CREAT. MATH. INFORM.
Volume **26** (2017), No. 2,
Pages 153 - 162

Online version at `https://creative-mathematics.cunbm.utcluj.ro/`
Print Edition: ISSN 1584 - 286X; Online Edition: ISSN 1843 - 441X
DOI: https://doi.org/10.37193/CMI.2017.02.05

A proof of Garfunkel inequality and of some related results in inner product spaces

DAN ŞTEFAN MARINESCU and MIHAI MONEA

ABSTRACT. In this paper, we will present a inner product space proof of a geometric inequality proposed by J. Garfunkel in American Mathematical Monthly [Garfunkel, J., *Problem 2505*, American Mathematical Monthly, **81** (1974), No. 11] and consider some other similar results.

1. INTRODUCTION

In [3], Garfunkel proposed the following problem:

Problem 1.1. *Let a, b, c be the sides of a triangle ABC, and let m_a, m_b, m_c be the medians to sides a, b, c, respectively. Extend the medians so as to meet the circumcircle again, and these chords be M_a, M_b, M_c, respectively. Then:*
a) $M_a + M_b + M_c \geqslant \frac{4}{3}(m_a + m_b + m_c)$;
b) $M_a + M_b + M_c \geqslant \frac{2\sqrt{3}}{3}(a + b + c)$.

The readers can find two solutions to this problem in [2]. These geometrical inequalities raised the interest of more mathematicians. For example, the second solution is due Erdös and Klamkim. The references of this paper contains more results connected with Problem 1.1 as solutions, extensions or generalizations. Garfunkel (see [4]) himself, proposed another similar inequality.

Problem 1.2. *Triangle ABC is inscribed in a circle. The medians of the triangle intersect at G and are extended to the circle to points D, E and F. Then:*

$$GA + GB + GC \leqslant GD + GE + GF. \tag{1.1}$$

In fact, the relation $GA = \frac{2}{3}m_a$ and the analogues show that inequality (1.1) is equivalent with the previous. Boente ([1]) presented a proof to Problem 1.2., which includes the following generalization.

Proposition 1.3. *Let P_1,P_2, .., P_m be points of an n-sphere S, and G their center of gravity. For any $i \in \{1, 2, ..., m\}$ the extensions of P_iG intersect S at Q_i. Then:*

$$\sum_{i=1}^{n} P_iG \leqslant \sum_{i=1}^{n} Q_iG.$$

Tsintsifas ([10]) completed the inequality (1.1) with another two inequalities. These results are included in the following problem.

Received: 18.01.2017. In revised form: 29.05.2017. Accepted: 05.06.2017
2010 *Mathematics Subject Classification.* 51M16, 46C15, 26D05.
Key words and phrases. *Inner product space, Garfunkel inequality*.
Corresponding author: Mihai Monea; mihaimonea@yahoo.com

Problem 1.4. *Let G be the centroid of a triangle ABC and and suppose that AG, BG, CG meet the circumcircle of the triangle again in A', B' , C', respectively. Then:*

a) $GA' + GB' + GC' \geqslant GC + GB + GC$;

b) $\frac{GA}{GA'} + \frac{GB}{GB'} + \frac{GC}{GC'} = 3$;

c) $GA' \cdot GB' \cdot GC' \geqslant GA \cdot GB \cdot GC$.

In 1985, Klamkin moved this problem to a higher level. He presented a generalization for a n-simplex from $\mathbb{R}^n$ (see [5]).

Proposition 1.5. *The medians of a n-dimensional simplex $A_0A_1A_2...A_n$ from $\mathbb{R}^n$ intersect at the centroid G and are extended to meet the circumsphere again, in the points $B_0, B_1, B_2, ..., B_n$, respectively.*

a) *Prove:* $GA_0 + GA_1 + GA_2 + ... + GA_n \leqslant GB_0 + GB_1 + GB_2 + ... + GB_n$;

b) *Prove:* $\frac{GA_0}{GB_0} + \frac{GA_1}{GB_1} + ... + \frac{GA_n}{GB_n} = n + 1$;

c) *Prove:* $GA_0 \cdot GA_1 \cdot GA_2 \cdot ... \cdot GA_n \leqslant GB_0 \cdot GB_1 \cdot GB_2 \cdot ... \cdot GB_n$.

Supplementary, he proposed an open question.

Open Question 1.6. *The medians of an n-dimensional simplex $A_0A_1A_2...A_n$ from $\mathbb{R}^n$ intersect at the centroid G and are extended to meet the circumsphere again in the points $B_0, B_1, B_2, ..., B_n$, respectively. Determine all other points P such that*

$$PA_0 + PA_1 + PA_2 + ... + PA_n \leqslant PB_0 + PB_1 + PB_2 + ... + PB_n.$$

We found a partial answer in [11]. The authors of this paper investigated all the previous results and concluded that these inequalities are more general.

In this context, the aim of this paper is to prove the inequality 1.1 in inner product spaces. We completed with new inequalities related to 1.1. The main results are represented by the Proposition 3.1 from Section 3. Moreover, we present a similar open question for inner product space. We offer only a partial answer. We start with some preliminary results which are included in the second section.

2. Some preliminary results

In this section, X denote a real or complex inner product space. First, we recall a useful result.

Proposition 2.1. *Let $n \in \mathbb{N}$, $n \geqslant 2$ and $x_1, x_2, ..., x_n, y \in X$. For any $\alpha_1, \alpha_2, ..., \alpha_n \in \mathbb{R}$ with $\alpha_1 + \alpha_2 + ... + \alpha_n = 1$, we have:*

$$\left\| y - \sum_{k=1}^{n} \alpha_k x_k \right\|^2 = \sum_{k=1}^{n} \alpha_k \left\| y - x_k \right\|^2 - \sum_{1 \leqslant k < j \leqslant n} \alpha_i \alpha_j \left\| x_k - x_j \right\|^2. \tag{2.2}$$

In [7], it showed that this identity characterizes an inner product space. We apply 2.2 to prove the following proposition.

Proposition 2.2. *Let $x_0, x_1, x_2 \in X$ and $r \in [0, \infty)$ such that $\|x_1 - x_0\| = \|x_2 - x_0\| = r$. For any $\alpha \in [0, 1]$, we denote $w = \alpha x_1 + (1 - \alpha) x_2$. Then*

$$\|w - x_1\| \cdot \|w - x_2\| = r^2 - \|w - x_0\|^2 .$$

Proof. Tha case $n = 2$ in 2.2 led us to

$$\begin{aligned} \|w - x_0\|^2 &= \|x_0 - \alpha x_1 - (1 - \alpha) x_2\|^2 \\ &= \alpha \|x_0 - x_1\|^2 + (1 - \alpha) \|x_0 - x_2\|^2 - \alpha (1 - \alpha) \|x_1 - x_2\|^2 \\ &= r^2 - \alpha (1 - \alpha) \|x_1 - x_2\|^2 . \end{aligned}$$

It follows

$$\begin{aligned} \|w - x_1\| \cdot \|w - x_2\| &= \|\alpha x_1 + (1 - \alpha) x_2 - x_2\| \cdot \|\alpha x_1 + (1 - \alpha) x_2 - x_1\| \\ &= \alpha \|x_1 - x_2\| \cdot (1 - \alpha) \|x_1 - x_2\| \\ &= \alpha (1 - \alpha) \|x_1 - x_2\|^2 , \end{aligned}$$

which concludes our proof. □

The previous proposition represent a version for an inner product space of a *"point power theorem"*.

Proposition 2.3. *Let $x_0 \in X$ and $r \in [0, \infty)$. Then, for any $x_1, x \in X$ with $\|x_1 - x_0\| = r$ and $\|x - x_0\| < r$, it exists just one pair $(y_1; \alpha)$ with $y_1 \in X$ and $\alpha \in (0, 1)$ so that $x = \alpha y_1 + (1 - \alpha) x_1$ and $\|y_1 - x_0\| = r$.*

Proof. First, we want to find $t \in \mathbb{R} \setminus \{0,\ 1\}$ such that $y_1 = t x_1 + (1 - t) x$ and $\|y_1 - x_0\| = r$. Then

$$y_1 - x_0 = t x_1 + (1 - t) x - x_0$$
$$\Rightarrow \|y_1 - x_0\|^2 = \|t x_1 + (1 - t) x - x_0\|^2 .$$

By using 2.2, we have

$$t \|x_1 - x_0\|^2 + (1 - t) \|x - x_0\|^2 - t (1 - t) \|x_1 - x\|^2 = r^2$$
$$\Rightarrow t (1 - t) \|x_1 - x\|^2 = (1 - t) \|x - x_0\|^2 - (1 - t) r^2$$
$$\Rightarrow t = \frac{\|x - x_0\|^2 - r^2}{\|x_1 - x\|^2} < 0,$$

so $t \in \mathbb{R} \setminus \{0,\ 1\}$. Then

$$y = \frac{\|x - x_0\|^2 - r^2}{\|x_1 - x\|^2} x_1 + \frac{r^2 + \|x_1 - x\|^2 - \|x - x_0\|^2}{\|x_1 - x\|^2} x.$$

It follows that

$$\left(r^2 + \|x_1 - x\|^2 - \|x - x_0\|^2\right) x = \left(r^2 - \|x - x_0\|^2\right) x_1 + \|x_1 - x\|^2 y_1$$

$$\Rightarrow x = \frac{\|x_1 - x\|^2}{r^2 + \|x_1 - x\|^2 - \|x - x_0\|^2} y_1 + \frac{r^2 - \|x - x_0\|^2}{r^2 + \|x_1 - x\|^2 - \|x - x_0\|^2} x_1 .$$

If we denote

$$a = \frac{\|x_1 - x\|^2}{r^2 + \|x_1 - x\|^2 - \|x - x_0\|^2},$$

we have $x = a y_1 + (1 - a) x_1$ and $a \in (0, 1)$.

Now, let (y_2, β) another pair in the same conditions as the pair $(y_1; \alpha)$. Then

$$\alpha y_1 + (1-\alpha) x_1 = \beta y_2 + (1-\beta) x_1$$

$$\begin{aligned} &\Rightarrow \quad (\alpha - \beta) x_1 = \alpha y_1 - \beta y_2 \\ &\Rightarrow \quad (\alpha - \beta) x_1 - (\alpha - \beta) x_0 = \alpha y_1 - \beta y_2 - (\alpha - \beta) x_0. \end{aligned}$$

If we suppose that $\alpha \neq \beta$, then

$$x_1 - x_0 = \frac{\alpha}{\alpha - \beta} y_1 + \frac{-\beta}{\alpha - \beta} y_2 - x_0$$

$$\Rightarrow r^2 = \|x_1 - x_0\|^2 = \left\| \frac{\alpha}{\alpha - \beta} y_1 + \frac{-\beta}{\alpha - \beta} y_2 - x_0 \right\|^2 .$$

Proposition 2.1 led us to

$$\begin{aligned} r^2 \quad &= \quad \frac{\alpha}{\alpha - \beta} \|y_1 - x_0\|^2 - \frac{\beta}{\alpha - \beta} \|y_2 - x_0\|^2 + \frac{\alpha\beta}{(\alpha - \beta)^2} \|y_1 - y_2\|^2 \\ &\Rightarrow \quad r^2 = \frac{\alpha}{\alpha - \beta} r^2 - \frac{\beta}{\alpha - \beta} r^2 + \frac{\alpha\beta}{(\alpha - \beta)^2} \|y_1 - y_2\|^2 \\ &\Rightarrow \quad \|y_1 - y_2\| = 0 \\ &\Rightarrow \quad y_1 = y_2. \end{aligned}$$

The equality $(\alpha - \beta) x_1 = \alpha y_1 - \beta y_2$ becames $(\alpha - \beta) x_1 = (\alpha - \beta) y_1$, also $x_1 = y_1$ which is false. The assumption $\alpha \neq \beta$ is not true and we have $\alpha = \beta$. From $(\alpha - \beta) x_1 = \alpha y_1 - \beta y_2$, we obtain again $y_1 = y_2$ and the pairs $(y_1; \alpha)$ and (y_2, β) are identical. □

Remarks. In fact, y_1 represents the "*intersect*" of $\overline{B}(x_0, r)$ with "*lines*" determined by x_1 and x.

Proposition 2.4. *Let $n \in \mathbb{N}$, $n \geqslant 2$, $r \in (0, \infty)$ and $x_0 \in X$. Let $x_1, x_2, ..., x_n \in X$, at least two distinct, such that $\|x_1 - x_0\| = \|x_2 - x_0\| = ... = \|x_n - x_0\| = r$. For any $\alpha_1, \alpha_2, ..., \alpha_n \in (0,1)$ with $\alpha_1 + \alpha_2 + ... + \alpha_n = 1$, we have*

$$\|\alpha_1 x_1 + \alpha_2 x_2 + ... + \alpha_n x_n - x_0\| < r.$$

Proof. We are using 2.2 and we obtain

$$\begin{aligned} \|\alpha_1 x_1 + \alpha_2 x_2 + ... + \alpha_n x_n - x_0\|^2 \quad &= \quad \left\| x_0 - \sum_{k=1}^{n} \alpha_k x_k \right\|^2 \\ &= \quad \sum_{k=1}^{n} \alpha_k \|x_0 - x_k\|^2 - \sum_{1 \leqslant k < j \leqslant n} \alpha_k \alpha_j \|x_k - x_j\|^2 \\ &< \quad \sum_{k=1}^{n} \alpha_k \|x_0 - x_k\|^2 \\ &= \quad \sum_{k=1}^{n} \alpha_k r^2 \end{aligned}$$

and the conclusion follows now. □

We conclude this section by recalling another useful result (see page 76 from [9]).

Proposition 2.5. **(Chebyshev Inequality)** *Let $n \in \mathbb{N}$, $n \geqslant 2$. Let $a_1, a_2, ..., a_n \in \mathbb{R}$, $b_1, b_2, ..., b_n \in \mathbb{R}$ and $p_1, p_2, ..., p_n \in [0,1]$ such that $a_1 \leqslant a_2 \leqslant ... \leqslant \alpha_n$, $b_1 \geqslant b_2 \geqslant ... \geqslant b_n$ and $p_1 + p_2 + ... + p_n = 1$. Then*

$$\sum_{k=1}^{n} p_k a_k b_k \leqslant \left(\sum_{k=1}^{n} p_k a_k\right)\left(\sum_{k=1}^{n} p_k b_k\right).$$

3. THE MAIN RESULTS

For the results from this section, we establish the following general conditions and notations. Let $n \in \mathbb{N}$, $n \geqslant 2$, $r \in (0, \infty)$ and $x_0 \in X$. Let $x_1, x_2, ..., x_n \in X$, at least two distinct, such that

$$\|x_1 - x_0\| = \|x_2 - x_0\| = ... = \|x_n - x_0\| = r.$$

Let $\alpha_1, \alpha_2, ..., \alpha_n \in (0,1)$ with $\alpha_1 + \alpha_2 + ... + \alpha_n = 1$. Denote

$$x = \alpha_1 x_1 + \alpha_2 x_2 + ... + \alpha_n x_n.$$

Proposition 2.4 shows that $\|x - x_0\| < r$. For any $k \in \{1, 2, .., n\}$ we denote $y_k \in X$, the elements defined by Proposition 2.3 for x_k and x.

Proposition 3.1. *The following assertions are true:*

a) $\sum\limits_{k=1}^{n} \alpha_k \frac{\|x_k - x\|}{\|y_k - x\|} = 1$;

b) $\sum\limits_{k=1}^{n} \alpha_k \|x_k - x\| \leqslant \sum\limits_{k=1}^{n} \alpha_k \|y_k - x\|$;

c) $\prod\limits_{k=1}^{n} \|y_k - x\|^{\alpha_k} \geqslant \prod\limits_{k=1}^{n} \|x_k - x\|^{\alpha_k}$;

d) $\|x_k - y_k\| \cdot \|x_k - x\| = \|x_k - x\|^2 + \sum\limits_{1 \leqslant i < j \leqslant n} \alpha_i \alpha_j \|x_i - x_j\|^2$, *for any* $k \in \{1, 2, ..., n\}$;

e) $\|x_k - y_k\| \geqslant 2\sqrt{\sum\limits_{1 \leqslant i < j \leqslant n} \alpha_i \alpha_j \|x_i - x_j\|^2}$, *for any* $k \in \{1, 2, ..., n\}$;

f) $\sum\limits_{k=1}^{n} \|x_k - y_k\| \geqslant \frac{2\sqrt{2n}}{\sqrt{n-1}} \cdot \sum\limits_{1 \leqslant i < j \leqslant n} \sqrt{\alpha_i \alpha_j} \|x_i - x_j\|$.

Proof. $a)$ We have

$$\sum_{k=1}^{n} \alpha_k \frac{\|x_k - x\|}{\|y_k - x\|} = \sum_{k=1}^{n} \alpha_k \frac{\|x_k - x\|^2}{\|y_k - x\| \cdot \|x_k - x\|}.$$

Proposition 2.2 goes to

$$\sum_{k=1}^{n} \alpha_k \frac{\|x_k - x\|}{\|y_k - x\|} = \frac{1}{r^2 - \|x - x_0\|^2} \sum_{k=1}^{n} \alpha_k \|x_k - x\|^2.$$

Now, we apply 2.2 to $y = x$ and $y = x_0$. Then

$$\begin{aligned}\|x_0 - x\|^2 &= \left\| x_0 - \sum_{k=1}^{n} \alpha_k x_k \right\|^2 \\ &= r^2 - \sum_{1 \leqslant k < j \leqslant n} \alpha_k \alpha_j \|x_k - x_j\|^2\end{aligned}$$

and

$$0 = \sum_{k=1}^{n} \alpha_k \|x - x_k\|^2 - \sum_{1 \leqslant k < j \leqslant n} \alpha_k \alpha_j \|x_k - x_j\|^2.$$

We obtain

$$\sum_{k=1}^{n} \alpha_k \|x - x_k\|^2 = r^2 - \|x - x_0\|^2$$

and

$$\sum_{k=1}^{n} \alpha_k \frac{\|x_k - x\|}{\|y_k - x\|} = \frac{1}{r^2 - \|x - x_0\|^2} \sum_{k=1}^{n} \alpha_k \|x_k - x\|^2 = 1.$$

$b)$ For any $k \in \{1, 2, ..., n\}$, we have

$$\frac{\|x_k - x\|}{\|y_k - x\|} = \frac{r^2 - \|x - x_0\|^2}{\|y_k - x\|^2}.$$

Then, we can apply Proposition 2.5 for the systems $\left(\frac{\|x_1 - x\|}{\|y_1 - x\|}, \frac{\|x_2 - x\|}{\|y_2 - x\|}, ..., \frac{\|x_n - x\|}{\|y_n - x\|} \right)$ and $(\|y_1 - x\|, \|y_2 - x\|, ..., \|y_n - x\|)$ and we obtain

$$\begin{aligned}\sum_{k=1}^{n} \alpha_k \|x_k - x\| &= \sum_{k=1}^{n} \alpha_k \frac{\|x_k - x\|}{\|y_k - x\|} \cdot \|y_k - y\| \\ &\leqslant \sum_{k=1}^{n} \alpha_k \frac{\|x_k - x\|}{\|y_k - x\|} \cdot \sum_{k=1}^{n} \alpha_k \cdot \|y_k - y\|.\end{aligned}$$

Using the previous assertion, we have

$$\sum_{k=1}^{n} \alpha_k \|x_k - x\| \leqslant \sum_{k=1}^{n} \alpha_k \|y_k - x\|.$$

c) By applying the generalized means inequality, we obtain

$$\sum_{k=1}^{n} \alpha_k \frac{\|x_k - x\|}{\|y_k - x\|} \geqslant \prod_{k=1}^{n} \left(\frac{\|x_k - x\|}{\|y_k - x\|} \right)^{\alpha_k}.$$

The assertion $a)$ give now the conclusion.

$d)$ From the definition of y_k, we have

$$\|x_k - y_k\| = \|x_k - x\| + \|x - y_k\|,$$

for any $k \in \{1, 2, .., n\}$. Then

$$\|x_k - y_k\| \cdot \|x_k - x\| = \|x_k - x\|^2 + \|x - y_k\| \cdot \|x_k - x\|.$$

By using Propositions 2.1 and 2.2 , we obtain

$$\begin{aligned}\|x_k - y_k\| \cdot \|x_k - x\| &= \|x_k - x\|^2 + r^2 - \|x - x_0\|^2 \\ &= \|x_k - x\|^2 + \sum_{1 \leqslant i < j \leqslant n} \alpha_i \alpha_j \|x_i - x_j\|^2.\end{aligned}$$

$e)$ For any $k \in \{1,2,...,n\}$, we have $\|x_k - x\| > 0$ due to Proposition 2.4. Then

$$\begin{aligned}\|x_{\mathrm{k}} - y_k\| &\geqslant \|x_k - x\| + \frac{1}{\|x_k - x\|} \sum_{1 \leqslant i<j \leqslant n} \alpha_i \alpha_j \|x_i - x_j\|^2 \\ &\geqslant 2\sqrt{\sum_{1 \leqslant i<j \leqslant n} \alpha_i \alpha_j \|x_i - x_j\|^2}.\end{aligned}$$

$f)$ Cauchy's inequality give us

$$\left(\sum_{1 \leqslant i<j \leqslant n} \sqrt{\alpha_i \alpha_j} \|x_i - x_j\| \right)^2 \leqslant \frac{n(n-1)}{2} \left(\sum_{1 \leqslant i<j \leqslant n} \alpha_i \alpha_j \|x_i - x_j\|^2 \right).$$

Then

$$\begin{aligned}\sum_{k=1}^{n} \|x_k - y_k\| &\geqslant \sum_{k=1}^{n} 2\sqrt{\sum_{1 \leqslant i<j \leqslant n} \alpha_i \alpha_j \|x_i - x_j\|^2} \\ &\geqslant \frac{2\sqrt{2n}}{\sqrt{n-1}} \cdot \sum_{1 \leqslant i<j \leqslant n} \sqrt{\alpha_i \alpha_j} \|x_i - x_j\|\end{aligned}$$

and the proofs are complete. □

Particularly, for $\alpha_1 = \alpha_2 = ... = \alpha_n = \dfrac{1}{n}$, the Proposition 3.1 becomes:

Corollary 3.2. *The following assertions are true:*

a) $\sum\limits_{k=1}^{n} \dfrac{\|x_k - x\|}{\|y_k - x\|} = n$;

b) $\sum\limits_{k=1}^{n} \|x_k - x\| \leqslant \sum\limits_{k=1}^{n} \|y_k - x\|$;

c) $\prod\limits_{k=1}^{n} \|y_k - x\| \geqslant \prod\limits_{k=1}^{n} \|x_k - x\|$;

d) $\|x_k - y_k\| \cdot \|x_k - x\| = \|x_k - x\|^2 + \dfrac{1}{n^2} \sum\limits_{1 \leqslant i<j \leqslant n} \|x_i - x_j\|^2$, *for any* $k \in \{1,2,...,n\}$;

e) $\|x_k - y_k\| \geqslant \dfrac{2}{n}\sqrt{\sum\limits_{1 \leqslant i<j \leqslant n} \|x_i - x_j\|^2}$, *for any* $k \in \{1,2,...,n\}$;

f) $\sum\limits_{k=1}^{n} \|x_k - y_k\| \geqslant \dfrac{2\sqrt{2}}{\sqrt{n(n-1)}} \cdot \sum\limits_{1 \leqslant i<j \leqslant n} \|x_i - x_j\|$.

Remarks.

1. If we apply the assertion $b)$ from Corollary 3.2 for $\mathbb{R}^n$, we obtain the inequality from Proposition 1.3. More, this proof is not similar with the Boente's proof from [1].
2. If we apply Corollary 3.2 to a n-simplex from $\mathbb{R}^n$, we recover the results from Proposition 1.5.

Finally, we give a partial answer to Open Problem 1.6. With the conditions established to the beginning of this section, we denote

$$\overline{x} = \frac{1}{n}(x_1 + x_2 + ... + x_n).$$

Proposition 3.3. *If* $\|\overline{x}-x_0\|^2 \geqslant \|x_0-x\|^2+\|\overline{x}-x\|^2$ *then*

$$\sum_{k=1}^{n}\|x_k-x\| \leqslant \sum_{k=1}^{n}\|y_k-x\|.$$

Proof. From 2.2, we obtain

$$\begin{aligned}\|\overline{x}-x_0\|^2 &= \frac{1}{n}\sum_{k=1}^{n}\|x_k-x_0\|^2-\frac{1}{n^2}\sum_{1\leqslant i<j\leqslant n}\|x_i-x_j\|^2\\ &= r^2-\frac{1}{n^2}\sum_{1\leqslant i<j\leqslant n}\|x_i-x_j\|^2.\end{aligned}$$

Then

$$r^2-\frac{1}{n^2}\sum_{1\leqslant i<j\leqslant n}\|x_i-x_j\|^2 \geqslant \|x_0-x\|^2+\|\overline{x}-x\|^2$$

$$\Rightarrow r^2-\|x_0-x\|^2 \geqslant \frac{1}{n^2}\sum_{1\leqslant i<j\leqslant n}\|x_i-x_j\|^2+\|\overline{x}-x\|^2.$$

From Proposition 2.2, we have

$$r^2-\|x_0-x\|^2=\|x_k-x\|\cdot\|y_k-x\|,$$

for any $k\in\{1,2,..,n\}$ and 2.2 give us

$$\|\overline{x}-x\|^2=\frac{1}{n}\sum_{k=1}^{n}\|x_k-x\|-\frac{1}{n^2}\sum_{1\leqslant i<j\leqslant n}\|x_i-x_j\|^2.$$

Then

$$\|x_k-x\|\cdot\|y_k-x\| \geqslant \frac{1}{n}\sum_{k=1}^{n}\|x_k-x\|^2,$$

for any $k\in\{1,2,..,n\}$. Hence

$$\frac{1}{n}\sum_{k=1}^{n}\|x_k-x\|^2 \geqslant \left(\frac{1}{n}\sum_{k=1}^{n}\|x_k-x\|\right)^2,$$

so that

$$n^2\|x_k-x\|\cdot\|y_k-x\| \geqslant \left(\sum_{k=1}^{n}\|x_k-x\|\right)^2,$$

for any $k\in\{1,2,..,n\}$. We obtain

$$n^2\cdot\|y_k-x\| \geqslant \frac{1}{\|x_k-x\|}\left(\sum_{k=1}^{n}\|x_k-x\|\right)^2.$$

Now we use Cauchy inequality and we have

$$\begin{aligned} n^2 \cdot \sum_{k=1}^{n} \|y_k - x\| &\geqslant \left(\sum_{k=1}^{n} \|x_k - x\|\right)^2 \left(\sum_{k=1}^{n} \frac{1}{\|x_k - x\|}\right) \\ &\geqslant \left(\sum_{k=1}^{n} \|x_k - x\|\right)^2 \cdot \frac{n^2}{\sum\limits_{k=1}^{n} \|x_k - x\|} \\ &= n^2 \cdot \sum_{k=1}^{n} \|x_k - x\|, \end{aligned}$$

which concludes our proof. □

Final Remarks.

1. In fact, the equality

$$\|\overline{x} - x_0\|^2 \geqslant \|x_0 - x\|^2 + \|\overline{x} - x\|^2 ,$$

from Proposition 3.3 is equivalent with

$$\begin{aligned} & \|x_0 - x\|^2 + \|\overline{x} - x\|^2 - \frac{1}{2} \|\overline{x} - x_0\|^2 \leqslant \frac{1}{2} \|\overline{x} - x_0\|^2 \\ \Leftrightarrow\ & \frac{1}{2} \|x_0 - x\|^2 + \frac{1}{2} \|\overline{x} - x\|^2 - \frac{1}{4} \|\overline{x} - x_0\|^2 \leqslant \frac{1}{4} \|\overline{x} - x_0\|^2 \\ \Leftrightarrow\ & \left\|x - \frac{x_0 + \overline{x}}{2}\right\|^2 \leqslant \frac{1}{4} \|\overline{x} - x_0\|^2 \\ \Leftrightarrow\ & \left\|x - \frac{x_0 + \overline{x}}{2}\right\| \leqslant \frac{1}{2} \|\overline{x} - x_0\|^2 \\ \Leftrightarrow\ & x \in \overline{B}\left(\frac{x_0 + \overline{x}}{2},\ \frac{1}{2} \|x_0 - \overline{x}\|\right). \end{aligned}$$

It means that x is situated in the *"closed ball"* with the center $\dfrac{x_0 + \overline{x}}{2}$ and the *"diameter"* $\|x_0 - \overline{x}\|$.

2. If we apply Proposition 3.2 for a n-simplex from $\mathbb{R}^n$ and taking into account by the previous remark, we recover the answer proposed to Open Question 1.6 by G. Tsintsifas (see [11]).

3. An elementary version of the results from Section 3 of this paper can be found in [8].

REFERENCES

[1] Boente, P., *Solution of Problem 2959*, American Mathematical Monthly, **92** (1985), No. 5

[2] Erdos, P. and Klamkin, M. S., *Solution of Problem 250*, American Mathematical Monthly, **83** (1976), No. 1

[3] Garfunkel, J., *Problem 2505*, American Mathematical Monthly, **81** (1974), No. 11

[4] Garfunkel, J., *Problem 2959*, American Mathematical Monthly, **89** (1982), No. 7

[5] Klamkin, M. S., *Solution to Problem 723*, Crux Mathematicorum, **9** (1983), No. 5

[6] Klamkin, M. S., *Problem 1086*, Crux Mathematicorum, **11** (1985), No. 9

[7] Marinescu, D. Ş., Monea, M., Opincariu, M. and Stroe, M., *Some Equivalent Characterizations Of Inner Product Spaces And Their Consequences, Mathematical Inequalities and Applications*, **18** (2015), No. 4, 1375–1385

[8] Marinescu, D. Ş. and Monea, M., *Asupra inegalităţii lui Garfunkel*, Gazeta Matematică B, **CXIX** (2014), nr. 12

[9] Steele, J. M.,*The Cauchy–Schwarz Master Class: An Introduction to the Art of Mathematical Inequalities*, Cambridge University Press, 2004

[10] Tsintsifas, G., *Problem 723*, Crux Mathematicorum, **8** (1982), No. 3

[11] Tsintsifas, G., *Solution to Problem 1086*, Crux Mathematicorum, **13** (1987), No. 5

NATIONAL COLLEGE "IANCU DE HUNEDOARA"
HUNEDOARA, ROMANIA
Email address: marinescuds@gmail.com

NATIONAL COLLEGE "DECEBAL",
DEVA, ROMANIA
Email address: mihaimonea@yahoo.com