Utumi quotient ring U, and so any derivation of R can be defined on the whole U.*

Fact 3.2 ([6, p.38]). *If R is semiprime, then so is its left Utumi quotient ring. The extended centroid C of a semiprime ring coincides with the center of its left Utumi quotient ring.*

Fact 3.3 ([6, p.42]). *Let B be the set of all the idempotents in C, the extended centroid of R. Suppose that R is an orthogonally complete B-algebra. For any maximal ideal P of B, PR forms a minimal prime ideal of R, which is invariant under any derivation of R.*

Now we are ready to prove the following:

Theorem 3.3. *Let R be a semiprime ring, U the left Utumi quotient ring of R and k be a fixed positive integer. If R admits a nonzero derivation d such that $d([x,y]_k)^n = [x,y]_k$ for all $x, y \in R$, then there exists a central idempotent element e in U such that on the direct sum decomposition $U = eU \oplus (1-e)U$, d vanishes identically on eU and the ring $(1-e)U$ is commutative.*

Proof. Since R is semiprime and d is a derivation of R, we have given that $d([x,y]_k)^n = [x,y]_k$ for all $x, y \in R$. By Fact 3.2, $Z(U) = C$, the extended centroid of R, and, by Fact 3.1, the derivation d can be uniquely extended on U. As we know that R and U satisfy the same differential identities [16], therefore R satisfies $d([x,y]_k)^n = [x,y]_k$. Let B be the complete Boolean algebra of idempotents in C and M be any maximal ideal of B. Since U is an orthogonally complete B-algebra [6, p.42], thus by Fact 3.3, MU is a prime ideal of U, which is d-invariant. Denote $\overline{U} = U/MU$ and $\overline{d}$ the derivation induced by d on $\overline{U}$, i.e., $\overline{d}(\overline{u}) = \overline{d(u)}$ for all $u \in U$. For any $\overline{x}, \overline{y} \in \overline{U}$, $\overline{d}([\overline{x},\overline{y}]_k)^n = [\overline{x},\overline{y}]_k$. It is obvious that $\overline{U}$ is prime. Therefore, by Theorem 2.1, we have either $\overline{U}$ is commutative or $\overline{d} = 0$ in $\overline{U}$. This implies that, for any maximal ideal M of B, $d(U) \subseteq MU$ or $[U,U] \subseteq MU$, where MU runs over all minimal prime ideals of U. In any case $d(U)[U,U] \subseteq MU = 0$, for all M. Therefore $d(U)[U,U] \subseteq \bigcap_M MU = 0$.

By using the theory of orthogonal completion for semiprime rings [3, Chapter 3], it is clear that there exists a central idempotent element e in U such that on the direct sum decomposition $U = eU \oplus (1-e)U$, d vanishes identically on eU and the ring $(1-e)U$ is commutative. With this completes the proof. □

We come now to our last result of this section:

Theorem 3.4. *Let R be a semiprime ring of characteristic different from 2 with center $Z(R)$, U the left Utumi quotient ring of R and k be a fixed positive integer. If R admits a nonzero derivation d such that $d([x,y]_k) - [x,y]_k \in Z(R)$ for all $x, y \in R$, then there exists a central idempotent element e in U such that on the direct sum decomposition $U = eU \oplus (1-e)U$, d vanishes identically on eU and the ring $(1-e)U$ satisfies s_4, the standard identity in four variables.*

Proof. By Fact 3.2, $Z(U) = C$, the extended centroid of R, and by Fact 3.1, the derivation d can be uniquely extended on U. Since R and U satisfy the same differential identities, then $d([x,y]_k)^n - [x,y]_k \in C$ for all $x, y \in U$. Let B be the complete Boolean algebra of idempotents in C and M be any maximal ideal of B. As already pointed out in the proof of Theorem 3.3, U is an orthogonally complete B-algebra, and by Fact 3.3, MU is a prime ideal of U, which is d-invariant. Let $\overline{d}$ be the derivation induced by d on $\overline{U} = U/MU$. Since $Z(\overline{U}) = (C + MU)/MU = C/MU$, then $d([x, y]_k)^n - [x, y]_k \in (C + MU)/MU$, for all $x, y \in \overline{U}$. Moreover $\overline{U}$ is prime, hence we may conclude, by Theorem 2.2, either $\overline{d} = 0$ in $\overline{U}$ or $\overline{U}$ satisfies s_4. This implies that, for any maximal ideal M of B, either $d(U) \subseteq MU$ or $s_4(x_1, x_2, x_3, x_4) \subseteq MU$, for all $x_1, x_2, x_3, x_4 \in U$. In any case $d(U)s_4(x_1, x_2, x_3, x_4) \subseteq \bigcap_M MU = 0$. From [3, Chapter 3], there exists a central idempotent element e of U, the left Utumi quotient ring of R such that on the direct sum decomposition $U = eU \oplus (1 - e)U$, $d(eU) = 0$ and the ring $(1 - e)U$ satisfies s_4. This completes the proof of the theorem. □

According to Theorem 2.1 and Theorem 2.2, we conclude with the following conjecture.

Conjecture 3.1. *Let R be a prime or semiprime ring with suitable torsion free restriction, $Z(R)$ be the center of R, I be a nonzero ideal of R, and n, k be the fixed positive integers. If R admits a derivation d such that $d([x, y]_k)^n - [x, y]_k \in Z(R)$ for all $x, y \in I$, then R is commutative (or satisfies s_4).*

Acknowledgement. The authors wish to thank the referee for his/her suggestions which improve the quality of the paper.

REFERENCES

[1] Ali, S. and Huang, S., *On derivations in semiprime rings*, Algebr. Represent. Theory, **15** (2012), 1023–1033
[2] Ashraf, M. and Rehman, N., *On commutativity of rings with derivations*, Result. Math., **42** (2002), 3–8
[3] Beidar, K. I., Martindale III, W. S. and Mikhalev, A. V., *Rings with Generalized Identities*, Pure and Applied Mathematics, Marcel Dekker 196, New York, 1996
[4] Bell, H. E. and Daif, M. N., *On derivations and commutativity in prime rings*, Acta Math. Hung., **66** (1995), 337–343
[5] Chuang, C. L., *GPIs having coefficients in Utumi quotient rings*, Proc. Amer. Math. Soc., **103** (1988), 723–728
[6] Chuang, C. L., *Hypercentral derivations*, J. Algebra, **166** (1994), 34–71
[7] Daif, M. N. and Bell, H. E., *Remarks on derivations on semiprime rings*, Int. J. Math. Math. Sci., **15** (1992), 205–206
[8] Dhara, B. and Sharma, R. K., *Vanishing power values of commutators with derivations*, Sib. Math. J., **50** (2009), 60–65
[9] Erickson, T. S., Martindale III, W. S. and Osborn J. M., *Prime nonassociative algebras*, Pacific J. Math., **60** (1975), 49–63
[10] Giambruno, A., Goncalves, J. Z. and Mandel, A., *Rings with algebraic n-Engel elements*, Comm. Algebra, **22** (1994), No. 5, 1685–1701
[11] Herstein, I. N., Topics in ring theory, *Univ. Chicago Press*, 1969
[12] Huang, S., *Derivations with Engel conditions in prime and semiprime rings*, Czechoslovak Math. J., **61** (2011), No. 136, 1135–1140
[13] Jacobson, N., Structure of Rings, *Colloquium Publications 37, Amer. Math. Soc. VII, Provindence, RI*, 1956
[14] Kharchenko, V. K., *Differential identities of prime rings*, Algebra Logic, **17** (1979), 155–168
[15] Lanski, C., *An Engel condition with derivation*, Proc. Amer. Math. Soc., **118** (1993), 731–734
[16] Lee, T. K., *Semiprime rings with differential identities*, Bull. Inst. Math. Acad. Sin., **20** (1992), 27–38
[17] Martindale III, W. S., *Prime rings satisfying a generalized polynomial identity*, J. Algebra, **12** (1969), 576–584
[18] Mayne, J. H., *Centralizing mappings of prime rings*, Cand. Math. Bull., **27** (1984), 122–126
[19] Posner, E. C., *Derivations in prime rings*, Proc. Amer. Math. Soc., **8** (1958), 1093–1100

DEPARTMENT OF MATHEMATICS
ALIGARH MUSLIM UNIVERSITY
ALIGARH-202002, INDIA
Email address: `arifraza03@gmail.com`

DEPARTMENT OF MATHEMATICS
FACULTY OF SCIENCE
TAIBAH UNIVERSITY, AL-MADINAH, KSA
Email address: `rehman100@gmail.com`