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A convergence of a Steffensen-like method for solving nonlinear equations in a Banach space

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ABSTRACT. We present a local as well as a semilocal convergence analysis of a Steffensen-like method in order to approximate a locally unique solution of a nonlinear equation in a Banach space setting. This method generalizes and improves the sufficient convergence conditions of earlier methods. In particular, a numerical example is presented to show the advantages of our approach.

1. INTRODUCTION

In this paper we study the convergence of Steffensen-like method defined for each $n = 0, 1, \ldots$ by [1–4]

$$\begin{aligned} y_n &= x_n - aF(x_n) \\ z_n &= x_n + bF(x_n) \\ x_{n+1} &= x_n - A_n^{-1}F(x_n), \end{aligned} \tag{1.1}$$

where x_0 is an initial point $a, b \in \mathbb{R}^+$, $A_n = \delta F(y_n, z_n)$ and δF is a consistent approximation to the derivative F' [5, 6] (see also conditions $(\mathcal{A})$ that follow) for approximating the solution of a nonlinear equation

$$F(x) = 0, \tag{1.2}$$

where $F : D \subseteq X \longrightarrow X$ is a continuously Fréchet-differentiable operator, X is a Banach space and D is a convex subset of the Banach space X. Due to the wide applications, finding solution for the equation (1.2) is an important problem in mathematics.

Observe that method (1.1) generalizes the method

$$\begin{aligned} y_n &= x_n - aF(x_n) \\ z_n &= x_n + bF(x_n) \\ x_{n+1} &= x_n - [y_n, z_n; F]^{-1}F(x_n), \end{aligned} \tag{1.3}$$

studied in [1–4] for $a, b \in \mathbb{R}^+$. The following semilocal convergence result of method (1.3) was shown in [2] (see also [1, 3–6, 8]) using divided difference of order one given by

$$[x, y; F] = \int_0^1 F'(\theta x + (1-\theta)y)d\theta. \tag{1.4}$$

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Theorem 1.1. *Let $F : D \subseteq X \longrightarrow X$ be a Fréchet-differentiable operator. Suppose that*

$$\|F(x_0)\| \leq d_1 \tag{1.5}$$
$$\|F'(x_0)^{-1}\| \leq d_2 \tag{1.6}$$
$$\|F'(x) - F'(y)\| \leq d_3\|x-y\| \text{ for each } x, y \in D \tag{1.7}$$
$$(a+b)d_3d_2d_1 < 2, \tag{1.8}$$
$$d_4d_1d_5^2 \leq \frac{1}{2} \tag{1.9}$$

and

$$U(x_0, u^* + d_6d_1) \subseteq D, \tag{1.10}$$

where $a > 0, b > 0, d_4 = d_3(1 + \frac{a+b}{d_5}), d_5 = \frac{2d_2}{2-(a+b)d_3d_2d_1}, d_6 = \max\{a, b\}$ *and*

$$u^* = \frac{1 - \sqrt{1 - 2d_4d_1d_5^2}}{d_4d_5}. \tag{1.11}$$

Define, scalar sequence $\{u_n\}$ *by*

$$u_0 = 0, u_{n+1} = u_n - \frac{q(u_n)}{q'(u_n)} \text{ for each } n = 0, 1, 2, \ldots \tag{1.12}$$

where

$$q(t) = \frac{d_4}{2}t^2 - \frac{t}{d_5} + d_1. \tag{1.13}$$

Then, method (1.3) converges to solution x^* of the equation $F(x) = 0$. starting at x_0, and $x_n, y_n, x^* \in \bar{U}(x_0, u^* + d_1d_6)$ for all $n = 0, 1, 2, \ldots$. Moreover, the following estimates hold

$$\|x_{n+1} - x_n\| \leq u_{n+1} - u_n$$

and

$$\|x_n - x^*\| \leq u^* - u_n.$$

Furthermore, the solution x^* is unique in $U(x_0, r) \cap D$, where $r = \frac{2}{d_3d_2} - (u^* + d_6d_1)$ provided that $d_3d_2(u^* + d_6d_1) < 2$.

In the present study we avoid the restrictive choice of the divided difference given by (1.5) and use more flexible conditions than (1.4)–(1.10) in our semilocal convergence analysis. This way, we expand the applicability of method (1.3).

The conditions are denoted by $(\mathcal{A})$:

- $(\mathcal{A}_1)$ $F : D^0 \subseteq X \longrightarrow X$ is Fréchet-differentiable and there exists a mapping $\delta F : D^0 \times D^0 \longrightarrow L(X)$ such that for some $x_0 \in D^0, F'(x_0)^{-1}, A_0^{-1} \in L(X)$ and there exist $a, b \in \mathbb{R}, K > 0, M_1 > 0, M_2 > 0, L_1 > 0, L_2 > 0, N > 0, N_0 > 0, N_1 > 0, \eta_0 > 0$ and $\eta_1 \geq 0$ such that for each $x, y, z \in D^0$
- $(\mathcal{A}_2)$ $\|F(x_0)\| \leq \eta_0, \ \|A_0^{-1}F(x_0)\| \leq \eta_1,$
- $(\mathcal{A}_3)$ $\|F'(x_0)^{-1}(\delta F(x, z) - F'(z))\| \leq K\|x - z\|,$
- $(\mathcal{A}_4)$ $\|F'(x_0)^{-1}(\delta F(x, y) - F'(z))\| \leq M_1\|x - z\| + M_2\|y - z\|,$
- $(\mathcal{A}_5)$ $\|F'(x_0)^{-1}(\delta F(x, y) - F'(x_0))\| \leq L_1\|x - x_0\| + L_2\|y - x_0\|,$
- $(\mathcal{A}_6)$ $\|\delta F(y_0, z_0)\| \leq N_0, \ \|\delta F(x, y) \leq N,$
- $(\mathcal{A}_7)$ $\|F'(x_0)\| \leq N_1,$

The rest of the paper is organized as follows. In Section 2 we present the convergence of a majorizing sequence and in Section 3 we present semilocal convergence analysis of method (1.1). Numerical examples are given in the last Section.

2. MAJORIZING SEQUENCES

We need some auxiliary results on majorizing sequences for method (1.1).

Lemma 2.1. *Let $\beta > 0, \gamma > 0, \delta > 0$ and $\eta > 0$ be given parameters. Denote by α the smallest root in the interval $(0, 1)$ of the polynomial p defined by*

$$p(t) = \delta\eta t^3 + (\gamma - \delta\eta)t^2 + \beta t - \beta. \tag{2.14}$$

Suppose that

$$\eta < \frac{2}{\gamma + \sqrt{\gamma + 4\delta}} \tag{2.15}$$

and

$$\frac{\beta\eta}{1 - (\gamma + \beta\eta)\eta} \leq \alpha < 1 - \gamma\eta. \tag{2.16}$$

Then, the scalar sequence $\{t_n\}$ defined by

$$t_0 = 0, t_1 = \eta, t_{n+2} = t_{n+1} + \frac{\beta(t_{n+1} - t_n)^2}{1 - (\gamma t_{n+1} + \delta(t_{n+1} - t_n)^2)} \text{ for each } n = 0, 1, 2, \ldots, \tag{2.17}$$

is well defined, nondecreasing, bounded from above by

$$t^{**} = \frac{\eta}{1 - \alpha} \tag{2.18}$$

and converges to its unique least upper bound denoted by t^* which satisfies

$$\eta \leq t^* \leq t^{**}. \tag{2.19}$$

Moreover, the following estimates hold

$$0 \leq t_{n+1} - t_n \leq \alpha^n(t_n - t_{n-1}) \leq \alpha^n(t_1 - t_0) \tag{2.20}$$

and

$$0 \leq t^* - t_n \leq \frac{\alpha^n}{1 - \alpha}\eta. \tag{2.21}$$

Proof. The polynomial p defined by (2.14) has roots in the interval $(0, 1)$. Indeed, we have that $p(0) = -\beta < 0$ and $p(1) = \gamma > 0$. Then, it follows from the intermediate value theorem that polynomial p has roots in the interval $(0, 1)$. Denote by α the smallest such root. Next, we shall show estimate (2.20) using mathematical induction. Set

$$\alpha_k = \frac{\beta(t_{k+1} - t_k)}{1 - (\gamma t_{k+1} + \delta(t_{k+1} - t_k)^2)}, \quad \text{for each } \ k = 0, 1, 2, \ldots. \tag{2.22}$$

Then, to show (2.20), we must have $\alpha_k \leq \alpha$. We get by (2.22) for $k = 0$, (2.17) and the left hand side inequality of (2.16) that $\alpha_0 \leq \alpha$ from which it follows that $0 \leq t_2 - t_1 \leq \alpha(t_1 - t_0)$ and $t_2 \leq \frac{1-\alpha^2}{1-\alpha}(t_1 - t_0) < t^{**}$. Hence, estimate (2.20) holds for $k = 0$. Suppose that $t_{k+1} - t_k \leq \alpha(t_k - t_{k-1}) \leq \alpha^k(t_1 - t_0)$ holds for all integers $k \leq n$. Evidently, (2.20) holds, if

$$\beta\alpha^k(t_1 - t_0) + \gamma\alpha\frac{1 - \alpha^{k+1}}{1 - \alpha}(t_1 - t_0) + \delta\alpha(\alpha^k(t_1 - t_0))^2 - \alpha \leq 0.$$

or since $2k \geq k + 1$,

$$\beta\alpha^{k-1}(t_1 - t_0) + \gamma\frac{1 - \alpha^{k+1}}{1 - \alpha}(t_1 - t_0) + \delta\alpha^{k+1}(t_1 - t_0)^2 - 1 \leq 0. \tag{2.23}$$

Estimate (2.23) motivates us to introduce recurrent functions f_k defined on the interval $[0, 1)$ by

$$f_k(t) = \delta\eta^2 t^{k+1} + \beta t^{k-1}\eta + \gamma\frac{1 - t^{k+1}}{1 - t}\eta - 1. \tag{2.24}$$

Then, (2.23) can be written as

$$f_k(\alpha) \leq 0. \tag{2.25}$$

We need a relationship between two consecutive functions f_k. In view of the definition of polynomial p and function f_k, we can write

$$\begin{aligned} f_{k+1}(t) &= f_k(t) + \delta t^{k+2}\eta^2 + \beta t^k \eta + \gamma \frac{1-t^{k+2}}{1-t}\eta - 1 \\ &\quad -\delta t^{k+1}\eta^2 - \beta t^{k-1}\eta - \gamma \frac{1-t^{k+1}}{1-t}\eta + 1 \\ &= f_k(t) + p(t)t^{k-1}\eta. \end{aligned} \tag{2.26}$$

In particular, we have by (2.26) that

$$f_{k+1}(\alpha) = f_k(\alpha), \ \text{ since } \ p(\alpha) = 0. \tag{2.27}$$

Define function f_∞ on the interval $[0, 1)$ by

$$f_\infty(t) = \lim_{t \longrightarrow \infty} f_k(t). \tag{2.28}$$

Then, we have by (2.24) and (2.28) that

$$f_\infty(t) = \frac{\gamma\eta}{1-t} - 1. \tag{2.29}$$

It follows from (2.27) and (2.29) that (2.25) holds, if $f_\infty(\alpha) \leq 0$, which is true by the right hand side inequality of (2.16). The induction for (2.25) (i.e., for (2.20)) is complete. Then, we get from (2.20) that $t_{k+2} \leq \frac{1-\alpha^{k+2}}{1-\alpha}\eta < t^{**}$. Hence, sequence $\{t_k\}$ is nondecreasing bounded from above by t^{**} and as such it converges to its unique least upper bound t^* which satisfies (2.19). Finally, let $m = 0, 1, \ldots$. Then, we get by (2.20) that

$$\begin{aligned} t_{k+m} - t_k &= (t_{k+m} - t_{k+m-1}) + \ldots + (t_{k+1} - t_k) \\ &\leq \alpha^{k+m-1}\eta + \ldots + \alpha^k \eta = \alpha^k \frac{1-\alpha^m}{1-\alpha}\eta. \end{aligned} \tag{2.30}$$

By letting $m \longrightarrow \infty$ in (2.30), we deduce (2.21). □

Lemma 2.2. *Let $\beta_0 > 0, \beta > 0, \gamma_0 > 0, \gamma > 0, \delta_0 > 0, \delta > 0$ and $\eta \geq 0$ be given parameters.Let α, p be as in Lemma 2.1. Define parameters s_0, s_1, s_2 by*

$$s_0 = 0, s_1 = \eta, \ s_2 = \eta + \frac{\beta_0\eta^2}{1-(\gamma_0\eta + \delta_0\eta^2)}. \tag{2.31}$$

Suppose that

$$\eta < \max\{\frac{2}{\gamma_0 + \sqrt{\gamma_0^2 + 4\delta_0}}, \frac{1}{\gamma}\}, \tag{2.32}$$

$$\gamma s_2 + \delta(s_2 - s_1)^2 < 1 \tag{2.33}$$

and

$$\bar{\alpha}_1 \leq \alpha < 1 - \frac{\gamma(s_2 - s_1)}{1-\gamma\eta}, \tag{2.34}$$

where

$$\bar{\alpha}_1 = \frac{\beta(s_2 - s_1)^2}{1-(\gamma s_2 + \delta(s_2 - s_1)^2)}. \tag{2.35}$$

Then, the scalar sequence $\{s_n\}$ defined by

$$s_{n+2} = s_{n+1} + \frac{\beta(s_{n+1} - s_n)^2}{1-(\gamma s_{n+1} + \delta(s_{n+1} - s_n)^2)} \ \text{ for each } n = 1, 2, \ldots \tag{2.36}$$

is well defined, nondecreasing, bounded from above by

$$s^{**} = s_1 + \frac{s_2 - s_1}{1 - \alpha} \tag{2.37}$$

and converges to its unique least upper bound s^* *which satisfies*

$$s_2 \leq s^* \leq s^{**}. \tag{2.38}$$

Moreover, the following estimates hold

$$0 \leq s_{n+1} - s_{n+1} \leq \alpha^n (s_2 - s_1) \tag{2.39}$$

and

$$0 \leq s^* - s_{n+1} \leq \frac{\alpha^n}{1 - \alpha}(s_2 - s_1). \tag{2.40}$$

Proof. Set

$$\bar{\alpha}_k = \frac{\beta(s_{k+1} - s_k)^2}{1 - (\gamma s_{k+1} + \delta(s_{k+1} - s_k)^2)} \tag{2.41}$$

for each $k = 1, 2, \ldots$. We must show that

$$\bar{\alpha}_k \leq \alpha. \tag{2.42}$$

Estimate (2.42) holds for $k = 1$ by the left hand side inequality of (2.34). Then, we have by (2.36) that

$$\begin{aligned} 0 &\leq s_3 - s_2 \leq \alpha(s_2 - s_1) \Longrightarrow s_3 \leq s_2 + \alpha(s_2 - s_1) \\ &\leq s_2 + (1 + \alpha)(s_2 - s_1) - (s_2 - s_1) \\ &= s_1 + \frac{1 - \alpha^2}{1 - \alpha}(s_2 - s_1) < s^{**}. \end{aligned} \tag{2.43}$$

Suppose that (2.42) holds, then

$$0 < s_{k+2} - s_{k+1} \leq \alpha^k (s_2 - s_1) \text{ and } s_{k+2} \leq s_1 + \frac{1 - \alpha^{k+1}}{1 - \alpha}(s_2 - s_1). \tag{2.44}$$

Estimate (2.42) shall be true if k is replaced by $k + 1$ provided that

$$\beta\alpha^k (s_2 - s_1) + \alpha\gamma[s_1 + \frac{1 - \alpha^{k+1}}{1 - \alpha}(s_2 - s_1)] + \alpha\delta(\alpha^k (s_2 - s_1))^2 - \alpha \leq 0. \tag{2.45}$$

Define recurrent functions $\bar{f}_k$ on the interval $[0, 1)$ by

$$\bar{f}_k = \beta(s_2 - s_1)t^k + \gamma t \frac{1 - t^{k+1}}{1 - t}(s_2 - s_1) + \delta t^{k+1}(s_2 - s_1)^2 - (1 - \gamma\eta)t.$$

Then, we have that

$$\bar{f}_{k+1}(t) = \bar{f}_k(t) + p(t)t^k (s_2 - s_1)$$

and

$$\bar{f}_{k+1}(\alpha) = \bar{f}_k(\alpha).$$

Define function $\bar{f}_\infty$ on the interval $[0, 1)$ by

$$\bar{f}_\infty(t) = \lim_{k \longrightarrow \infty} \bar{f}_k(t).$$

Then, we have that

$$\bar{f}_\infty(t) = [\frac{\gamma}{1 - t}(s_2 - s_1) - (1 - \gamma\eta)]t.$$

Estimate (2.45) certainly holds, if $\bar{f}_\infty(\alpha) \leq 0$, which is true by (2.32)–(2.35). $\square$

A simple inductive argument on sequences $\{t_n\}$ and $\{s_n\}$ leads to the following comparison result between these sequences.

Lemma 2.3. *Suppose that the hypotheses of Lemma 2.1, Lemma 2.2 and*

$$\beta_0 \leq \beta \text{ or } \gamma_0 \leq \gamma \text{ or } \delta_0 \leq \delta \tag{2.46}$$

hold. Then, the following estimates hold

$$s_0 = t_0, s_1 = t_1$$

$$s_n \leq t_n \text{ for each } n = 2, 3, \ldots \tag{2.47}$$

$$s_{n+1} - s_n \leq t_{n+1} - t_n \text{ for each } n = 1, 2, \ldots \tag{2.48}$$

and

$$s^* \leq t^*. \tag{2.49}$$

Moreover, if strict inequality holds in any of the inequalities in (2.46), then strict inequality holds in (2.47) and (2.48).

3. Semilocal convergence

We present the semilocal convergence analysis of method (1.1) in this section under the $(\mathcal{A})$ conditions. First, we use the sequence $\{t_n\}$ given in Lemma 2.1 as majorizing for method (1.1). The technique of proof for the next result is due to Kantorovich [10].

Theorem 3.2. *Suppose that: hypotheses $(\mathcal{A})$ and hypotheses of Lemma 2.1 hold with*

$$\begin{aligned} \eta &= \max\{\eta_0, \eta_1\}, \\ \beta &= K + (M_1|a| + M_2|b|)N, \\ \gamma &= L_1 + L_2, \\ \delta &= (L_1|a| + L_2|b|)N_1\beta, \ \rho = c\max\{\eta_0, \beta\eta^2\} \end{aligned} \tag{3.50}$$

where $c = \max\{|a|, |b|\}$ and

$$U_0 = U(x_0, t^* + \rho) \subseteq D. \tag{3.51}$$

Then, the sequence $\{x_n\}$ generated by method (1.1) is well defined, remains in $\bar{U}_0$ for each $n = 0, 1, 2, \ldots$ and converges to a solution x^* of the equation $F(x) = 0$ in $\bar{U}_0$. Moreover, the following estimates hold

$$\|x_{n+1} - x_n\| \leq t_{n+1} - t_n \tag{3.52}$$

and

$$\|x_n - x^*\| \leq t^* - t_n, \tag{3.53}$$

where the sequence $\{t_n\}$ is defined by (2.17) and $t^* = \lim_{n \longrightarrow \infty} t_n$. Furthermore, the solution x^* is unique in $U(x_0, r_0) \cap D$, where

$$r_0 = \frac{1}{L_1} - \frac{L_2}{L_1}(t^* + \rho),$$

if

$$L_2(t^* + \rho) < L_1. \tag{3.54}$$

Proof. We shall show estimate

$$\|x_{n+1} - x_n\| \leq t_{n+1} - t_n \tag{3.55}$$

holds, using mathematical induction. Estimate (3.55) holds for $k = 0$ by $(\mathcal{A}_2)$. We also have by $(\mathcal{A}_2)$

$$\|y_0 - x_0\| \leq \|x_0 - x_0\| + |a|\|F(x_0)\| \leq c\eta_0 < t^* + \rho,$$
$$\|z_0 - x_0\| \leq \|x_0 - x_0\| + |b|\|F(x_0)\| \leq c\eta_0 < t^* + \rho.$$

It follows that $y_0, z_0, x_1 \in U_0 \subset D$. Hence, x_2 is well defined by method (1.1) for $n = 0$. Let us note that condition $(\mathcal{A}_4)$ implies the following Lipschitz condition for F'

$$\|F'(x_0)^{-1}(F'(x) - F'(y))\| \leq (M_1 + M_2)\|x - y\| \text{ for each } x, y \in D^\circ. \tag{3.56}$$

Using the integral representation

$$F(x)-F(y)=\int_0^1 F'(y+\theta(x-y))d\theta(x-y), \tag{3.57}$$

we deduce by ($\mathcal{A}_4$) and ($\mathcal{A}_5$), respectively that

$$\|F'(x_0)^{-1}(F(x)-F(y)-F'(u)(x-y))\| \le (M_1(\|x-u\|+M_2\|y-u\|)\|x-y\| \tag{3.58}$$

and

$$\|F'(x_0)^{-1}(F(x)-F(y)-F'(y)(x-y))\| \le K\|x-y\|^2. \tag{3.59}$$

Then, using method (1.1) for $n=0$, we can write

$$F(x_1)=F(x_1)-F(x_0)-F'(x_0)(x_1-x_0)+(F'(x_0)-A_0)(x_1-x_0). \tag{3.60}$$

In view of ($\mathcal{A}_6$), (3.50), (3.58)–(3.60), we get in turn that

$$\begin{aligned}\|F'(x_0)^{-1}F(x_1)\| &\le \|F'(x_0)^{-1}(F(x_1)-F(x_0)-F'(x_0)(x_1-x_0))\| \\ &\quad +\|F'(x_0)^{-1}(A_0-F'(x_0))(x_1-x_0)\| \\ &\le K\|x_1-x_0\|^2+(M_1|a|+M_2|b|)\|F(x_0)\| \\ &\le K\|x_1-x_0\|^2+(M_1|a|+M_2|b|)\|A_0\|\|x_1-x_0\|^2 \\ &\le (K+(M_1|a|+M_2|b|)N)\|x_1-x_0\|^2 \\ &= \beta\|x_1-x_0\|^2 \le \beta(t_1-t_0)^2. \end{aligned} \tag{3.61}$$

We also have that

$$\begin{aligned}\|y_1-x_0\| &\le \|x_1-x_0\|+|a|\|F(x_1)\| \\ &\le t_1-t_0+c\|F'(x_0)\|\|F'(x_0)^{-1}F(x_1)\| \\ &\le t_1+c\eta_0\beta(t_1-t_0)^2 \\ &= t^*+c\eta_0\beta\eta^2 \le t^*+\rho \end{aligned}$$

and

$$\begin{aligned}\|z_1-x_0\| &\le \|x_1-x_0\|+|b|\|F(x_1)\| \\ &\le t_1-t_0+c\|F'(x_0)\|\|F'(x_0)^{-1}F(x_1)\| \le t^*+\rho. \end{aligned}$$

Next, we shall show the existence of A_1^{-1} to define x_2. We have by ($\mathcal{A}_5$), (3.50), (3.61) and the proof of Lemma 2.1 that

$$\begin{aligned}\|F'(x_0)^{-1}(A_1-F'(x_0))\| &\le L_1\|y_1-x_0\|+L_2\|z_1-x_0\| \\ &\le (L_1+L_2)\|x_1-x_0\|+(L_1|a|+L_2|b|)\|F(x_1)\| \\ &\le (L_1+L_2)(t_1-t_0)+(L_1|a|+L_2|b|)N_1\beta\|x_1-x_0\|^2 \\ &\le \gamma t_1+\delta(t_1-t_0)^2<1. \end{aligned} \tag{3.62}$$

It follows from (3.62) and the Banach Lemma on invertible operators [5, 10, 11] that A_1^{-1} exists and

$$\|A_1^{-1}F'(x_0)\| \le \frac{1}{1-(\gamma+\delta\|x_1-x_0\|)\|x_1-x_0\|} \le \frac{1}{1-(\gamma t_1+\delta(t_1-t_0)^2)}. \tag{3.63}$$

Consequently,

$$\begin{aligned}\|x_2-x_1\| &\le \|A_1^{-1}F'(x_0)\|\|F'(x_0)^{-1}F(x_1)\| \\ &\le \frac{\beta\|x_1-x_0\|^2}{1-(\gamma+\delta(\|x_1-x_0\|)\|x_1-x_0\|} \\ &\le \frac{\beta(t_1-t_0)^2}{1-(\gamma t_1+\delta(t_1-t_0)^2)}=t_2-t_1, \end{aligned} \tag{3.64}$$

which shows (3.55) for $k=1$. We also have that

$$\|x_2-x_0\|\le\|x_2-x_1\|+\|x_1-x_0\|\le t_2-t_0<t^*<t^*+\rho,$$

so $x_2\in U_0$. Hence, y_2 and z_2 are well defined. Next, from the approximation

$$F(x_{k+1})=F(x_{k+1})-F(x_k)-F'(x_k)(x_{k+1}-x_k)+(F'(x_k)-A_k)(x_{k+1}-x_k) \tag{3.65}$$

as in (3.60) and (3.61), we get that

$$\|F'(x_0)^{-1}F(x_{k+1})\|\le\beta\|x_{k+1}-x_k\|^2\le\beta(t_{k+1}-t_k)^2. \tag{3.66}$$

In addition, we get that

$$\begin{aligned}\|y_{k+1}-x_0\| &\le \|x_{k+1}-x_0\|+|a|\|F(x_{k+1})\|\\ &\le t_{k+1}-t_0+c\|F'(x_0)\|\|F'(x_0)^{-1}F(x_{k+1})\|\\ &\le t_{k+1}+cN_1\beta(t_{k+1}-t_k)^2\le t^*+cN_1\beta\eta^2\\ &\le t^*+\rho\end{aligned}$$

and similarly

$$\|z_{k+1}-x_0\|\le t^*+\rho.$$

That is, $y_{k+1},z_{k+1}\in U_0$. We must show that A_{k+1}^{-1} exists. Using ($\mathcal{A}_5$), (3.50) and the induction hypotheses as in (3.62) we get that

$$\begin{aligned}\|F'(x_0)^{-1}(A_{k+1}-F'(x_0))\| &\le L_1\|y_{k+1}-x_0\|+L_2\|z_{k+1}-x_0\|\\ &\le (L_1+L_2)\|x_{k+1}-x_0\|\\ &\quad +(L_1|a|+L_2|b|)\|F'(x_0)F'(x_0)^{-1}F(x_{k+1})\|\\ &\le \gamma\|x_{k+1}-x_0\|+\delta\|x_{k+1}-x_k\|^2\\ &\le \gamma(t_{k+1}-t_0)+\delta(t_{k+1}-t_k)^2\\ &= \gamma t_{k+1}+\delta(t_{k+1}-t_k)^2<1.\end{aligned} \tag{3.67}$$

It follows from (3.67) that

$$\begin{aligned}\|A_{k+1}^{-1}F'(x_0)\| &\le \frac{1}{1-(\gamma\|x_{k+1}-x_0\|+\delta\|x_{k+1}-x_k\|^2)}\\ &\le \frac{1}{1-(\gamma t_{k+1}+\delta(t_{k+1}-t_k)^2)}.\end{aligned} \tag{3.68}$$

Hence, using method (1.1), (2.17), (3.66) and (3.68), we get that

$$\begin{aligned}\|x_{k+2}-x_{k+1}\| &\le \|A_{k+1}^{-1}F'(x_0)\|\|F'(x_0)^{-1}F(x_{k+1})\|\\ &\le \frac{\beta(t_{k+1}-t_k)^2}{1-(\gamma t_{k+1}+\delta(t_{k+1}-t_k)^2)}=t_{k+2}-t_{k+1},\end{aligned} \tag{3.69}$$

which completes the induction for (3.55). We also have that

$$\begin{aligned}\|x_{k+2}-x_0\| &\le \|x_{k+2}-x_{k+1}\|+\|x_{k+1}-x_0\|\\ &\le t_{k+2}-t_{k+1}+t_{k+1}-t_0\\ &\le t^*-t_0<t^*+\rho.\end{aligned} \tag{3.70}$$

That is, $x_{k+2}\in U_0$. By Lemma 2.1 sequence $\{t_n\}$ is complete. If follows from (3.55) that sequence $\{x_n\}$ is also complete in a Banach space X and as such it converges to some $x^*\in\bar{U}_0$ (since $\bar{U}_0$ is a closed set). By letting $k\longrightarrow\infty$ in (3.66) we deduce that $F(x^*)=0$. Estimate (3.53) follows from (3.52) by using standard majorization techniques [5, 8–11].

Finally, to show the uniqueness part, let $y^* \in U(x_0, r_0) \cap D$ with $F(y^*) = 0$. Define linear operator $Q = \int_0^1 F'(y^* + \theta(x^* - y^*))d\theta$. Using ($\mathcal{A}_5$) and (3.54) we get that

$$\begin{aligned} \|F'(x_0)^{-1}(Q - F'(x_0))\| &\leq L_1\|y^* - x_0\| + L_2\|x^* - x_0\| \\ &\leq L_1 r_0 + L_2(t^* + \rho) < 1. \end{aligned} \tag{3.71}$$

That is, Q^{-1} exists. Then, in view of the identity $0 = F(y^*) - F(x^*) = Q(y^* - x^*)$, we conclude that $x^* = y^*$. □

Remark 3.1. (a) If $a = 0$ and $b = 1$, Theorem 3.1 extends and improves the result of Steffensen's method in [1, 2] (see also the numerical examples). Moreover, if $a = b = 0$, we obtain the result for Newton's method by t^{**} given in closed form by (2.18).

(b) The limit point t^* can be replaced by t^{**} given in closed form by (2.18) in Theorem 3.2.

(c) In view of (3.61), we can arrive instead (using ($\mathcal{A}_5$) instead of ($\mathcal{A}_3$)) at

$$\|x_2 - x_1\| \leq \beta_0\|x_1 - x_0\|^2 \tag{3.72}$$

where $\beta_0 = L_1 + (M_1|a| + M_2|b|)$. Notice also that in view of ($\mathcal{A}_5$) there exist $\bar{L}_1 \leq L_1$ and $\bar{L}_2 \leq L_2$ such that

$$\|F'(x_0)^{-1}(\delta F(y_1, z_1) - F'(x_0))\| \leq \bar{L}_1\|y_1 - x_0\| + \bar{L}_2\|z_1 - x_0\|.$$

Notice that $y_1 = x_1 + aF(x_1) = x_0 - A_0^{-1}F(x_0) + aF(x_0 - A_0^{-1}F(x_0))$ and $z_1 = x_1 + bF(x_1) = x_0 - A_0^{-1}F(x_0) + bF(x_0 - A_0^{-1}F(x_0))$ which depend on the initial data. Set $\gamma_0 = \bar{L}_1 + \bar{L}_2$ and $\delta_0 = (\bar{L}_1|a| + \bar{L}_2|b|)N_0\beta_0$.

Notice that $N_0 \leq N, L_1 \leq K, \beta_0 \leq \beta, \gamma_0 \leq \gamma$ and $\delta_0 \leq \delta$. These observations motivate us to introduce second majorizing sequence $\{s_n\}$ defined by (2.36).

(d) The results obtained here can be improved further as follows: Suppose that ($\mathcal{A}_5$) holds but the rest of the ($\mathcal{A}$) conditions hold for $x, y, z \in D_1 := D^\circ \cap U(x_0, \frac{1}{L_1+L_2})$. Notice also that the iterates x_n lie in D_1 which is a more precise location than D°. Then, since $D_1 \subseteq D^\circ$, we have that $\bar{K} \leq K, \bar{M}_1 \leq M_1, \bar{M}_2 \leq M_2, \bar{N}_0 \leq N_0$ and $\bar{N} \leq N$, where the "bar" constants correspond to the new ($\mathcal{A}$) conditions. Then, in case any of the preceding inequalities is strict, we obtain weaker sufficient convergence conditions and tighter error bounds on the distances $\|x_{n+1} - x^*\|$.

(e) If there exists $\bar{t}^* \in [t^*, \frac{1}{L_1+L_2})$, then the limit point x^* is the only solution of equation $F(x) = 0$ in $D^* := D^\circ \cap \bar{U}(x_0, \bar{t}^*)$. Indeed, let $y^* \in D^*$ with $F(y^*) = 0$. Define $Q = \delta F(x^*, y^*)$, where δF is defined by (1.5). Then, we have by ($\mathcal{A}_5$) that

$$\|F'(x_0)^{-1}(Q - F'(x_0))\| \leq L_1\|x^* - x_0\| + L_2\|y^* - x_0\| \leq (L_1 + L_2)\bar{t}^* < 1,$$

so $Q^{-1} \in L(X)$. Then, from the identity $0 = F(x^*) - F(y^*) = Q(x^* - y^*)$, we conclude that $x^* = y^*$.

Hence, we arrive at:

Theorem 3.3. *Suppose that the hypotheses ($\mathcal{A}$) and the hypotheses of Lemma 2.2 hold. Then, the conclusions of Theorem 3.2 hold with $s^*, \{s_n\}$ replacing $t^*, \{t_n\}$, respectively.*

Remark 3.2. It follows from Lemma 2.3 that sequence $\{s_n\}$ is more precise than $\{t_n\}$. However, the conditions to verify in Lemma 2.2 are more than in Lemma 2.1 (see also the numerical examples).

Concerning the local convergence of method (1.1), let us introduce the conditions ($\mathcal{A}^*$):

($\mathcal{A}_1^*$) $F : D^0 \subseteq X \longrightarrow X$ is Fréchet-differentiable and there exists $x^* \in D$ and a mapping $\delta F : D^0 \times D^0 \longrightarrow L(X)$ such that $F(x^*) = 0, F'(x^*)^{-1} \in L(X)$.

($\mathcal{A}_2^*$) $\|F(x^*)^{-1}(\delta F(x,z) - F'(z))\| \leq K\|x - z\|$

($\mathcal{A}_3^*$) $\|F'(x^*)^{-1}(\delta F(x,y) - F'(z))\| \leq M_1\|x - z\| + M_2\|y - z\|$,

($\mathcal{A}_4^*$) $\|F'(x^*)^{-1}(\delta F(x,y) - F'(x^*))\| \leq L_1\|x - x^*\| + L_2\|y - x^*\|$,

($\mathcal{A}_5^*$) $\|\delta F(x,y)\| \leq N$

($\mathcal{A}_6^*$) $\|F'(x^*)\| \leq N_1$,

($\mathcal{A}_7^*$) $\|F'(x^*)^{-1}F'(x)\| \leq M$

($\mathcal{A}_8^*$) $\bar{U}(x^*, R^*) \subseteq D$, where

$$R^* = (1 + cN_1M)r^*$$

and

$$r^* = \frac{1}{K + \gamma + (L_1|a| + L_2|b|)N_1M + 2N(M_1|a| + M_2|b|)}.$$

Next, we present the local convergence analysis of method (1.1) using the preceding notation.

Theorem 3.4. *Suppose that the condition ($\mathcal{A}^*$) hold. Then, the sequence $\{x_n\}$ generated for $x_0 \in U(x^*, R^*) - \{x^*\}$ by method (1.1) is well defined, remains in $U(x^*, R^*)$ for each $n = 0, 1, 2, \ldots$ and converges to x^*. Moreover, the following estimates hold*

$$\|x_{n+1} - x^*\| \leq e_n\|x_n - x^*\| < \|x_n - x^*\| < R^* \tag{3.73}$$

where

$$e_n = \frac{[K\|x_n - x^*\| + N(M_1|a| + M_2|b|)\|x_{n+1} - x_n\|]\|x_n - x^*\|}{1 - (\gamma + (L_1|a| + L_2|b|)N_1M)\|x_n - x^*\|}.$$

Proof. We use the proof of Theorem 3.2, induction and the approximation

$$\begin{aligned} x_{n+1} - x^* &= -(A_n^{-1}F'(x^*)) \\ &\quad \times[F'(x^*)^{-1}(F(x_n) - F(x^*) - F'(x_n)(x_n - x^*)) + (F'(x_n) - A_n)(x_n - x^*))] \end{aligned}$$

and the estimate

$$\begin{aligned} \|F'(x^*)^{-1}F(x_n)\| &= \|F'(x^*)^{-1}(F(x_n) - F(x^*))\| \\ &= \|\int_0^1 F'(x^*)^{-1}F'(x^* + \theta(x_n - x^*))(x_n - x^*)d\theta\| \end{aligned}$$

to arrive at

$$\begin{aligned} & \|F'(x^*)^{-1}(F(x_k) - F(x^*) - F'(x_k)(x_k - x^*)) + F'(x^*)^{-1}(F'(x_k) - A_k)(x_k - x^*)\| \\ \leq & \|F'(x^*)^{-1}(F(x_k) - F(x^*) - F'(x_k)(x_k - x^*))\| \\ & + \|F'(x^*)^{-1}(F'(x_k) - A_k)(x_k - x^*)\| \\ \leq & K\|x_k - x^*\|^2 + (M_1|a| + M_2|b|)N\|x_{k+1} - x_k\|\|x_k - x^*\| \\ \leq & g(R^*)\|x_k - x^*\| \end{aligned}$$

and

$$\|A_k^{-1}F'(x^*)\| \leq \frac{1}{1 - (\gamma + (L_1|a| + L_2|b|)N_1M)\|x_k - x^*\|},$$

where

$$g(t) = [K + 2N(M_1|a| + M_2|b|)]t,$$

which leads to estimate (3.73). Then, it follows from

$$\|x_{k+1} - x^*\| \leq c\|x_k - x^*\| < R^*$$

that $\lim_{k \longrightarrow \infty} x_k = x^*$ and $x_{k+1} \in U(x^*, R^*)$, where $c = g_1(R^*) \in [0, 1)$ and $g_1(t) = \frac{g(t)}{1-(\gamma+(L_1|a|+L_2|b|)N_1M)t}$. □

Remark 3.3. Remarks similar to Remark 3.1 can now follows for the local convergence case in an analogous way.

4. Numerical examples

We present one example in this section. We define for simplicity $[x, y; F] = \frac{1}{2}(F'(x) + F'(y))$ for each $x, y \in D$ with $x \neq y$ and $[x, x; F] = F'(x)$ for each $x \in D$.

Example 4.1. Let $X = Y = \mathbb{R}^3, D = \bar{U}(0, 1), x^* = (0, 0, 0)^T$. Define function F on D for $w = (x, y, z)^T$ by

$$F(w) = (e^x - 1, \frac{e-1}{2}y^2 + y, z)^T.$$

Then, the Fréchet-derivative is given by

$$F'(v) = \begin{bmatrix} e^x & 0 & 0 \\ 0 & (e-1)y+1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

We obtain for $a = b = 0.5$, $K = M_1 = M_2 = N = M = \frac{e}{2}, L_1 = L_2 = \frac{e-1}{2}$ and $N_1 = 1$. Then,

$$r^* = 0.1607.$$

If we use Remark 2.6 (d), we have $\bar{K} = \bar{M}_1 = \bar{M}_2 = \bar{N} = \bar{M} = \frac{e^{\frac{1}{e-1}}}{2}, \bar{L}_1 = \bar{L}_2 = \frac{e-1}{2}$ and $\bar{N}_1 = 1$. Then,

$$\bar{r}^* = 0.3063.$$

Moreover, with the approaches in [1–8], $\bar{K} = \bar{M}_1 = \bar{M}_2 = \bar{M} = \bar{L}_1 = \bar{L}_2 = \frac{e}{2}$ and $\bar{N}_1 = 1$. Then, we get

$$\tilde{r} = 0.1449.$$

Notice that $\tilde{r} < r^* < \bar{r}^*$ as expected, since that "bar" constants are smaller (see also Remark 3.1 (d) or Remark 3.3). This example justifies the claims made in the abstract of this study.

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