Reverse degree distance of some graph operations

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ABSTRACT. In this paper, we present the exact formulae for the reverse degree distance of some graph operations, such as corona product, splice, link and composition of two connected graphs. Using the results obtained here, the reverse degree distance of some important classes of graphs are obtained.

1. INTRODUCTION

Let $G$ be a simple connected graph with vertex set $V(G)$ and edge set $E(G)$. For $u \in V(G)$, let $d_G(u)$ be the degree of $u$ in $G$ and $d_G(u, v)$ is the distance between the vertices $u$ and $v$ in $G$. For a vertex $v$ of $G$, the eccentricity $e(v)$ is the distance between $v$ and a vertex farthest from $v$. The maximum eccentricity is its diameter, denoted by $d(G)$. The degree distance of $G$ is defined as $\sum_{u,v\in V(G)} d_G(u) d_G(v)$. It is a useful molecular descriptor [28]. Earlier as noted in [23, 27], this graph invariant appeared to be part of the molecular topological index (or Schultz index) [26], which may be expressed as $DD(G) + \sum_{u \in V(G)} d_G(u)^2$, see [19, 22], where the latter part $\sum_{u \in V(G)} d_G(u)^2$ is known as the first Zagreb index [20]. Thus the degree distance is also called the true Schultz index in chemical literature [12]. Tomescu [30] showed that the star is the unique graph with minimum degree distance in the class of connected graphs with $n$ vertices. Further work on the minimum degree distance (especially for unicyclic and bicyclic graphs) may be found in A.I. Tomescu [29], Tomescu [31] and Bucicovschi and Cioaba [2]. Dankelmann et al. [11] gave asymptotically sharp upper bounds for the degree distance.

The Wiener index of $G$ is denoted by $W(G)$, defined as $W(G) = \frac{1}{2} \sum_{u,v\in V(G)} d_G(u,v)$. Gutman [19] showed that if $G$ is a tree with $n$ vertices, then $DD(G) = 4W(G) - n(n-1)$. Thus there is no need to study the degree distance for trees because this is equivalent to the study of the Wiener index, see [13]. Balaban et al. [1] introduced the concept of reverse Wiener index, which is defined to be $\Lambda(G) = \frac{|V(G)|(|V(G)|-1)d(G)}{2} - W(G)$. Let $\Lambda'(G) = \frac{(|V(G)|-1)^2d(G)}{2} - W(G)$, which is a revised version of the reverse Wiener index of $G$. The reverse Wiener index of unicyclic graphs are obtained by Du and Zhou [15].

The reverse degree distance of a connected graph $G$ is defined in discrete mathematical chemistry as $rD'(G) = 2(|V(G)| - 1) |E(G)| d(G) - \frac{1}{2} \sum_{u,v \in V(G)} (d_G(u) + d_G(v))d_G(u,v)$.

Since some of the chemical graphs are derived from the graph operation for instance zig-zag nanotube is obtained from the generalized hierarchical product and 1,3- tetrameric is obtained using link of two graphs and so on. For more information refer [3, 4, 5, 6]. This

Received: 28.01.2016. In revised form: 20.09.2016. Accepted: 04.10.2016
2010 Mathematics Subject Classification. 05C12, 05C76.
Key words and phrases. Reverse degree distance, degree distance, composite graphs.
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motivate us to consider such a graph operation related to Wiener, reverse Wiener index is defined in [1] and its relation with other indices has been studied. Also the reverse degree distance and its relation with degree distance and more indices has been studied [35]. This motivate us to consider the reverse degree distance and obtained the relations between this index and Wiener index, degree distance, Zagreb index and its coindex using various graph operations such as corona product, splice, link and composition.

Some basic properties of the reverse degree distance have been established by Zhou and Trinajstic [33], and in particular, it was shown that the reverse degree distance satisfies the basic requirement to be a branching index usable in chemistry. In continuation to the study of the reverse degree distance, a natural starting point is the reverse degree distances of unicyclic graphs. In [16] the graphs with maximum reverse degrees distance in the class of unicyclic graphs with given girth, number of pendant vertices and maximum degree are determined. The degree distance and reverse degree distance of one tetragonal carbon nanocones are determined by Momen and Alaeiyan in [25].

2. MAIN RESULTS

The first Zagreb index and second Zagreb index are defined as \( M_1(G) = \sum_{u \in V(G)} d_G(u)^2 \) and \( M_2(G) = \sum_{uv \in E(G)} d_G(u)d_G(v) \). Similarly, the first Zagreb coindex and second Zagreb coindex are defined as \( \overline{M}_1(G) = \sum_{uv \notin E(G)} (d_G(u) + d_G(v)) \) and \( \overline{M}_2(G) = \sum_{uv \notin E(G)} d_G(u)d_G(v) \).

2.1. Corona product. Corona product were introduced by Frucht and Harary in 1970 [18]. The corona product of two graphs \( G \) and \( H \), denoted as \( G \circ H \), is defined as the graph obtained by taking one copy of \( G \) and \( |V(G)| \) copies of \( H \), and joining by an edge the \( i \)th vertex of \( G \) to every vertex in the \( i \)th copy of \( H \). Let the vertices of \( G \) and \( H \) are labeled by \( v_1, v_2, \ldots, v_n \) and \( u_1, u_2, \ldots, u_m \), respectively. For \( 1 \leq i \leq n \), denote by \( H_i \) the \( i \)th copy of \( H \) joined to the vertex \( v_i \) and let \( V(H_i) = \{u_{i1}, u_{i2}, \ldots, u_{im} \} \). The following lemmas are follows from the structure of \( G \circ H \).

Lemma 2.1. Let \( G \) and \( H \) be two connected graphs. Then the distance between two vertices of \( G \circ H \) are given as follows:

(i) \( d_{G \circ H}((u_{ij}, u_{pq}) = \begin{cases} 
1 & \text{if } i = p \text{ and } u_{ij}u_{pq} \in E(H), \\
2 & \text{if } i = p \text{ and } u_{ij}u_{pq} \notin E(H), \\
d_G(v_i, v_p) + 2 & \text{if } i \neq p.
\end{cases} \)

(ii) \( d_{G \circ H}(v_i, v_p) = d_G(v_i, v_p), \text{ if } v_i, v_p \in V(G). \)

(iii) \( d_{G \circ H}(v_i, u_{pq}) = d_G(v_i, u_{pq}) + 1, \text{ if } u_i \in V(G) \text{ and } u_{pq} \in V(H_i). \)

Lemma 2.2. Let \( G \) be graph on \( n_1 \) vertices. If \( H \) is a connected graph, then the degree of a vertex of \( G \circ H \) is given by \( d_{G \circ H}(x) = \begin{cases} 
d_G(x) + |V(H)| & \text{if } x \in V(G), \\
d_H(x) + 1 & \text{if } x \in V(H_i) \text{ for some } i \in \{1, 2, \ldots, n_1\}. \end{cases} \)

Now we obtain the reverse degree distance of corona product of two connected graphs.

Theorem 2.1. Let \( G \) and \( H \) be two connected graphs with \( n_1, n_2 \) vertices and \( m_1, m_2 \) edges, respectively. Then the reverse degree distance of \( G \circ H \) is \( rD'(G \circ H) = 2(n_1(n_2 + 1) - 1)(m_1 + n_1(n_2 + m_2))(d(G) + 2) - (n_2 + 1)DD(G) - 4(n_2 + m_2)(n_2 + 1)W(G) - n_1(M_1(H) + 2M_1(H) - [2n_1(n_1 - 1)(2n_2m_2 + n_2^2) + 2n_1(n_2^2 + n_2 + 2m_2) + n_1(n_2^2 - n_2 - m_2 + 2n_2m_1)]). \)
Proof. Let $T = G \circ H$. By the definition of reverse degree distance,

\[
\begin{align*}
S_1 &= \frac{1}{2} \sum_{v_i, v_p \in V(G), i \neq p} (d_T(v_i) + d_T(v_p)) d_T(v_i, v_p) \\
&= \frac{1}{2} \sum_{i, p = 1}^{n_1} (d_G(v_i) + n_2 + d_G(v_p) + n_2) d_G(v_i, v_p), \text{ by Lemmas 2.1 and 2.2} \\
&= DD(G) + 2n_2W(G).
\end{align*}
\]

\[
\begin{align*}
S_2 &= \frac{1}{2} \sum_{i = 1}^{n_1} \sum_{u_{ij}, u_{iq} \in V(H_i), j \neq q} (d_T(u_{ij}) + d_T(u_{iq})) d_T(u_{ij}, u_{iq}) \\
&= \frac{1}{2} \sum_{i = 1}^{n_1} \sum_{j, q = 1}^{n_2} (d_H(u_j) + d_H(u_q) + 2)(1) \\
&\quad + \frac{1}{2} \sum_{i = 1}^{n_1} \sum_{j, q = 1}^{n_2} (d_H(u_j) + d_H(u_q) + 2)(2), \text{ by Lemmas 2.1 and 2.2} \\
&= n_1M_1(H) + 2n_1M_1(H) + n_1n_2(n_2 - 1) - n_1m_2.
\end{align*}
\]

\[
\begin{align*}
S_3 &= \sum_{p = 1}^{n_1} \sum_{i = 1}^{n_1} \sum_{q = 1}^{n_2} (d_T(v_i) + d_T(u_{pq})) d_T(v_i, u_{pq}) \\
&= \sum_{p = 1}^{n_1} \sum_{i = 1}^{n_1} \sum_{q = 1}^{n_2} (d_G(v_i) + n_2 + d_H(u_q) + 1)(d_G(v_i, v_p) + 1), \text{ by Lemmas 2.1 and 2.2} \\
&= n_2DD(G) + 2n_2(n_2 + 1)W(G) + 4m_2W(G) + 2n_1n_2m_1 + 2n_1^2n_2(n_2 + 1) + 4n_1^2m_2.
\end{align*}
\]

\[
\begin{align*}
S_4 &= \frac{1}{2} \sum_{u_{ij} \in V(H_i), u_{pq} \in V(H_p), i \neq p} (d_T(u_{ij}) + d_T(u_{pq})) d_T(u_{ij}, u_{pq}) \\
&= \frac{1}{2} \sum_{i, p = 1}^{n_1} \sum_{j, q = 1}^{n_2} (d_T(u_{ij}) + d_T(u_{pq})) d_T(u_{ij}, u_{pq})
\end{align*}
\]
By Lemmas 2.1 and 2.2, we have

\[ S_4 = \frac{1}{2} \sum_{i,p=1,i\neq p}^{n_1} \sum_{j,q=1,j\neq q}^{n_2} (d_H(v_j) + d_H(v_q) + 2)(d_G(u_i, u_p) + 2) \]

\[ = \frac{1}{2} \sum_{i,p=1,i\neq p}^{n_1} 2(d_G(u_i, u_p) + 2) \sum_{j,q=1}^{n_2} (d_H(v_j) + 1) \]

\[ = (2n_2m_2 + n_2^2) \sum_{i,p=1,i\neq p}^{n_1} (d_G(u_i, u_p) + 2) \]

\[ = (2n_2m_2 + n_2^2)(2W(G) + 2n_1(n_1 - 1)). \]

Using the sums \( S_1 \) to \( S_4 \) in (2.1), we obtain:

\[ rD'(T) = 2(n_1(n_2 + 1) - 1)(m_1 + n_1(n_2 + m_2))(d(G) + 2) - (n_2 + 1)DD(G) \]

\[-4(n_2 + m_2)(n_2 + 1)W(G) - n_1(M_1(H) + 2M_1(H)) - [2n_1(n_1 - 1)(2n_2m_2 + n_2^2) + 2n_1^2(n_2^2 + n_2 + 2m_2) + n_1(n_2^2 - n_2 - m_2 + 2n_2m_1)] . \]

\[ \square \]

Let \( C_n, P_n \) and \( K_n \) denote the cycle, path and complete graph on \( n \) vertices, respectively. It is known that \( W(P_n) = \frac{n(n^2 - 1)}{6} \), \( PI(C_n) = \frac{n^3}{8} \) when \( n \) is even, and \( \frac{n(n^2 - 1)}{2} \) otherwise and \( W(K_n) = \frac{n(n - 1)}{2} \). It can be easily verified that \( DD(P_n) = \frac{n(n - 1)(2n - 1)}{4} \), \( DD(C_n) = \frac{n^3}{2} \) when \( n \) is even, and \( \frac{n(n^2 - 1)}{2} \) otherwise and \( DD(K_n) = n(n - 1)^2 \).

By direct calculations we obtain the first and second Zagreb indices of \( P_n \) and \( C_n \).

\( M_1(C_n) = 4n, \ n \geq 3, \ M_1(P_n) = 0, \ M_1(P_n) = 4n - 6, \ n > 1 \) and \( M_1(K_n) = n(n - 1)^2 \). \( M_2(C_n) = 4(n - 2) \) and \( M_2(C_n) = 4n \). Similarly, \( M_1(P_n) = 2(n^2 - 4n + 4), \ M_1(C_n) = 2n(n - 3) \) and \( M_1(K_n) = 0 \).

For a given graph \( G \), its \( t \)-fold bristled graph \( Brs_t(G) \) is obtained by attaching \( t \) vertices of degree 1 to each vertex of \( G \). This graph can be represented as the corona product of \( G \) and complement of a complete graph on \( t \) vertices.

**Example 2.1.** Let \( G \) be a graph with \( n \) vertices. Then \( rD'(G \circ K_t) = 2(nt + n - 1)(m + nt)(d(G) + 2) - (t + 1)DD(G) - 4t(t + 1)W(G) - 2mnt - nt(3nt + n - 2) \).

**Example 2.2.** (i) \( rD'(P_n \circ K_t) = 2(nt + n - 1)^2(n + 1) - \frac{n}{3}(n - 1)(t + 1)(2nt + 2n + 2t - 1) + nt(3nt + 3n - 4) \).

(ii) \( rD'(C_n \circ K_t) = \begin{cases} n(n + 4)(t + 1)(nt + n - 1) - \frac{n^3(t + 1)^2}{2} \\ -2n^2t - nt(3nt + n - 2) \text{ if } n \text{ is even} \\ n(n + 3)(t + 1)(nt + n - 1) - \frac{n(n^2 - 1)(t + 1)^2}{2} \\ -2n^2t - nt(3nt + n - 2) \text{ if } n \text{ is odd} \end{cases} \)

(iii) A special corona graph \( C_n \circ K_1 \) is called a sunlet graph on \( 2n \) vertices. \( rD'(C_n \circ K_1) = 2n(n^2 + 4n - 3) \) if \( n \) is even and \( 2n(3n^2 + 2n - 1) \) if \( n \) is odd.

The star graph \( S_{t+1} \) on \( t + 1 \) vertices is the corona product of \( K_1 \) and \( K_t \). The fan graph \( F_{t+1} \) and the wheel graph \( W_{t+1} \) on \( t + 1 \) vertices are also corona product of \( K_1 \) and \( P_t \) and \( C_t \). From the above formula the reverse degree distance of these graphs are obtained.

**Example 2.3.** (i) \( rD'(K_1 \circ K_t) = t(7t - 1) \). (ii) \( rD'(K_1 \circ P_t) = 3t^2 - 22t + 25 \).

(iii) \( rD'(K_1 \circ C_t) = 3t^2 - 20t \).
Example 2.4. Using Theorem 2.1, we have (i) $\tauD'(P_n \circ K_2) = 12n^3 + 27n^2 - 15n + 2$.
(ii) $\tauD'(P_n \circ K_3) = 8n^3 - 2n^2 - 2n^3 + 5nt + 2nt^2 + 3nt^3 - 6n^2 - 3n - 6t^2 - 12t - 4$.
(iii) $\tauD'(P_n \circ C_m) = 4n^3 + 4n^3 + 7n^2 - \frac{7n}{3} - 4n^2 + \frac{17nt}{3} - 7nt^2 + nt^2 + 2$.

2.2. Splice and link. Let $G$ and $H$ be two simple connected graphs with the vertex sets $V(G)$ and $V(H)$ and the edge sets $E(G)$ and $E(H)$, respectively. For given vertices $y \in V(G)$ and $z \in V(H)$, a splice of $G$ and $H$ by vertices $y$ and $z$ is denoted by $S(G, H)(y, z)$ and defined by identifying the vertices $y$ and $z$ in the union of $G$ and $H$. In the following Lemma, the distance between vertices of $S(G, H)(y, z)$ is computed. The proof can be easily obtained from the definition of splice of graphs, so is omitted.

Lemma 2.3. Let $G$ and $H$ be two graphs. Then

$$d_{S(G, H)(y, z)}(u, v) = \begin{cases} d_G(u, v) & \text{if } u, v \in V(G), \\ d_H(u, v) & \text{if } u, v \in V(H), \\ d_G(u, y) + d_H(z, v) & \text{if } u \in V(G), v \in V(H). \end{cases}$$

Theorem 2.2. Let $G$ and $H$ be two graphs with $n_1, n_2$ vertices and $m_1, m_2$ edges. Then

$$\tauD'(S(G, H)(y, z)) = 2(n_1 + n_2 - 1)(m_1 + m_2)(d(y) + d(z)) - DD(G) - DD(H) - (n_2 - 1) \sum_{y \neq u \in V(G)} d_G(u) d_G(u, y) - (n_1 - 1) \sum_{z \neq v \in V(H)} d_H(v) d_H(v, z) - 2m_1 D(z) - 2m_2 D(y),$$

where $D(y) = \sum_{x \in V(G)} d_G(x, y)$ and $D(z) = \sum_{z \in V(H)} d_H(x, z)$.

Proof. Let $S = S(G, H)(y, z)$. By the definition of reverse degree distance,

$$\tauD'(S) = 2(n_1 + n_2 - 1)(m_1 + m_2)(d(y) + d(z)) - \frac{1}{2} \sum_{u, v \in V(S)} (d_G(u) + d_H(v)) d_S(u, v). \tag{2.2}$$

We partition the set of pairs of vertices of $S$ into five subsets, namely, $V_1, V_2, V_3, V_4,$ and $V_5$, where $V_1 = \{u, v \in V(S) | u, v \neq y \in V(G)\}$, $V_2 = \{u, v \in V(S) | u = y, v \in V(G)\}$, $V_3 = \{u, v \in V(S) | u, v \neq z \in V(H)\}$, $V_4 = \{u, v \in V(S) | u = z, v \in V(H)\}$ and $V_5 = \{u, v \in V(S) | y \neq u \in V(G), z \neq v \in V(H)\}$.

Therefore by Lemma 2.3, we have

$$\begin{align*}
\frac{1}{2} \sum_{u, v \in V(S)} (d_G(u) + d_H(v)) d_S(u, v) &= \frac{1}{2} \sum_{u, v} (d_G(u) + d_H(v)) d_G(u, v) \\
&= \frac{1}{2} \sum_{u, v \neq y, u, v \in V(G)} (d_G(u) + d_G(v)) d_G(u, v) + \frac{1}{2} \sum_{u = y, v \in V(G)} (d_G(y) + d_H(z) + d_G(v)) d_G(u, y) \\
&+ \frac{1}{2} \sum_{u, v \neq z, u, v \in V(H)} (d_H(u) + d_H(v)) d_H(u, v) + \frac{1}{2} \sum_{u = z, v \in V(H)} (d_G(y) + d_H(z) + d_H(v)) d_H(u, z) \\
&+ \frac{1}{2} \sum_{y \neq u \in V(G), z \neq v \in V(H)} (d_G(u) + d_H(v))(d_G(u, y) + d_H(v, z)) \\
&= DD(G) + DD(H) + (n_2 - 1) \sum_{y \neq u \in V(G)} d_G(u) d_G(u, y) \\
&+ (n_1 - 1) \sum_{z \neq v \in V(H)} d_H(v) d_H(v, z) + 2m_1 D(z) + 2m_2 D(y). \tag{2.3} \end{align*}$$

Using (2.3) in (2.2), we obtain:

$$\tauD'(S) = 2(n_1 + n_2 - 1)(m_1 + m_2)(d(y) + d(z)) - DD(G) - DD(H) - (n_2 - 1) \sum_{y \neq u \in V(G)} d_G(u) d_G(u, y) - (n_1 - 1) \sum_{z \neq v \in V(H)} d_H(v) d_H(v, z) - 2m_1 D(z) - 2m_2 D(y).$$
Let $G$ and $H$ be two simple connected graphs with the vertex sets $V(G)$ and $V(H)$ and the edge sets $E(G)$ and $E(H)$, respectively. For vertices $y \in V(G)$ and $z \in V(H)$, a link of $G$ and $H$ by vertices $y$ and $z$ is denoted by $L(G \bullet H)(y, z)$ and obtained by joining $y$ and $z$ by an edge in the union of these graphs. In the following Lemma, the distance between vertices of $L(G \bullet H)(y, z)$ is computed. The proof can be easily obtained from the definition of link of graphs, so is omitted.

**Lemma 2.4.** Let $G$ and $H$ be two graphs. Then

$$d_{L(G \bullet H)}(y, z)(u, v) = \begin{cases} d_G(u, v) & \text{if } u, v \in V(G), \\ d_H(u, v) & \text{if } u, v \in V(H), \\ d_G(u, y) + d_H(z, v) + 1 & \text{if } u \in V(G) \text{ and } v \in V(H). \end{cases}$$

**Theorem 2.3.** Let $G$ and $H$ be two graphs with $n_1$, $n_2$ vertices and $m_1$, $m_2$ edges. Then

$$rD'(L(G \bullet H))(y, z) = 2(n_1 + n_2)(m_1 + m_2 + 1)(d(y) + d(z) + 1) - 2\sum_{y \neq u \in V(G)} d_G(u)d_G(u, y) - n_1 \sum_{y \neq v \in V(H)} d_H(v)d_H(v, z) - \left(2m_1 + 2\right)D(z) + \left(2m_2 + 2\right)D(y) - \left(2m_1 + 1\right) + \left(n_1 - 1\right)(2m_2 + 1) + 2m_1 + 2m_2 + 2,$$

where $D(y) = \sum_{u \in V(G)} d_G(x, y)$ and $D(z) = \sum_{z \in V(H)} d_H(x, z)$.

**Proof.** Let $L = L(G \bullet H)(y, z)$. By the definition of reverse degree distance,

$$rD'(L) = 2(n_1 + n_2)(m_1 + m_2 + 1)(d(y) + d(z) + 1) - \frac{1}{2} \sum_{u, v \in V(L)} (d_L(u) + d_L(v))d_L(u, v). \quad (2.4)$$

By Lemma 2.4, we have

$$\frac{1}{2} \sum_{u, v} (d_S(u) + d_S(v))d_S(u, v) = \frac{1}{2} \sum_{u, v \neq y, u, v \in V(G)} (d_G(u) + d_G(v))d_G(u, v) + \frac{1}{2} \sum_{u, v \in V(G)} (d_G(u, y) + d_G(v))d_G(v, y)$$

$$+ \frac{1}{2} \sum_{u, v \neq z, u, v \in V(H)} (d_H(u) + d_H(v))d_H(u, v) + \frac{1}{2} \sum_{u, v \in V(H)} (1 + d_H(z) + d_H(v))d_H(v, z)$$

$$+ \sum_{v = z, u, v \in V(H)} (1 + d_H(z) + d_G(u))(1 + d_G(v, y))$$

$$+ \sum_{y \neq u \in V(G), z \neq v \in V(H)} \left((d_G(u) + d_G(u))(d_G(u, y) + d_H(v, z) + 1) + d_G(y) + d_H(z) + 2\right).$$

Using (2.5) in (2.4), we have

$$rD'(L) = 2(n_1 + n_2)(m_1 + m_2 + 1)(d(y) + d(z) + 1) - DD(G) - DD(H)$$

$$- n_2 \sum_{y \neq u \in V(G)} d_G(u)d_G(u, y) - n_1 \sum_{z \neq v \in V(H)} d_H(v)d_H(v, z) - \left(2m_1 + 2\right)D(z)$$

$$+ (2m_2 + 2)D(y) - \left(2m_2 + 2\right)D(y) - \left((n_2 - 1)(2m_1 + 1) + (n_1 - 1)(2m_2 + 1) + 2m_1 + 2m_2 + 2\right).$$

\[\square\]
2.3. Composition. The composition of the graphs $G$ and $H$, denoted by $G[H]$, has vertex set $V(G) \times V(H)$ in which $(u_1, v_1)(u_2, v_2)$ is an edge whenever $u_1u_2$ is an edge in $G$ or, $u_1 = u_2$ and $v_1v_2$ is an edge in $H$. The following lemma gives the distance and degree of vertices of $G[H]$.

Lemma 2.5. Let $G$ and $H$ be two connected graphs with $V(G) = \{u_1, u_2, \ldots, u_{n_1}\}$ and $V(H) = \{v_1, v_2, \ldots, v_{n_2}\}$. If $x_{ij}$ denote the vertex $(u_i, v_j)$ of $G[H]$, then (i) the distance between two vertices of $G[H]$ is given by

$$d_G(x_{ij}, x_{pq}) = \begin{cases} d_G(u_i, v_p) & \text{if } j = q \text{ or } i \neq p \\ 1 & \text{if } i = p \text{ and } v_jv_q \in E(H) \\ 2 & \text{if } i = p \text{ and } v_jv_q \notin E(H). \end{cases}$$

(ii) the degree of a vertex $x_{ij}$ of $G[H]$ is $n_2d_G(u_i) + d_H(v_j)$.

Theorem 2.4. Let $G$ and $H$ be two connected graphs with $n_1$ and $n_2$ vertices, respectively. Then

$$rD'(G[H]) = 2(n_1n_2 - 1)(m_1n_2^2 + m_2n_1)d(G) - n_1\Delta D(G) - 2(2m_2 + M_1(H) + \overline{M_1(H)})W(G) - n_1M_1(H) - 2n_2M_1(n_2 - n_2 - m_2).$$

Proof. Let $V(G) = \{u_1, u_2, \ldots, u_{n_1}\}$, similarly for $V(H) = \{v_1, v_2, \ldots, v_{n_2}\}$. Let $x_{ij}$ denote the vertex $(u_i, v_j)$ of $G[H]$. By the definition of reverse degree distance,

$$rD'(G[H]) = 2(n_1n_2 - 1)(m_1n_2^2 + m_2n_1)d(G)$$

$$- \frac{1}{2} \sum_{i=1}^{n_1-1} \sum_{j=1}^{n_2} (d_G(x_{ij}) + d_G(x_{i\ell}))d_G(x_{ij}, x_{i\ell})$$

$$- \frac{1}{2} \sum_{i,j=1}^{n_1} \sum_{j \neq \ell} (d_G(x_{ij}) + d_G(x_{i\ell}))d_G(x_{ij}, x_{i\ell})$$

$$- \frac{1}{2} \sum_{i,k=1}^{n_1} \sum_{j=1}^{n_2} (d_G(x_{ij}) + d_G(x_{kj}))d_G(x_{ij}, x_{kj})$$

$$- \frac{1}{2} \sum_{i,k=1}^{n_1} \sum_{j \neq k} (d_G(x_{ij}) + d_G(x_{kj}))d_G(x_{ij}, x_{kj}).$$

(2.6)

We partition the sums into three sums, $S_1, S_2$ and $S_3$ as follows.

$$S_1 = \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} (d_G(x_{ij}) + d_G(x_{i\ell}))d_G(x_{ij}, x_{i\ell})$$

$$= \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} (2n_2d_G(u_i) + d_H(v_j) + d_H(v_\ell))d_G(x_{ij}, x_{i\ell})$$

$$= 2n_2 \sum_{i=1}^{n_1} d_G(u_i) \left( \sum_{v_j,v_\ell \in E(H)} d_H(v_j,v_\ell) + \sum_{v_j,v_\ell \notin E(H)} d_H(v_j, v_\ell) \right)$$

$$+ \sum_{i=1}^{n_1} \left( \sum_{v_j,v_\ell \in E(H)} (d_H(v_j) + d_H(v_\ell))d_H(v_j, v_\ell) + \sum_{v_j,v_\ell \notin E(H)} (d_H(v_j) + d_H(v_\ell))d_H(v_j, v_\ell) \right),$$

since $d_G(x_{ij}, x_{i\ell}) = 1$ if $v_jv_\ell \in E(H)$ and $2$ if $v_jv_\ell \notin E(H)$. 

\[\Box\]
Using Example 2.5.

\[ S_2 = \sum_{i,k=1, j=1}^{n_1, n_2} (d(x_{ij}) + d(x_{kj}))d_{G[H]}(x_{ij}, x_{kj}) \]
\[ = \sum_{i,k=1, j=1}^{n_1, n_2} (n_2(d(u_i) + d(u_k)) + 2d(v_j))d_G(u_i, u_k) \]
\[ = \sum_{i,k=1, j=1}^{n_1, n_2} n_2(d(u_i) + d(u_k))d_G(u_i, u_k) + \sum_{i,k=1, j=1}^{n_1, n_2} 2d(v_j)d_G(u_i, u_k) \]
\[ = 2n_2^2DD(G) + 8m_2W(G). \]

\[ S_3 = \sum_{i,k=1, j=1}^{n_1, n_2} (d(x_{ij}) + d(x_{kj}))d_{G[H]}(x_{ij}, x_{kj}) \]
\[ = \sum_{i,k=1, j=1}^{n_1, n_2} (n_2d(u_i) + d(v_j) + n_2d(u_k) + d(v_j))d_G(u_i, u_k), \]

since \( d_{G[H]}(x_{ij}, x_{kj}) = d_G(u_i, u_k) \) for all \( j \) and \( k \) and further the distance between the corresponding vertices of the layers is counted in \( S_2 \)
\[ = 2n_2^2((n_2 - 1)DD(G) + 4W(G)(M_1(H) + \overline{M_1(H)})). \]

Using \( S_1 \) to \( S_3 \) in (2.6), we have
\[ rD'(G[H]) = 2(n_1n_2 - 1)(m_1n_2^2 + m_2n_1)d(G) - n_2^3DD(G) - 2(2m_2 + M_1(H)) + \overline{M_1(H)})W(G) - n_1M_1(H) - 2n_1\overline{M_1(H)} - 4n_2m_1(n_2^2 - n_2 - m_2). \]

As an application we present formulae for reverse degree distance of open and closed fences, \( P_n[K_2] \) and \( C_n[K_2] \).

**Example 2.5.**

(i) \( rD'(P_n[K_2]) = \frac{2}{3}(6n^2 - 9n - 1) - \frac{4}{3}n(n - 1)(n + 7). \)

(ii) \( rD'(C_n[K_2]) = \begin{cases} 
  n(5n^2 - 5n - 4) & \text{if } n \text{ is even} \\
  5n^2(n^2 - 3) & \text{if } n \text{ is odd} 
\end{cases} \)

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