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On the Stancu operators and their applications

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ABSTRACT. The aim of this paper is to pay a modest homage to the world class researcher Professor Dimitrie D. Stancu, well known for the classes of linear positive operators he introduced and studied, which have influenced, are still influencing and will influence the future development of approximation theory.

1. INTRODUCTION

Dimitrie D. Stancu
(1927–2014)

Let α, β, γ be real non-negative parameters satisfying the following relations: $\alpha \geq 0$ may depend on the natural number n and $0 \leq \beta \leq \gamma$. We recall that, for any function $f : [0,1] \to \mathbb{R}$, the Stancu operators are defined [24] by

$$S_n^{\langle \alpha,\beta,\gamma\rangle}(f;x) = \sum_{k=0}^{n} s_{n,k}^{\langle\alpha\rangle}(x) f\left(\tfrac{k+\beta}{n+\gamma}\right) = \sum_{k=0}^{n} \tbinom{n}{k} \frac{x^{[k,-\alpha]}(1-x)^{[n-k,-\alpha]}}{1^{[n,-\alpha]}} f\left(\tfrac{k+\beta}{n+\gamma}\right). \tag{1.1}$$

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The notation $x^{[k,-\alpha]}$ represents the factorial power of x with increment $(-\alpha)$, defined by

$$x^{[k,-\alpha]} := \begin{cases} \prod_{i=0}^{k-1}(x+i\alpha), & \text{if } k \geq 1 \text{ and } x \neq 0 \\ 1, & \text{elsewhere.} \end{cases} \tag{1.2}$$

In the paper [10], the Stancu operators given by the relation (1.1) are presented as a special case. By choosing suitable values for the parameters α, β, γ, from Stancu operators (1.1) one can obtain the most important linear positive operators in approximation theory.

The aim of this paper is to present a survey of some results stemming from several classes of Stancu linear positive operators, thus paying a modest tribute to the great mathematician D. D. Stancu, who was the first author's PhD supervisor.

2. Stancu operators based on factorial powers

In 1923, Eggenberger and Pólya [8] considered an urn model which contains w white balls and b black balls. A ball is drawn randomly and then replaced together with s balls of the same color. This procedure is repeated n times by noting the distribution of the random variable X representing the number of times a white ball is drawn. The distribution of X is given by

$$Pr(X=k) = \binom{n}{k}\frac{w(w+s)\cdot\ldots\cdot(w+\overline{k-1}s)b(b+s)\cdot\ldots\cdot(b+\overline{n-k-1}s)}{(w+b)(w+b+s)\cdot\ldots\cdot(w+b+\overline{n-1}s)}. \tag{2.3}$$

Based on Pólya-Eggenberger distribution (2.3), Stancu [20] introduced a new class of linear positive operators associated to a real-valued function $f:[0,1]\to\mathbb{R}$, given by

$$S_n^{\langle\alpha\rangle}(f;x) = \sum_{k=0}^{n} s_{n,k}^{\langle\alpha\rangle}(x) f\left(\tfrac{k}{n}\right) = \sum_{k=0}^{n}\binom{n}{k}\frac{x^{[k,-\alpha]}(1-x)^{[n-k,-\alpha]}}{1^{[n,-\alpha]}} f\left(\tfrac{k}{n}\right), \tag{2.4}$$

where α is a non-negative parameter which may depend only on the natural number n. If we choose the value 0 for the parameters β and γ in the relation (1.1), then we obtain the Stancu operators (2.4). We note that Stancu operators (2.4) are of Bernstein type because if we take $\alpha = 0$ (2.4), one obtains the well-known Bernstein operators [5], given by

$$B_n(f;x) = \sum_{k=0}^{n} s_{n,k}(x) f\left(\tfrac{k}{n}\right) = \sum_{k=0}^{n}\binom{n}{k}x^k(1-x)^{n-k} f\left(\tfrac{k}{n}\right). \tag{2.5}$$

In the papers [20], [21], Stancu had proved that the classical Mirakjan-Favard-Szász operators could be obtained as limiting case from Stancu operators (2.4). For $\alpha = n^{-2}$ and the change of variable $x = my/n$, where m is a natural number not depending on n, the right hand side of relation (2.4) becomes

$$\sum_{k=0}^{n}\binom{n}{k}\frac{\prod_{i=0}^{k-1}\left(\frac{my}{n}+\frac{i}{n^2}\right)\cdot\prod_{j=0}^{n-k-1}\left(1-\frac{my}{n}+\frac{j}{n^2}\right)}{\left(1+\frac{1}{n^2}\right)\left(1+\frac{2}{n^2}\right)\cdot\ldots\cdot\left(1+\frac{n-1}{n^2}\right)} f\left(\frac{k}{m}\right) =$$

$$= \sum_{k=0}^{n}\frac{1}{k!}\left(1-\frac{1}{n}\right)\cdot\ldots\cdot\left(1-\frac{k-1}{n}\right)\frac{\prod_{i=0}^{k-1}\left(my+\frac{i}{n}\right)\cdot\prod_{j=0}^{n-k-1}\left(1-\frac{my}{n}+\frac{j}{n^2}\right)}{\left(1+\frac{1}{n^2}\right)\left(1+\frac{2}{n^2}\right)\cdot\ldots\cdot\left(1+\frac{n-1}{n^2}\right)} f\left(\frac{k}{m}\right).$$

If we let $n\to\infty$ in this last expression, we get

$$\lim_{n\to\infty}\left(1-\frac{1}{n}\right)\cdot\ldots\cdot\left(1-\frac{k-1}{n}\right) = 1 \text{ and } \lim_{n\to\infty}\prod_{i=0}^{k-1}\left(my+\frac{i}{n}\right) = (my)^k.$$

In order to find the limit of

$$E_1(n)=\prod_{j=0}^{n-k-1}\left(1-\frac{my}{n}+\frac{j}{n^2}\right)=\frac{\prod\limits_{j=0}^{n}\left[1+\frac{1}{n}\left(\frac{j}{n}-my\right)\right]}{\prod\limits_{j=n-k}^{n}\left[1+\frac{1}{n}\left(\frac{j}{n}-my\right)\right]}=\frac{\prod\limits_{j=0}^{n}\left[1+\frac{1}{n}\left(\frac{j}{n}-my\right)\right]}{\prod\limits_{l=0}^{k}\left[1+\frac{1}{n}\left(\frac{n-k+l}{n}-my\right)\right]}$$

and

$$E_2(n)=\prod_{i=0}^{n-1}\left(1+\frac{i}{n^2}\right)=\frac{n}{n+1}\prod_{i=0}^{n}\left(1+\frac{1}{n}\cdot\frac{i}{n}\right)$$

we shall use the following result: if f could be integrated on the interval $[a,b]$, then

$$\lim_{n\to\infty}\prod_{j=1}^{n}(1+\delta_n f_{jn})=\exp\left(\int_a^b f(x)dx\right),$$

where $\delta_n=\frac{b-a}{n}$ and $f_{jn}=f(a+j\delta_n)$. By setting $f=x-my$, $a=0$ and $b=1$, we get

$$\lim_{n\to\infty}\left(1-\frac{my}{n}\right)\prod_{j=1}^{n}\left(1+\frac{1}{n}\left(\frac{j}{n}-my\right)\right)=\exp\left(\int_0^1(x-my)dx\right)=e^{\frac{1}{2}-my},$$

and

$$\lim_{n\to\infty}\prod_{l=0}^{k}\left[1+\frac{1}{n}\left(\frac{n-k+l}{n}-my\right)\right]=1,$$

respectively. Therefore we have

$$\lim_{n\to\infty}E_1(n)=e^{\frac{1}{2}-ny}.$$

Similarly, by setting $f=x$, $a=0$ and $b=1$, we get

$$\lim_{n\to\infty}\frac{n}{n+1}\prod_{i=1}^{n}\left(1+\frac{1}{n}\cdot\frac{i}{n}\right)=\exp\left(\int_0^1 xdx\right)=e^{\frac{1}{2}},$$

which means that

$$\lim_{n\to\infty}E_2(n)=e^{\frac{1}{2}}.$$

Using the above results, we obtain

$$M_m(f;y)=e^{-my}\sum_{k=0}^{\infty}\tfrac{(my)^k}{k!}f\left(\tfrac{k}{m}\right), \tag{2.6}$$

which was introduced by Mirakjan [17] and then studied intensively by Favard [9] and Szász [25].

Taking $\alpha=n^{-1}$ one obtains a special case of the operators (2.4), introduced by Lupaş and Lupaş [12], defined by

$$S_n^{\left\langle\frac{1}{n}\right\rangle}(f;x)=\sum_{k=0}^{n}s_{n,k}^{\left\langle\frac{1}{n}\right\rangle}(x)f\left(\tfrac{k}{n}\right)=\sum_{k=0}^{n}\binom{n}{k}\frac{x^{\left[k,-\frac{1}{n}\right]}(1-x)^{\left[n-k,-\frac{1}{n}\right]}}{1^{\left[n,-\frac{1}{n}\right]}}f\left(\tfrac{k}{n}\right). \tag{2.7}$$

In what follows, we survey some important results for the Stancu operators (2.4), which were obtained in some previous works of the authors.

The computation of the test functions by Stancu operators was done long time ago and can be found in [20]. Based on the fact that many properties of Bernstein operators can be transferred to the Stancu operators (2.4), different mathematicians studied almost all results using this standpoint. In [13] we revised and gave new results concerning the computation of the test functions, and the moments for the Stancu operators, respectively, without using the properties of Bernstein operators.

Theorem 2.1. [13] *For any $j, n \in \mathbb{N}$ and $x \in [0,1]$, the following formula*

$$S_n^{\langle\alpha\rangle}(e_j;x) = \tfrac{1}{n^j}\sum_{i=0}^{j-1} S(j,j-i)(n)_{j-i}\tfrac{x^{[j-i,-\alpha]}}{1^{[j-i,-\alpha]}} \tag{2.8}$$

holds, where $(x)_n := \prod_{i=0}^{n-1}(x-i)$ denotes the falling factorial, with $(x)_0 := 1$.

The relation (2.8) represents a general formula for computation of images of the test functions by Stancu operators (2.4), and can be found also in [14] where the results are proved by using the properties of Bernstein operators.

Application 1. [13] By using (2.8), for $j \in \{1,2,3,4\}$, we get the images of the test functions by Stancu operators (2.4), such that

$$S_n^{\langle\alpha\rangle}(e_1;x) = x, \quad S_n^{\langle\alpha\rangle}(e_2;x) = x^2 + \tfrac{(\alpha n+1)x(1-x)}{n(1+\alpha)},$$
$$S_n^{\langle\alpha\rangle}(e_3;x) = x^3 + \tfrac{\left(2\alpha^2n^2+3\alpha n^2+3n-2\right)x^2(1-x)}{n^2(1+\alpha)(1+2\alpha)} + \tfrac{\left(2\alpha^2n^2+3\alpha n+1\right)x(1-x)}{n^2(1+\alpha)(1+2\alpha)},$$
$$S_n^{\langle\alpha\rangle}(e_4;x) = x^4 + \tfrac{\left(6\alpha^3n^3+11\alpha^2n^3+6\alpha n^3+6n^2-11n+6\right)x^3(1-x)}{n^3(1+\alpha)(1+2\alpha)(1+3\alpha)} +$$
$$+\tfrac{\left(6\alpha^3n^3+11\alpha^2n^3+18\alpha n^2-12\alpha n+7n-6\right)x^2(1-x)}{n^3(1+\alpha)(1+2\alpha)(1+3\alpha)} + \tfrac{\left(6\alpha^3n^3+12\alpha^2n^2-\alpha^2n+7\alpha n-\alpha+1\right)x(1-x)}{n^3(1+\alpha)(1+2\alpha)(1+3\alpha)}.$$

By taking $\alpha = n^{-1}$ in Application 1 we get the images of the test functions for the particular case (2.7) of Stancu operators. It follows that

$$S_n^{\left\langle\frac{1}{n}\right\rangle}(e_1;x) = x, \quad S_n^{\left\langle\frac{1}{n}\right\rangle}(e_2;x) = x^2 + \tfrac{2x(1-x)}{n+1}, \quad S_n^{\left\langle\frac{1}{n}\right\rangle}(e_3;x) = x^3 + \tfrac{6nx^2(1-x)}{(n+1)(n+2)} + \tfrac{6x(1-x)}{(n+1)(n+2)},$$

$$S_n^{\left\langle\frac{1}{n}\right\rangle}(e_4;x) = x^4 + \tfrac{12\left(n^2+1\right)x^3(1-x)}{(n+1)(n+2)(n+3)} + \tfrac{12(3n-1)x^2(1-x)}{(n+1)(n+2)(n+3)} + \tfrac{2(13n-1)x(1-x)}{n(n+1)(n+2)(n+3)}.$$

The computation of the moments of higher order for the Stancu operators (2.4) was done in [13], too, as follows:

$$S_n^{\langle\alpha\rangle}\left((e_1-x)^2;x\right) = \tfrac{(1+\alpha n)x(1-x)}{n(1+\alpha)}, \quad S_n^{\langle\alpha\rangle}\left((e_1-x)^3;x\right) = \tfrac{(1+\alpha n)(1+2\alpha n)x(1-x)(1-2x)}{n^2(1+\alpha)(1+2\alpha)},$$

$$S_n^{\langle\alpha\rangle}\left((e_1-x)^4;x\right) = \tfrac{\left((3n-18\alpha n)(1+\alpha n)^2-6(1+\alpha n)\right)(x(1-x))^2}{n^3(1+\alpha)(1+2\alpha)(1+3\alpha)} + \tfrac{\left(6\alpha n(1+\alpha n)^2+(1-\alpha)(1+\alpha n)\right)x(1-x)}{n^3(1+\alpha)(1+2\alpha)(1+3\alpha)}.$$

Similarly, for the particular case (2.7) of Stancu operators, we get

$$S_n^{\left\langle\frac{1}{n}\right\rangle}\left((e_1-x)^2;x\right) = \tfrac{2x(1-x)}{n+1}, \quad S_n^{\left\langle\frac{1}{n}\right\rangle}\left((e_1-x)^3;x\right) = \tfrac{6x(1-x)(1-2x)}{(n+1)(n+2)},$$

$$S_n^{\left\langle\frac{1}{n}\right\rangle}\left((e_1-x)^4;x\right) = \tfrac{12\left(n^2-7n\right)(x(1-x))^2+(26n-2)x(1-x)}{n(n+1)(n+2)(n+3)}.$$

In [20], Stancu established an important relationship between two consecutive terms of the sequence $\left(S_n^{\langle\alpha\rangle}(f;x)\right)_{n\in\mathbb{N}}$, which is useful for proving a monotonicity property of it, in the case of convex or concave functions of first order. To our best knowledge, the problem concerning the monotonicity property of the sequence of Stancu polynomials has been completely solved the paper [15], where the Popoviciu's Theorem [18] was applied to the Stancu operators in order to get an appropriate form of the remainder term.

Application 2. [15] Let $x \in (0,1)$ be arbitrary but fixed. Applying the Popoviciu's Theorem to Stancu operators, it follows that the linear functional $R_n^{\langle\alpha\rangle}$ on $C[0,1]$, defined by

$$R_n^{\langle\alpha\rangle}(f;x) = f(x) - S_n^{\langle\alpha\rangle}(f;x),$$

satisfies the following conditions:

i) $R_n^{\langle\alpha\rangle}(e_0;x) = R_n^{\langle\alpha\rangle}(e_1;x) = 0$, $R_n^{\langle\alpha\rangle}(e_2;x) = -\frac{(1+\alpha n)x(1-x)}{n(1+\alpha)} \neq 0$;

ii) for any convex function f of the first order, $R_n^{\langle\alpha\rangle}(f;x) \neq 0$; then, there exist three points $0 \leq \xi_0 < \xi_1 < \xi_2 \leq 1$ such that

$$R_n^{\langle\alpha\rangle}(f;x) = R_n^{\langle\alpha\rangle}(e_2;x)[\xi_0,\xi_1,\xi_2;f] = -\frac{(1+\alpha n)x(1-x)}{n(1+\alpha)}[\xi_0,\xi_1,\xi_2;f]. \tag{2.9}$$

For any $f \in C[0,1]$, $x \in [0,1]$ and $n \in \mathbb{N}$, the following

$$f(x) = S_n^{\langle\alpha\rangle}(f;x) + R_n^{\langle\alpha\rangle}(f;x) \tag{2.10}$$

is called Stancu approximation formula, where $R_n^{\langle\alpha\rangle}$ is the remainder operator associated to the Stancu operator $S_n^{\langle\alpha\rangle}$. The study of the remainder term associated to the Stancu operators was done in [20], in terms of divided differences of first, respectively second order of the function f. Three years later, in [23], Stancu established an expression of the remainder term by using only divided differences of second order. In [13] we refined the result established by Stancu for the remainder term.

Theorem 2.2. [13] *The representation of the remainder term associated to Stancu operators is given by*

$$R_n^{\langle\alpha\rangle}(f;x) = -\frac{x(1-x)(1+\alpha n)}{n(1+\alpha)} \sum_{k=0}^{n-1} p_{n-1,k}^{\langle\alpha\rangle}(x+\alpha)\left[x, \frac{k}{n}, \frac{k+1}{n}; f\right], \tag{2.11}$$

for $x \in [0,1] \setminus \left\{ \frac{k}{n} \mid k = \overline{0,n} \right\}$, *where* $p_{n-1,k}^{\langle\alpha\rangle}(x) = \binom{n}{k} \frac{x^{[k,-\alpha]}(1-x+2\alpha)^{[n-k,-\alpha]}}{(1+2\alpha)^{[n,-\alpha]}}$, $x \geq 0$, $\alpha \geq 0$.

Remark 2.1. The relation (2.11) could be also found in [16], where a complete study on the bivariate approximation formula of Stancu operators was done.

Remark 2.2. The asymptotic behavior of the Stancu operators as well as various quantitative forms of Voronovskaja's result [26] could be found in [13]. For other recent results concerning Stancu operators and their generalizations, the reader is referred to [7].

3. Stancu operators without factorial powers

Let β, γ be real non-negative parameters satisfying the following relation $0 \leq \beta \leq \gamma$. In 1969, Stancu [22] introduced new linear positive operators associated to a real-valued function $f : [0,1] \to \mathbb{R}$, defined by

$$P_n^{\langle\beta,\gamma\rangle}(f;x) = \sum_{k=0}^{n} s_{n,k}(x) f\left(\frac{k+\beta}{n+\gamma}\right) = \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} f\left(\frac{k+\beta}{n+\gamma}\right), \tag{3.12}$$

for any positive integer n. In the well-known monograph of Altomare and Campiti [2], the operators (3.12) are called *"the operators of Bernstein-Stancu"*, because in the particular case $\beta = \gamma = 0$, they reduce to the classical Bernstein operators (2.5). If we choose the value 0 for the parameter α in relation (1.1), then the Stancu operators given by (3.12) are obtained. The equality

$$f(x) = P_n^{\langle\beta,\gamma\rangle}(f;x) + R_n^{\langle\beta,\gamma\rangle}(f;x) \tag{3.13}$$

is called the Bernstein-Stancu approximation formula, $R_n^{\langle\beta,\gamma\rangle}$ being its remainder term. Concerning the remainder term of the relation (3.13), in [22] were established the following results.

Theorem 3.3. [22] *If the function $f \in C[0,1]$ possesses divided differences of first and second order, finite on $[0,1]$, then the remainder term of* (3.13) *can be expressed under the following form*

$$R_n^{\langle \beta,\gamma \rangle}(f;x) = \tfrac{\gamma x-\beta}{n+\gamma} \sum_{k=0}^{n} s_{n,k}(x) \left[x, \tfrac{k+\beta}{n+\gamma}; f\right] - \tfrac{nx(1-x)}{(n+\gamma)^2} \sum_{k=0}^{n-1} s_{n-1,k}(x) \left[x, \tfrac{k+\beta}{n+\gamma}, \tfrac{k+\beta+1}{n+\gamma}; f\right], \tag{3.14}$$

for any $x \in [0,1] \backslash \left\{ \frac{k+\beta}{n+\gamma} \mid k = \overline{0,n} \right\}$.

The result given by (3.14) is the Stancu representation for the remainder term of the Bernstein-Stancu approximation formula (3.13) and for $\beta = \gamma = 0$ it reduces to the well-known Stancu representation [19] for the remainder term of classical Bernstein approximation formula. Using the hypotheses of the above theorem, we can state

Theorem 3.4. [22] *The remainder term of* (3.13) *can be expressed by*

$$R_n^{\langle \beta,\gamma \rangle}(f;x) = \tfrac{\gamma x-\beta}{n+\gamma} [\xi_1, \xi_2; f] - \tfrac{nx(1-x)}{(n+\gamma)^2} [\eta_1, \eta_2, \eta_3; f], \tag{3.15}$$

for any $x \in [0,1] \backslash \left\{ \frac{k+\beta}{n+\gamma} \mid k = \overline{0,n} \right\}$, where $0 \le \xi_1 < \xi_2 \le 1$, $0 \le \eta_1 < \eta_2 < \eta_3 \le 1$.

The result expressed by (3.15) represents a type of Aramă mean value theorem [3] for the Bernstein-Stancu approximation formula, while the next result is a Voronovskaja-type theorem.

Theorem 3.5. [22] *If the function $f \in C[0,1]$ is differentiable in some neighborhood of $x \in [0,1]$ and has the second order derivative $f''(x)$, the following formula*

$$\lim_{n \to \infty} n \left(f(x) - P_n^{\langle \beta,\gamma \rangle}(f;x) \right) = (\gamma x - \beta) f'(\xi) - \tfrac{1}{2} x(1-x) f''(\eta) \tag{3.16}$$

holds, for $0 \le \xi \le 1$ and $0 \le \eta \le 1$.

An important particular case of Bernstein-Stancu operators (3.12) is obtained by chossing $\beta = 0$ and $\gamma = 1$ and is given by

$$P_n^{\langle 0,1 \rangle}(f;x) = \sum_{k=0}^{n} s_{n,k}(x) f\left(\tfrac{k}{n+1}\right) = \sum_{k=0}^{n} \tbinom{n}{k} x^k (1-x)^{n-k} f\left(\tfrac{k}{n+1}\right). \tag{3.17}$$

In 1980, Bleimann, Butzer and Hahn [6] introduced a sequence of linear operators defined for real-valued functions $f : [0,+\infty) \to \mathbb{R}$ by

$$L_n(f;x) = \tfrac{1}{(1+x)^n} \sum_{k=0}^{n} \tbinom{n}{k} x^k f\left(\tfrac{k}{n+1-k}\right). \tag{3.18}$$

They proved that, for all functions $f \in C[0,+\infty)$ satisfying a growth condition as $x \to \infty$, it follows that $\lim_{n\to\infty} L_n f = f$, pointwisely on $[0,+\infty)$, the convergence being uniform on each compact subset of $[0,+\infty)$. In [11] Ivan and in [1] Abel and Ivan discovered the close connection between Bernstein operators and BBH operators which allowed them to transfer some approximation properties from Bernstein to BBH operators. Following their ideas, we consider the space of functions

$$C_*[0,+\infty) = \left\{ f \in C[0,+\infty) \,:\, \lim_{x\to\infty} f(x) = 0 \right\}.$$

For a function $f \in C_*[0,+\infty)$, $x \in [0,+\infty)$ and $y = \frac{x}{1+x}$, we define

$$F(y) = \begin{cases} f\left(\frac{y}{1-y}\right), & y \in [0,1), \\ 0, & y = 1. \end{cases} \tag{3.19}$$

It is clear that a function $f \in C_*[0,+\infty)$ implies $F \in C[0,1]$, such that in [4] we proved the following equality

$$L_n(f;x) = P_n^{\langle 0,1\rangle}\left(F; \tfrac{x}{1+x}\right). \tag{3.20}$$

The equality (3.20) allows to transfer some approximation properties of Bernstein-Stancu operators (3.17) to the BBH operators (3.18). An example in this direction could be the following

Theorem 3.6. [4] *Suppose that $f \in C_*[0,+\infty)$ is differentiable in a neighborhood of $x \in [0,+\infty)$ and has the second order derivative $f''(x)$, then the following Voronovskaja-type formula*

$$\lim_{n\to\infty} n\left(f(x) - L_n(f;x)\right) = -\tfrac{1}{2}x(1+x)^2 f''(x)$$

holds.

In a similar way, it is possible to get the Stancu type representation, respectively Aramă mean value type theorem of the remainder term for Bleimann, Butzer and Hahn approximation formula.

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