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A note on the convergence of Mann iteration

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ABSTRACT. The main result in this note slightly generalizes Theorem 3.4 in [Şt. Măruster and I. A. Rus, *Kannan contractions and strongly demicontractive mappings*, Creat. Math. Inform., **24** (2015), No. 3, 171–180]. We also significantly improve its proof. Both results in the paper are based on the concepts of demicontractivity and quasi-expansivity and involve relatively weak conditions that guarantee the convergence of Krasnoselskij method (the same conditions are not sufficient for the convergence of Picard iteration).

Thus, our note is a satisfactory answer to the following question: if for a given mapping some specific conditions for the convergence of Picard iteration are not satisfied (and presumptively the Picard iteration fails to converge), what are the conditions which still ensure the convergence of the Mann iteration?

Let C be a closed convex subset of a real Hilbert space $\mathcal{H}$ with scalar product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$, and let $T : C \to C$ be a (nonlinear) mapping. Recall that T is said to be (a, k)-strongly demicontractive [6] if the set of fixed points of T is nonempty, i. e., $Fix(T) \neq \emptyset$, and

$$\|Tx - p\|^2 \leq a\|x - p\|^2 + k\|x - Tx\|^2,\ \forall x \in \mathcal{C},\ p \in Fix(T), \tag{1.1}$$

where $a \in (0, 1),\ k \geq 0$. If $a = 1$, from (1.1) we obtain the weaker condition of demicontractivity [4]. The inequality (1.1) is equivalent to

$$\langle x - Tx, x - p\rangle \geq \frac{1-a}{2}\|x - p\|^2 + \frac{1-k}{2}\|x - Tx\|^2. \tag{1.2}$$

If T is strongly demicontractive then the set of fixed points in C is a singleton. In this case, the requirement that (1.1) (or (1.2)) to be satisfied *for all* p in $Fix(T)$ is superfluous.

The demicontractivity (or even the strongly demicontractivity) together with demiclosedness at zero and some restrictions on control sequence ensure the weak convergence of the Mann iteration [5, 4]. Obviously, in finite dimensional spaces these conditions are sufficient. In order to get strong convergence, some additional conditions are needed.

Recall that the Mann iteration is defined by

$$x_{n+1} = (1 - t_n)x_n + t_n T(x_n), \tag{1.3}$$

where $\{t_n\}$ is the control sequence and we have usually $t_n \in (0, 1)$.

In the case of a constant control sequence, $t_n = t,\ n = 0, 1, \ldots$, it is used the term *Krasnoselskij* or *Krasnoselskij-Mann* iteration for the iterative process obtained from (1.3). In fact, the Krasnoselskij iteration is a particular Picard iteration with the iteration function $T_t = (1 - t)I + tT$, where I is the identity mapping.

Remark 1.1. In his original paper [3], Krasnoselskij considered a still more particular case, $x_{n+1} = (T(x_n) + x_n)/2$, i.e., the Mann iteration with $t_n = 1/2,\ n = 0, 1, \ldots$. Supposing that T is non-expansive ($\|T(x) - T(y)\| \leq \|x - y\|$) he proved that the generated sequence converges to some fixed point of T. It is mentioned that, in the same condition, the Picard iteration is not necessarily convergent.

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In [7] a condition of expansive type, called *quasi-expansive*, was proposed in order to get convergence in norm of the Mann iteration. A mapping T is said to be β*-quasi-expansive* if

$$\|x-p\| \le \beta\|x-T(x)\|,\ x\in C,$$

where $0<\beta<1$. It is easy to see that this implies

$$\|x-p\| \le \frac{\beta}{1-\beta}\|T(x)-p\|,$$

which motivates the term *quasi-expansive*. Note that the strongly demicontractivity and the quasi-expansivity are not contradictory.

The two conditions (strongly demicontractivity and quasi-expansivity) together with some restrictions on constants a, k, β and on the control parameter t ensure the convergence in norm of the Krasnoselskij method.

Theorem 3.4 [7] is one of the first results in which the quasi-expansivity is used.

Theorem 1.1. ([7], Theorem 3.4). *Suppose that* $T:\mathcal{C}\to\mathcal{C}$ *is* (a,k)*-strongly demicontractive with* $a,\ k\in[0.1,1)$ *and* β*-quasi-expansive with* $\beta=(1-k)/1.8$. *Then*

$$\|T_tx-T_ty\| \le \delta(\|x-T_tx\|-\|y-T_ty\|),\ \forall x,y\in\mathcal{C},$$

where $\delta=0.43588985...$, $t\in(t_1,t_2)\cap(0,1)$ *and* t_1,t_2 *are the roots of the polynomial*

$$P(t)=(1-\delta^2)t^2-[1-k+(1-\alpha)\beta^2]t+\beta^2. \tag{1.4}$$

The theorem bellow is a slight generalization of Theorem 3.4 [7]. We also present a significantly improved proof of it.

Theorem 1.2. *Let* $T:\mathcal{C}\to\mathcal{C}$ *be an* (a,k)*-strongly demicontractive and* β*-quasi-expansive mapping. Let* δ *be such that* $0<\delta<0.5$ *and suppose that* a,k *satisfy the conditions* $a,k\in(0,1)$ *and* $\beta=(1-k)/(2\sqrt{1-\delta^2})$. *Then the sequence* $\{x_n\}$ *given by the Krasnoselskij iteration method converges to the unique fixed point* p *of* T, *provided that* $t\in(t_1,t_2)\cap(0,1)$ *and* t_1,t_2 *are the roots of the polynomial*

$$P(t)=(1-\delta^2)t^2-[1-k+(1-a)\beta^2]t+\beta^2.$$

Proof. Using the notation $d=1-k+(1-a)\beta^2$ we have

$$d^2-4(1-\delta^2)\beta^2=d^2-(1-k)^2>0,$$

which shows that P has two distinct real roots t_1,t_2. The lowest root, say t_1, is

$$t_1=\frac{d-\sqrt{d^2-(1-k)^2}}{2(1-\delta^2)}.$$

Obvious, $t_1>0$. We will show that $t_1<1$. Using the fact that $a,k\in(0,1)$ and $\delta<0.5$, we have

$$(1-a)\beta^2=\frac{(1-a)(1-k)^2}{4(1-\delta^2)}<\frac{1}{4(1-\delta^2)}<1-2\delta^2<1+k-2\delta^2=2(1-\delta^2)-1+k.$$

Thus

$$d-2(1-\delta^2)=1-k+(1-a)\beta^2-2(1-\delta^2)<0<\sqrt{d^2-(1-k)^2},$$

and

$$d-\sqrt{d^2-(1-k)^2}<2(1-\delta^2).$$

Therefore $0<t_1<1$ and $(t_1,t_2)\cap[0,1]\neq\emptyset$. For $t\in(t_1,t_2)\cap[0,1]$, we have $P(t)<0$, which implies

$$\frac{(1-t+ta)\beta^2+t^2-t+tk}{t^2}<\delta^2.$$

Now, using (1.2), we have

$$\begin{aligned}\|T_t x - p\|^2 &= \|x - p - t(x - Tx)\|^2 \\ &= \|x - p\|^2 - 2t\langle x - Tx, x - p\rangle + t^2\|x - Tx\|^2 \\ &\leq \|x - p\|^2 - t(1 - a)\|x - p\|^2 - t(1 - k)\|x - Tx\|^2 + t^2\|x - Tx\|^2 \\ &= (1 - t + ta)\|x - p\|^2 + (t^2 - t + tk)\|x - Tx\|^2.\end{aligned}$$

Then, taking into account that $\|x-p\| \leq \beta\|x-Tx\|$ and $\|x-Tx\| = \|x-T_t x\|/t$ we obtain

$$\|T_t x - p\|^2 \leq \frac{(1 - t + ta)\beta^2 + t^2 - t + tk}{t^2}\|x - T_t x\|^2.$$

Therefore, for $t \in (t_1, t_2) \cap (0, 1)$ it results

$$\|T_t x - p\| \leq \delta\|x - T_t x\|, \ \forall x \in C.$$

Finally we have

$$\|T_t x - T_t y\| \leq \|T_t x - p\| + \|T_t y - p\| \leq \delta(\|x - T_t x\| + \|y - T_t y\|), \ \forall x, y \in C.$$

That is T_t is a Kannan contraction. □

Remark 1.2. For the sake of simplicity, we considered here a particular case of Mann iteration with constant control sequence, $t_n = t$, i.e., the Krasnoselskij method, but we can obtain the same result in the general case of Mann iteration.

The real function in the example below fulfils the conditions of Theorem 1.1

Example 1.1. Let $f : [-0.375, 0.375] \to [-0.375, 0.375]$ be given by $f(x) = 1.5x^3 - 1.2x$, for all $x \in [-0.375, 0.375]$. We can take $\delta = 0.49$. Then, the function f is strongly demicontractive with $p = 0$, $a = 0.9$, $k = 0.12$ and quasi-expansive with $\beta = (1 - k)/(2\sqrt{(1 - \delta^2)}) = 0.504....$ The two roots of P are $t_1 = 0.455...$ and $t_2 = 0.736....$

Therefore, for any $t \in (0.455..., 0.736...)$ and $x_0 \in [-0.375, 0.375]$, the Krasnoselskij iteration associated to f converges to 0.

For the function f in Example 1.1, in Table 1 it is shown the number of iterations needed to obtain a precision of 10^{-15} for various initial values of $x_0 \in [-0.375, 0.375]$ and parameter $t \in (t_1, t_2)$.

The symbol "$\star$" in the last column shows that the Picard iteration does not converge in all those cases.

x_0/t	t_1	0.5	0.6	0.7	t_2	1
-0.375	5	13	28	53	69	$\star$
-0.3	5	13	28	53	69	$\star$
-0,2	5	14	28	53	68	$\star$
-0.1	5	13	28	52	67	$\star$
0.1	5	13	28	52	67	$\star$
0.2	5	14	28	53	68	$\star$
0.3	5	13	28	53	69	$\star$
0.375	5	13	28	53	69	$\star$

TABLE 1. The behavior of Krasnoselskij method for various values of x_0 and t.

Note that the interval (t_1, t_2) defined in Theorem 1.2 does not cover all the good values of t. In our example the interval of t for which the Krasnoselskij iteration converges is actually $(0, 2/2.2 = 0.90909...)$.

For t in this interval, the graph of T_t is situated in the area limited by the first bisectrix and the second bisectrix of the coordinates axes, which explains the behavior of Krasnoselskij method.

Remark 1.3. Theorem 1.2 gives a satisfactory answer to the following question:

If, for a given mapping some specific conditions for the convergence of Picard iteration are not satisfied (and presumptively the Picard iteration fails to converge), what are the conditions which ensure instead the convergence of the Krasnoselskij iteration?

The function f in Example 1.1 is not a Berinde-Almost-Contraction, see [1] and [2], since it can be simply checked that f it is not a graphic contraction (orbital contraction) and, therefore, f is not a Banach, Kannan, Ciric-Reich-Rus, Chatterjea, Zamfirescu contraction etc. (see [8]), too.

Thus the convergence theorems concerning these kind of contractions cannot be applied for Picard iteration.

Various numerical experiments that we have performed but which are not presented here show that, indeed, the Picard iteration does not converge for f in Example 1.1. Instead, the Krasnoselskij iteration, with the mentioned condition on the control parameter, can be successfully applied.

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