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Coincidence and periodic point results in a modular metric space endowed with a graph and applications

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ABSTRACT. In this paper, we present some results on the existence of coincidence and periodic point of F-contractive mappings in the framework of modular metric spaces endowed with a graph. We also present an application to partial differential equations in order to support the theoretical results.

1. INTRODUCTION

In 2007, by using the language of graph theory, Jachymski [12] introduced the concept of a G-contraction on a metric space endowed with a graph and proved a fixed point theorem which extends the results of Ran and Reurings [20]. In 2012, Wardowski [23] introduced the concept of an F-contraction and proved a fixed point theorem which generalizes Banach contraction principle in many ways. These two results have become of recent interest of many authors (see [3, 4, 9, 10, 11, 17, 18, 19, 21] and references therein).

On the other hand, Chistyakov [8] introduced the notion of modular metric space and gave some fundamental results on this topic, whereas in [2] the authors introduced the analogue of the Banach contraction principle theorem in modular metric spaces and described some important aspects and applications of fixed points of mappings in this framework.

Following this direction of research, in this paper, we establish some coincidence and periodic point theorems concerning F-contractive mappings in modular metric space endowed with a graph. Our main result is a generalization of Gopal et al [11] theorem and others. We also give an application of our main results to establish the existence of solution for a nonhomogenuous linear parabolic partial differential equation.

Consistent with Chistyakov [8] and Abdou [2], we begin with some basic definitions and results which will be used in the sequel.

Throughout the article $\mathbb{N}$, $\mathbb{R}_+, \mathbb{R}^+$ and $\mathbb{R}$ will denote the set of natural numbers, nonnegative real numbers, positive real numbers and real numbers, respectively.

Let X be a nonempty set. Throughout this paper, for a function $\omega : (0,\infty) \times X \times X \to [0,\infty]$, we write

$$\omega_\lambda(x,y) = \omega(\lambda, x, y)$$

for all $\lambda > 0$ and $x, y \in X$.

Definition 1.1. [8, 2] Let X be a nonempty set. A function $\omega : (0,\infty) \times X \times X \to [0,\infty]$ is said to be a metric modular on X if, for all $x, y, z \in X$, the following conditions hold:

(i) $\omega_\lambda(x,y) = 0$, for all $\lambda > 0$, if and only if $x = y$,
(ii) $\omega_\lambda(x,y) = \omega_\lambda(y,x)$, for all $\lambda > 0$,

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(iii) $\omega_{\lambda+\mu}(x,y) \leq \omega_\lambda(x,z) + \omega_\mu(z,y)$, for all $\lambda, \mu > 0$.

If, instead of (i) we have only the condition (i')

$$\omega_\lambda(x,x) = 0 \text{ for all } \lambda > 0,\ x \in X,$$

then ω is said to be a pseudomodular (metric) on X. A modular metric ω on X is said to be regular if the following weaker version of (i) is satisfied:
$x = y$ if and only if $\omega_\lambda(x,y) = 0$ for some $\lambda > 0$.

Note that for a metric (pseudo)modular ω on a set X, and any $x, y \in X$, the function $\lambda \mapsto \omega_\lambda(x,y)$ is nonincreasing on $(0,\infty)$. Indeed, if $0 < \mu < \lambda$, then

$$\omega_\lambda(x,y) \quad \leq \omega_{\lambda-\mu}(x,x) + \omega_\mu(x,y) \quad = \omega_\mu(x,y).$$

Note that every modular metric is regular but the converse may not necessarily be true.

Example 1.1. Let $X = \mathbb{R}$ and w defined by $w_\lambda(x,y) = \infty$ if $\lambda < 1$, and $w_\lambda(x,y) = |x-y|$ if $\lambda \geq 1$. It is easy to verify that w is a regular modular metric but not a modular metric.

Definition 1.2. [8, 2] Let X_ω be a (pseudo)modular on X. Fix $x_0 \in X$. The two sets

$$X_\omega = X_\omega(x_0) = \{x \in X : \omega_\lambda(x,x_0) \to 0 \text{ as } \lambda \to \infty\}$$

and

$$X^*_\omega = X^*_\omega(x_0) = \{x \in X : \exists \lambda = \lambda(x) > 0 \text{ such that } \omega_\lambda(x,x_0) < \infty\}$$

are said to be modular spaces (around x_0).

Definition 1.3. [2] Let X_ω be a modular metric space.

(i) The sequence $(x_n)_{n\in\mathbb{N}}$ in X_ω is said to be ω-convergent to $x \in X_\omega$ if and only if $\omega_\lambda(x_n,x) \to 0$, as $n \to \infty$ for some $\lambda > 0$.

(ii) The sequence $(x_n)_{n\in\mathbb{N}}$ in X_ω is said to be ω-Cauchy if $\omega_\lambda(x_m,x_n) \to 0$, as $m, n \to \infty$ for some $\lambda > 0$.

(iii) A subset C of X_ω is said to be ω-complete if any ω-Cauchy sequence in C is a convergent sequence and its limit is in C.

(iv) A subset C of X_ω is said to be ω-bounded if, for some $\lambda > 0$, we have $\delta_\omega(C) = \sup\{\omega_\lambda(x,y); x, y \in C\} < \infty$.

Definition 1.4. [18] Let X_ω be a modular metric space and C is a nonempty subset of X_ω. The sequence $(x_n)_{n\in\mathbb{N}}$ in C is said to satisfy Δ_M-condition if

$$\lim_{m,n\to\infty} \omega_{m-(n+1)}(x_n,x_m) = 0, \text{ for } (m,n \in \mathbb{N},\ m \geq n)$$

implies $\lim\limits_{m,n\to\infty} \omega_\lambda(x_n,x_m) = 0$ for all $\lambda > 0$.

Following Wardowski [23], we denote by $\mathcal{F}$ the family of all functions $F : \mathbb{R}^+ \to \mathbb{R}$ satisfying the following conditions:

$(F1)$ F is strictly increasing on $\mathbb{R}^+$,

$(F2)$ for every sequence $\{s_n\}$ in $\mathbb{R}^+$, we have $\lim\limits_{n\to\infty} s_n = 0$ if and only if

$$\lim_{n\to\infty} F(s_n) = -\infty,$$

$(F3)$ there exists a number $k \in (0,1)$ such that $\lim\limits_{s\to 0^+} s^k F(s) = 0$.

Example 1.2. The following functions $F : \mathbb{R}^+ \to \mathbb{R}$ belongs to $\mathcal{F}$:

(i) $F(s) = \ln s$, with $s > 0$,

(ii) $F(s) = \ln s + s$, with $s > 0$.

Let (X,ω) be a modular metric space and D be a nonempty subset of X_ω. Let Δ denote the diagonal of the Cartesian product $D \times D$. Let G_ω be a directed graph (digraph) such that the set $V(G_\omega)$ of its vertices coincides with D, and the set $E(G_\omega)$ of its edges contains all loops, i.e., $E(G_\omega) \supseteq \Delta$. We assume G_ω simple graph (opposite of multigraph), so we can identify G_ω with the pair $(V(G_\omega), E(G_\omega))$. Our graph theory notations and terminology are standard and can be found in all graph theory books, like [6] and [14]. Moreover, we may treat G_ω as a weighted graph (see [14], p. 309) by assigning to each edge the distance between its vertices. By G_ω^{-1} we denote the reverse of a graph G_ω, i.e., the graph obtained from G_ω by reversing the direction of edges. Thus we have

$$E(G_\omega^{-1}) = \{(y,x)|(x,y) \in E(G_\omega)\}.$$

A digraph G_ω is an oriented graph if whenever $(u,v) \in E(G_\omega)$, then $(v,u) \notin E(G_\omega)$. The letter $\tilde{G_\omega}$ denotes the undirected graph obtained from G_ω by ignoring the direction of edges. Actually, it will be more convenient for us to treat $\tilde{G_\omega}$ as a directed graph for which the set of its edges is symmetric. Under this convention,

$$E(\tilde{G_\omega}) = E(G_\omega) \cup E(G_\omega^{-1}).$$

We call (V', E') a subgraph of G_ω if $V' \subseteq V(G_\omega)$, $E' \subseteq E(G_\omega)$, and for any edge $(x,y) \in E'$, $x, y \in V'$.

If x and y are vertices in a graph G_ω, then a (directed) path in G_ω from x to y of length N is a sequence $(x_i)_{i=1}^{i=N}$ of $N+1$ vertices such that $x_0 = x$, $x_N = y$ and $(x_{n-1}, x_n) \in E(G_\omega)$ for $i = 1, \ldots, N$. A graph G_ω is connected if there is a directed path between any two vertices. G_ω is weakly connected if $\tilde{G_\omega}$ is connected. If G_ω is such that $E(G_\omega)$ is symmetric and x is a vertex in G_ω, then the subgraph $G_{\omega[x]}$ consisting of all edges and vertices which are contained in some path beginning at x is called the component of G_ω containing x. In this case $V(G_{\omega[x]}) = [x]_{G_\omega}$, where $[x]_{G_\omega}$ is the equivalence class of the following relation $\mathcal{R}$ defined on $V(G_\omega)$ by the rule: $y\mathcal{R}z$ if there is a (directed) path in G_ω from y to z. Clearly $G_{\omega[x]}$ is connected.

2. Periodic point results

Throughout this section we assume that (X,ω) is a modular metric space, D be a nonempty subset of X_ω and $\mathcal{G} := \{G_\omega$ is a directed graph with $V(G_\omega) = D$ and $\Delta \subseteq E(G_\omega)\}$.

Definition 2.5. [12, 3] The pair (D, G_ω) has Property (A) if for any sequence $\{x_n\}_{n\in\mathbb{N}}$ in D, with $x_n \to x$ as $n \to \infty$ and $(x_n, x_{n+1}) \in E(G_\omega)$, then $(x_n, x) \in E(G_\omega)$, for all n.

Definition 2.6. Let $F \in \mathcal{F}$ and $G_\omega \in \mathcal{G}$. A mapping $T : D \to D$ is said to be F-G_ω-contraction with respect to $R : D \to D$ if

(i) $(Rx, Ry) \in E(G_\omega) \Rightarrow (Tx, Ty) \in E(G_\omega)$ for all $x, y \in D$, i. e., T preserves edges w.r.t. R,

(ii) there exists a number $\tau > 0$ such that

$$\omega_1(Tx, Ty) > 0 \Rightarrow \tau + F(\omega_1(Tx, Ty)) \le F(\omega_1(Rx, Ry))$$

for all $x, y \in D$ with $(Rx, Ry) \in E(G_\omega)$.

Example 2.3. Let $F \in \mathcal{F}$ be arbitrary. Then every F-contractive mapping w.r.t. R is an F-G_ω-contraction w.r.t. R for the graph G_ω given by $V(G_\omega) = D$ and $E(G_\omega) = D \times D$.

We denote by $C(T,R) := \{x \in D : Tx = Rx\}$ the set of all coincidence points of two self-mappings T and R, defined on D.

Now, we state our first theorem which generalises the main theorem of Gopal et al. [11] for regular modular metric spaces.

Theorem 2.1. *Let (X,ω) be a regular modular metric space with a graph G_ω. Assume that $D = V(G_\omega)$ is a nonempty ω-bounded subset of X_ω and the pair (D, G_ω) has property (A) and also satisfy Δ_M-condition. Let $R,T : D \to D$ be two self mappings satisfying the following conditions:*

(i) *there exists $x_0 \in D$ such that $(Rx_0, Tx_0) \in E(G_\omega)$,*
(ii) *T is an F-G_ω-contraction w.r.t R,*
(iii) *$T(D) \subseteq R(D)$,*
(iii) *$R(D)$ is ω complete.*

Then $C(R,T) \neq \emptyset$.

Proof. Let $x_0 \in D$ such that $(Rx_0, Tx_0) \in E(G_\omega)$, since $T(D) \subseteq R(D)$, then there exists a point $x_1 \in D$ such that $Rx_1 = Tx_0$. From (i), we have $(Rx_0, Rx_1) \in E(G_\omega)$, and since T preserves edges w.r.t. R, we get $(Tx_0, Tx_1) \in E(G_\omega)$. By continuing this process, having chosen x_n in D, we obtain x_{n+1} in D such that

$$(Rx_n, Rx_{x+1}) = (Tx_{n-1}, Tx_n) \text{ for every } n \in \mathbb{N}.$$

Let $\kappa_n = \omega_1(Rx_n, Rx_{n+1})$ for all $n \in \mathbb{N} \cup \{0\}$. If there exists $n_0 \in \mathbb{N} \cup \{0\}$ such that $Rx_{n_0} = Rx_{n_0+1}$, then $Tx_{n_0} = Rx_{n_0+1}$ implies that $Tx_{n_0+1} = Rx_{n_0+1}$ that is $x_{n_0+1} \in C(T,R)$. Now, we assume $Rx_n \neq Rx_{n+1}$ for any $n \in \mathbb{N} \cup \{0\}$. Since T is an F-G_ω-contraction w.r.t R on $E(G_\omega)$, then we have

$$\begin{array}{rcl} F(\kappa_n) & = & F(\omega_1(Rx_n, Rx_{n+1})) \\ & = & F(\omega_1(Tx_{n-1}, Tx_n)) \\ & \leq & F(\omega_1(Rx_{n-1}, Rx_n)) - \tau \\ & = & F(\omega_1(Tx_{n-2}, Tx_{n-1})) - \tau \\ & \leq & F(\omega_1(Rx_{n-2}, Rx_{n-1})) - 2\tau \\ & & \vdots \\ & \leq & F(\omega_1(Rx_1, Rx_2)) - (n-1)\tau \\ & = & F(\kappa_1) - (n-1)\tau. \end{array}$$

Thus,

$$F(\kappa_n) \leq F(\kappa_1) - (n-1)\tau. \tag{2.1}$$

By letting $n \to \infty$ in (2.1) and since D is ω-bounded, we have $\lim_{n\to\infty} F(\kappa_n) = -\infty$. Thus, $\lim_{n\to\infty} \kappa_n = 0$, by $(F2)$. Now, by $(F3)$, there exists $k \in (0,1)$ such that $\lim_{n\to\infty} \kappa_n^k F(\kappa_n) = 0$. Note that

$$\begin{array}{rcl} \kappa_n^k F(\kappa_n) - \kappa_n^k F(\kappa_1) & \leq & \kappa_n^k (F(\kappa_1) - (n-1)\tau) - \kappa_n^k F(\kappa_1) \\ & = & -\kappa_n^k (n-1)\tau \leq 0. \end{array} \tag{2.2}$$

Therefore, by letting $n \to \infty$ in (2.2), we obtain $\lim_{n\to\infty} (n-1)\kappa_n^k = 0$. Consequently $\lim_{n\to\infty} n\kappa_n^k = 0$. Thus, there exists $n_1 \in \mathbb{N}$ such that $n\kappa_n^k \leq 1$ for all $n \geq n_1$, i.e. $\kappa_n \leq 1/n^{1/k}$ for all $n \geq n_1$. Now, for integers $m > n \geq 1$, we have

$$\begin{array}{rcl} \omega_{m-(n+1)}(Rx_n, Rx_m) & \leq & \omega_1(Rx_n, Rx_{n+1}) + \omega_1(Rx_{n+1}, Rx_{n+2}) + \cdots + \omega_1(Rx_{m-1}, Rx_m) \\ & < & \sum_{i=n}^{\infty} \kappa_i \\ & \leq & \sum_{i=n}^{\infty} \frac{1}{i^{\frac{1}{k}}} < \infty. \end{array}$$

Since w satisfies Δ_M-condition, we have

$$\lim_{m,n\to\infty} w_1(x_n, x_m) = 0.$$

This shows that $\{Rx_n\}_{n\in\mathbb{N}}$ is a ω-Cauchy sequence in $R(D)$. Since $R(D)$ is ω-complete then there exists $u \in R(D)$ such that $\lim_{n\to\infty} Rx_n = u$. Let $v \in D$ be such that $Rv = u$. By property (A), we have $(Rx_n, u = Rv) \in E(G_\omega)$ for all n, and hence by (ii), we get

$$\begin{aligned} F(\omega_1(Rx_n, Tv)) &= F(\omega_1(Tx_{n-1}, Tv)) \\ &\leq F(\omega_1(Rx_{n-1}, Rv)) - \tau. \end{aligned}$$

Since $\lim_{n\to\infty} \omega_1(Rx_{n-1}, Rv) = 0$, then by $(F2)$, we have $\lim_{n\to\infty} F(\omega_1(Rx_{n-1}, Rv)) = -\infty$. Hence $\lim_{n\to\infty} F(\omega_1(Rx_n, Tv)) = -\infty$, which implies that $\lim_{n\to\infty} \omega_1(Rx_n, Tv) = 0$, and since ω is regular, we have $Rv = u = \lim_{n\to\infty} Rx_n = Tv$ i.e. $v \in C(R, T)$.

□

3. Periodic point results

In this section we prove some periodic point results for self mappings on a modular metric space endowed with a graph.

Definition 3.7. [13] Let (X, ω) be a modular metric space and $T : D \to D$ be a mapping. Then T is said to have the property (P) if $Fix(T^n) = Fix(T)$ for every $n \in \mathbb{N}$ where $Fix(T) := \{x \in D : Tx = x\}$.

Again, let (X, ω) be a modular metric space and $T : D \to D$ be a mapping. The set $\mathcal{O}(x) = \{x, Tx, T^2x, \ldots, T^nx, \ldots\}$ is called the orbit of x under T.

Definition 3.8. A mapping $T : D \to D$ is called strong orbitally G_ω-continuous at x if

$$\lim_{n\to\infty} T^nx = x_* \text{ and } (T^nx, T^{n+1}x) \in E(G_\omega) \Rightarrow \lim_{n\to\infty} T^{n+1}x = Tx_*.$$

A mapping T is called strongly G_ω-orbitally continuous on D if T is strongly orbitally G_ω-continuous for all $x \in D$.

We denote $D^T := \{x \in D : (x, Tx) \in E(G_\omega) \text{ or } (Tx, x) \in E(G_\omega)\}$.

Definition 3.9. Let (X, ω) be a modular metric space. A mapping $T : D \to D$ is called an F-G_ω graphic contraction if

(i) T preserves edges, i.e. $(x, y) \in E(G_\omega) \Rightarrow (Tx, Ty) \in E(G_\omega)$,

(ii) there exists a number $\tau > 0$ such that

$$\omega_1(Tx, T^2x) > 0 \Rightarrow \tau + F(\omega_1(Tx, T^2x)) \leq F(\omega_1(x, Tx)) \tag{3.3}$$

for all $x \in D^T$ and $F \in \mathcal{F}$.

Remark 3.1. If we consider $F(s) = \ln s$ for all $s > 0$, then Definition 3.9 reduces to G_ω-graphic contractive given in [9].

Before stating the theorem of this section, we give the following lemma without proof.

Lemma 3.1. *Let (X, ω) be a modular metric space endowed with a graph G_ω. Let $T : D \to D$ be a G_ω-graphic contractive. Then T is a G_ω^{-1}-graphic contractive too.*

Theorem 3.2. *Let (X, w) be a regular modular metric space with a graph G_ω. Assume that $D = V(G_\omega)$ is w complete, ω-bounded (nonempty) subset of X_ω and the pair (D, G_ω) satisfy Δ_M-condition. Suppose that $T : D \to D$ is an F-G_ω-graphic contraction satisfying the following condition:*

$(*)$ $(x, Tx) \in E(G_\omega)$ or $(Tx, x) \in E(G_\omega)$ for all $x \in D$.

Then T has the property (P) provided that T is strongly G_ω-orbitally continuous on D.

Proof. Let x_0 be an arbitrary point in D. Define a sequence $\{x_n\}_{n\in\mathbb{N}}$ in D such that $x_{n+1} = Tx_n$ for all $n \in \mathbb{N}\cup\{0\}$ and denote $\kappa_n = \omega_1(x_n, x_{n+1})$ for all $n \in \mathbb{N}\cup\{0\}$. If there exists $n_0 \in \mathbb{N}\cup\{0\}$ for which $x_{n_0+1} = x_{n_0}$, then $Tx_{n_0} = x_{n_0}$ and the proof is finished. Then, we assume that $x_{n+1} \neq x_n$, for all $n \in \mathbb{N}\cup\{0\}$.

Since $(x_0, Tx_0) \in E(G_\omega)$ (or $(Tx_0, x_0) \in E(G_\omega)$) and T preserves edges, thus we get

$$(Tx_0, T(Tx_0)) = (x_1, x_2) = (x_1, Tx_1) \in E(G_\omega) \Rightarrow (Tx_1, T(Tx_1)) = (x_2, x_3) \in E(G_\omega).$$

By this process, we get that $(T^n x_0, T^{n+1}x_0) = (x_n, x_{n+1}) \in E(G_\omega)$ for all $n \in \mathbb{N}\cup\{0\}$. Now using (3.3), we get

$$\begin{array}{rcl} F(\kappa_n) & = & F(\omega_1(x_n, x_{n+1})) \\ & = & F(\omega_1(Tx_{n-1}, T^2x_{n-1})) \\ & \leq & F(\omega_1(x_{n-1}, Tx_{n-1})) - \tau \\ & = & F(\omega_1(Tx_{n-2}, T^2x_{n-2})) - \tau \\ & \leq & F(\omega_1(x_{n-2}, Tx_{n-2}) - 2\tau \\ & & \vdots \\ & = & F(\omega_1(Tx_0, T^2x_1)) - (n-1)\tau \\ & \leq & F(\omega_1(x_0, x_1)) - n\tau \\ & = & F(\kappa_0) - n\tau \end{array}$$

for every $n \in \mathbb{N}\cup\{0\}$. Therefore, proceeding as in the proof of Theorem 2.1, we get that $\{T^n x_0\}_{n\in\mathbb{N}}$ is ω-Cauchy sequence. Since $\{T^n x_0 : n \in \mathbb{N}\} \subseteq \mathcal{O}(x_0) \subseteq D$ and D is ω-complete, therefore $\{x_n\}$ ω-converges to some $x_* \in D$. Since T is strongly orbitally G_ω-continuous on D, then $x_* = \lim_{n\to\infty} T^n x_0 = T(\lim_{n\to\infty} T^{n-1}x_0) = Tx_*$. Thus T has a fixed point and $Fix(T^n) = Fix(T)$ is true for $n = 1$. Now assume that $n > 1$ and assume that $z \in Fix(T^n)$ but $z \notin Fix(T)$, then $\omega_1(z, Tz) = s > 0$. By $(*)$, we have $(z, Tz) \in E(G_\omega)$ or $(Tz, z) \in E(G_\omega)$. If we assume $(z, Tz) \in E(G_\omega)$, by (3.3), we get

$$\begin{array}{rcl} F(s) & = & F(\omega_1(z, Tz)) \\ & = & F(\omega_1(T(T^{n-1}z), T^2(T^{n-1}z))) \\ & \leq & F(\omega_1(T^{n-1}z, T^n z)) - \tau \\ & \leq & F(\omega_1(T^{n-2}z, T^{n-1}z)) - 2\tau \\ & & \vdots \\ & \leq & F(\omega_1(z, Tz)) - n\tau. \end{array}$$

Thus $F(s) \leq \lim_{n\to\infty}[F(\omega_1(z, Tz)) - n\tau] = -\infty$ and hence $F(s) = -\infty$, which is a contradiction until $\omega_1(z, Tz)) = 0$ and by the regularity of ω, we get that $z = Tz$. Hence, $Fix(T^n) = Fix(T)$ for all $n \in \mathbb{N}$.

□

4. EXISTENCE OF SOLUTION FOR A NONHOMOGENEOUS LINEAR PARABOLIC PARTIAL DIFFERENTIAL EQUATION

In this section, following the idea in [7], we discuss the application of coincidence (fixed) point techniques to the solution of the non-homogeneous linear parabolic partial differential equation satisfying a given initial condition.

More precisely, we consider the following initial value problem

$$\begin{cases} u_t(x,t) = u_{xx}(x,t) + H(x,t,u(x,t),u_x(x,t)), & -\infty < x < \infty, 0 < t \leq T \\ u(x,0) = \varphi(x) \geq 0, & -\infty < x < \infty, \end{cases} \tag{4.4}$$

where H is continuous and φ is assumed to be continuously differentiable and such that φ and φ' are bounded.

By a *solution* of the problem (4.4), we mean a function $u \equiv u(x,t)$ defined on $\mathbb{R} \times I$, where $I := [0,T]$, satisfying the following conditions:

(i) $u, u_t, u_x, u_{xx} \in C(\mathbb{R} \times I)$. { $C(\mathbb{R} \times I)$ denote the space of all continuous real valued functions },

(ii) u and u_x are bounded in $\mathbb{R} \times I$,

(iii) $u_t(x,t) = u_{xx}(x,t) + H(x,t,u(x,t),u_x(x,t))$ for all $(x,t) \in \mathbb{R} \times I$,

(iv) $u(x,0) = \varphi(x)$ for all $x \in \mathbb{R}$.

It is important to note that the initial value problem (4.4) is equivalent to the following integral equation

$$u(x,t) = \int_{-\infty}^{\infty} k(x-\xi,t)\varphi(\xi)d\xi + \int_0^t \int_{-\infty}^{\infty} k(x-\xi,t-\tau)H(\xi,\tau,u(\xi,\tau),u_x(\xi,\tau))d\xi d\tau \quad (4.5)$$

for all $x \in \mathbb{R}$ and $0 < t \leq T$, where

$$k(x,t) := \frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{4t}}.$$

The problem (4.4) admits a solution if and only if the corresponding integral equation (4.5) has a solution.

Let

$$\Omega := \{u(x,t); u, u_x \in C(\mathbb{R} \times I) \text{ and } \|u\| < \infty\},$$

where

$$\|u\| := \sup_{x\in\mathbb{R}, t\in I} |u(x,t)| + \sup_{x\in\mathbb{R}, t\in I} |u_x(x,t)|.$$

Obviously, the function $\omega : \mathbb{R}^+ \times \Omega \times \Omega \to \mathbb{R}_+$ given by

$$\omega_\lambda(u,v) := \frac{1}{\lambda}\|u-v\| = \frac{1}{\lambda}d(u,v)$$

is a metric modular on Ω. Clearly, the set Ω_ω is a complete modular metric space independent of generators.

Theorem 4.3. *Consider the problem (4.4) and assume the followings:*

(i) *for $c > 0$ with $|s| < c$ and $|p| < c$, the function $F(x,t,s,p)$ is uniformly Hölder continuous in x and t for each compact subset of $\mathbb{R} \times I$,*

(ii) *there exists a constant $c_H \leq (T + 2\pi^{-\frac{1}{2}}T^{\frac{1}{2}})^{-1} \leq q$, where $q \in (0,1)$ such that*

$$\begin{aligned} 0 &\leq \tfrac{1}{\lambda}[H(x,t,s_2,p_2) - H(x,t,s_1,p_1)] \\ &\leq c_H \left[\frac{s_2 - s_1 + p_2 - p_1}{\lambda}\right] \end{aligned}$$

for all $(s_1,p_1),(s_2,p_2) \in \mathbb{R} \times \mathbb{R}$ with $s_1 \leq s_2$ and $p_1 \leq p_2$,

(iii) *H is bounded for bounded s and p.*

Then the problem (4.4) admits a solution.

Proof. It is well known that $u \in \Omega_\omega$ is a solution (4.4) iff $u \in \Omega_\omega$ is a solution integral equation (4.5).

Consider the graph G with $V(G) = D = \Omega_\omega$ and $E(G) = \{(u,v) \in D \times D : u(x,t) \leq v(x,t)$ and $u_x(x,t) \leq v_x(x,t)$ at each $(x,t) \in \mathbb{R} \times I\}$. Clearly $E(G)$ is a partial ordered and

$(D, E(G))$ satisfy property (A).
Also, define a mapping $\Lambda : \Omega_\omega \to \Omega_\omega$ by

$$(\Lambda u)(x,t) := \int_{-\infty}^{\infty} k(x-\xi,t)\varphi(\xi)d\xi + \int_0^t \int_{-\infty}^{\infty} k(x-\xi,t-\tau)H(\xi,\tau,u(\xi,\tau),u_x(\xi,\tau))d\xi d\tau$$

for all $(x,t) \in \mathbb{R} \times I$. Then, finding solution of problem (4.5) is equivalent to the ensuring the existence of fixed point of Λ.

Since $(u,v) \in E(G), (u_x, v_x) \in E(G)$ and hence $(\Lambda u, \Lambda v) \in E(G), (\Lambda u_x, \Lambda v_x) \in E(G)$. Thus, from the definition of Λ and by (ii) we have

$$\begin{aligned}
&\tfrac{1}{\lambda}|(\Lambda v)(x,t) - (\Lambda u)(x,t)| \\
&\leq \tfrac{1}{\lambda} \int_0^t \int_{-\infty}^{\infty} k(x-\xi,t-\tau)|H(\xi,\tau,v(\xi,\tau),v_x(\xi,\tau)) - H(\xi,\tau,u(\xi,\tau),u_x(\xi,\tau))|d\xi d\tau \\
&\leq \int_0^t \int_{-\infty}^{\infty} k(x-\xi,t-\tau)c_H \left[\frac{1}{\lambda}|\big(v(\xi,\tau) - u(\xi,\tau) + v_x(\xi,\tau) - u_x(\xi,\tau)\big)|\right] d\xi d\tau \\
&\leq c_H \omega_\lambda(u,v)T. \qquad (4.6)
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
\tfrac{1}{\lambda}|(\Lambda v)_x(x,t) - (\Lambda u)_x(x,t)| &\leq c_H\omega_\lambda(u,v)\int_0^t \int_{-\infty}^{\infty} |k_x(x-\xi,t-\tau)|d\xi d\tau \\
&\leq 2\pi^{-\frac{1}{2}}T^{\frac{1}{2}}c_H\omega_\lambda(u,v). \qquad (4.7)
\end{aligned}$$

Therefore, from (4.6) and (4.7) we have

$$\omega_\lambda(\Lambda u, \Lambda v) \leq (T + 2\pi^{-\frac{1}{2}}T^{\frac{1}{2}})c_H\omega_\lambda(u,v)$$

i.e.

$$\omega_\lambda(\Lambda u, \Lambda v) \leq q\omega_\lambda(u,v), \ q \in (0,1)$$

i.e.

$$d(\Lambda u, \Lambda v) \leq e^{-\tau}d(u,v), \tau > 0$$

Now, by passing to logarithms, we can write this as

$$\ln(d(\Lambda u, \Lambda v)) \leq \ln(e^{-\tau}d(u,v))$$

$$\tau + \ln(d(\Lambda u, \Lambda v)) \leq \ln(d(u,v))$$

Now, from example 1.2(i) and taking $T = \Lambda$ and $R = \mathcal{I}$ (identity map), we deduce that the operator T satisfies all the hypothesis of Theorem 2.1.
Therefore, as an application of Theorem 2.1, we conclude the existence of $u^* \in \Omega_\omega$ such that $u^* = \Lambda u^*$ and so u^* is a solution of the problem 4.4. □

CONCLUSION

Taking into account its interesting applications, obtaining fixed point results in modular metric spaces has received considerable attention in recent years. In this connection, the main aim of this paper was to present some results on the existence of coincidence and periodic point of F-contractive mappings in the framework of modular metric spaces endowed with a graph. We also applied the obtained results to partial differential equation. The new concepts lead to further investigations and applications.

For instance, using the recent ideas in the literature [22, 5], it is possible to extend our results to the case of coupled as well as higher dimensional fixed points.

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