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Fixed point theorems and convergence theorems for monotone (α,β)-nonexpansive mappings in ordered Banach spaces

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ABSTRACT. In this paper, we introduce the notion of a monotone (α,β)-nonexpansive mapping in an ordered Banach space E with the partial order $\leq$ and prove some existence theorems of fixed points of a monotone (α,β)-nonexpansive mapping in a uniformly convex ordered Banach space. Also, we prove some weak and strong convergence theorems of Ishikawa type iteration under the control condition

$$\limsup_{n\to\infty} s_n(1-s_n)>0 \quad and \quad \liminf_{n\to\infty} s_n(1-s_n)>0.$$

Finally, we give an numerical example to illustrate the main result in this paper.

1. INTRODUCTION

Let T be a mapping with domain $D(T)$ and range $R(T)$ in an ordered Banach space E with the partial order $\leq$. A mapping $T : D(T) \to R(T)$ is said to be *monotone* if $Tx \leq Ty$ for all $x, y \in D(T)$ with $x \leq y$ and *monotone nonexpansive* if T is monotone and

$$\|Tx - Ty\| \leq \|x - y\|,$$

for all $x, y \in D(T)$ with $x \leq y$.

In 2010, Aoyama et al. [1] introduced a class of a λ-hybrid mapping, that is, a mapping $T : D(T) \to R(T)$ is called *λ-hybrid mapping* in a Hilbert space H if

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + 2(1-\lambda)\langle x - Tx, y - Ty\rangle,$$

for all $x, y \in D(T)$. They proved a fixed point theorem and an ergodic theorem for such a mapping. Clearly, a nonexpansive mapping is a 1-hybrid mapping. In 2011, Aoyama and Kohsaka [2] also introduced the concept of an α-nonexpansive mapping, that is, a mapping $T : D(T) \to R(T)$ is said to be *$\alpha-$nonexpansive* if $\alpha < 1$ and

$$\|Tx - Ty\|^2 \leq \alpha\|Tx - y\| + \alpha\|Ty - x\| + (1-2\alpha)\|x - y\|,$$

for all $x, y \in D(T)$. Obviously, a nonexpansive mapping is 0-nonexpansive and λ-hybrid mapping is $\frac{1-\lambda}{2-\lambda}$-nonexpansive if $\lambda < 2$ in a Hilbert space H (for more details, see [2]).

Recently, in 2015, Dehaish and Khamsi [3] proved some weak convergence theorems of Mann's iteration for finding some order fixed points of monotone nonexpansive mappings in uniformly convex ordered Banach spaces as follows:

Theorem DK1. *Let K be a nonempty closed convex and bounded subset of an ordered Banach space E. Let $T : K \to K$ be a monotone nonexpansive mapping. Assume that X is uniformly*

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convex and there exist $z \in F(T)$ and $x_1 \in K$ such that x_1 and z are comparable. Then we have

$$\lim_{n\to\infty} \|x_n - Tx_n\| = 0,$$

where $\{x_n\}$ is Mann's iteration generated by

$$x_{n+1} = \beta_n Tx_n + (1-\beta_n)x_n,$$

for each $n \geq 1$, where $\beta_n \in [a,b]$ for some $a > 0$ and $b < 1$.

Theorem DK2. *K be a nonempty closed convex and bounded subset of an ordered Banach space E. Let $T : K \to K$ be a monotone nonexpansive mapping. Let $x_1 \in K$ be such that x_1 and $T(x_1)$ are comparable. Let $\{x_n\}$ be Mann's iteration defined by generated by*

$$x_{n+1} = \beta_n Tx_n + (1-\beta_n)x_n,$$

for each $n \geq 1$, where $\beta_n \in [a,b]$ for some $a > 0$ and $b < 1$. Then we have

$$\lim_{n\to\infty} \|x_n - Tx_n\| = 0.$$

Theorem DK3. *Let E be an ordered Banach space which satisfies Opial's weakly condition. Let K be a nonempty weakly compact convex subset of E. Let $T : K \to K$ be a monotone nonexpansive mapping. Assume that there exists $x_1 \in K$ such that x_1 and Tx_1 are comparable. Let $\{x_n\}$ be Mann's iteration defined by generated by*

$$x_{n+1} = \beta_n Tx_n + (1-\beta_n)x_n,$$

for each $n \geq 1$, where $\beta_n \in [a,b]$ for some $a > 0$ and $b < 1$. Then $\{x_n\}$ converges weakly to a fixed point z of T, i.e., $T(z) = z$. Moreover, z and x_1 are comparable.

Note that, in Theorems DK1, DK2, DK3 of Dehaish and Khamsi, they gave the control condition $\{t_n\}$ in $[a,b]$ with $a > 0$ and $b < 1$, but we cannot apply the following control condition $\{\beta_n\}$ in their results: for each $n \geq 1$,

$$\beta_n = \frac{1}{n+1}.$$

Thus, to improve the results mentioned above, in 2016, Song et al. [9] we consider Mann's iteration $\{x_n\}$ for a monotone nonexpansive mapping $T : C \to C$ defined by

$$x_{n+1} = \beta_n x_n + (1-\beta_n)Tx_n,$$

for each $n \geq 1$, where $\{\beta_n\}$ in $(0,1)$ satisfies the following condition:

$$\sum_{n=1}^{\infty} \beta_n(1-\beta_n) = \infty.$$

Clearly, this control condition $\{\beta_n\}$ contains $\beta_n = \frac{1}{n+1}$ as a special case.

Recently, in 2016, Song et al. [10] considered the convergence of Mann's iteration for a monotone α-nonexpansive mapping T in an ordered Banach space E.

Theorem SPKC1. *Let K be a nonempty closed convex subset of an ordered Banach space $(E,\leq)$ and $T : K \to K$ be a monotone α-nonexpansive mapping. Assume that the sequence $\{x_n\}$ defined by the Mann iteration with $x_1 \leq Tx_1$ (or $Tx_1 \leq x_1$) and $F_{\geq}(T) \neq \emptyset$ (or $F_{\leq}(T) \neq \emptyset$). Then we have*

(1) *the sequence $\{x_n\}$ is bounded;*

(2) *$\|x_{n+1} - p\| \leq \|x_n - p\|$ and the limit $\lim_{n\to\infty} \|x_n - p\|$ exists for all $p \in F_{\geq}(T)$ (or $p \in F_{\leq}(T)$);*

(3) *$\liminf_{n\to\infty} \|x_n - Tx_n\| = 0$ provided $\limsup_{n\to\infty} \beta_n(1-\beta_n) > 0$;*

(4) *$\lim_{n\to\infty} \|x_n - Tx_n\| = 0$ provided $\liminf_{n\to\infty} \beta_n(1-\beta_n) > 0$.*

Theorem SPKC2. *Let K be a nonempty closed convex subset of a uniformly convex ordered Banach space $(E, \leq)$ and $T : K \to K$ be a monotone α-nonexpansive mapping. Assume that E satisfies Opial's condition and the sequence $\{x_n\}$ is defined by Mann's iteration with $x_1 \leq Tx_1$ (or $Tx_1 \leq x_1$). If $F_{\geq}(T) \neq \emptyset$ (or $F_{\leq}(T) \neq \emptyset$) and $\liminf_{n\to\infty} \beta_n(1-\beta_n) > 0$, then the sequence $\{x_n\}$ converges weakly to a fixed point z of T.*

Theorem SPKC3. *Let K be a nonempty compact and closed convex subset of a uniformly convex ordered Banach space $(E, \leq)$ and $T : K \to K$ be a monotone α-nonexpansive mapping. Assume that the sequence $\{x_n\}$ is defined by Mann's iteration with $Tx_1 \leq x_1$. If either $\liminf_{n\to\infty} \beta_n(1-\beta_n) > 0$ or $\limsup_{n\to\infty} \beta_n(1-\beta_n) > 0$, then the sequence $\{x_n\}$ converges strongly to a fixed point $y \in F_{\leq}(T)$.*

Motivated by the results mentioned above, in this paper, first, we introduce the concept of a monotone (α, β)-nonexpansive mapping in ordered Banach spaces. Second, we show the existence of a fixed point of the proposed mapping in ordered Banach spaces. Third, we proves some strong and weak convergence theorems of Ishikawa type iteration for a monotone (α, β)-nonexpansive mapping in ordered Banach spaces under the condition

$$\limsup_{n\to\infty} s_n(1-s_n) > 0, \quad \liminf_{n\to\infty} s_n(1-s_n) > 0.$$

Finally, we give a numerical example to illustrate the main result in this paper.

2. Preliminaries

Let P be a closed convex cone of a real Banach space E. A *partial order* "$\leq$" with respect to P in E is defined as follows:

$$x \leq y \ (x < y) \text{ if and only if } \ y - x \in P \ (y - x \in \mathring{P}),$$

for all $x, y \in E$, where $\mathring{P}$ is the interior of P.

Throughout this paper, let E be a Banach space with the norm $\|\cdot\|$ and the partial order $\leq$. Let $F(T) = \{x \in H : Tx = x\}$ denote the set of all fixed points of a mapping T. Also, we assume that the order intervals are closed and convex. Recall that an order interval is any of the subsets

$$[x, \to) = \{p \in E; x \leq p\} \text{ or } (\leftarrow, x] = \{p \in E; p \leq x\}$$

for any $a \in C$. An *order interval* $[x, y]$ for all $x, y \in E$ is given by

$$[x, y] = \{z \in E : x \leq z \leq y\} = [x, \to) \cap (\leftarrow, y]. \tag{2.1}$$

Then the convexity of the order interval $[x, y]$ implies that

$$x \leq tx + (1-t)y \leq y, \tag{2.2}$$

for all $x, y \in E$ with $x \leq y$. A Banach space E is said to be:

(1) *strictly convex* if $\|\frac{x+y}{2}\| < 1$ for all $x, y \in E$ with $\|x\| = \|y\| = 1$ and $x \neq y$;
(2) *uniformly convex* if, for all $\varepsilon \in (0, 2]$, there exists $\delta > 0$ such that $\frac{\|x+y\|}{2} < 1 - \delta$ for all $x, y \in E$ with $\|x\| = \|y\| = 1$ and $\|x - y\| \geq \varepsilon$.

The following inequality was shown by Xu [12] in a uniformly convex Banach space E, which is known as *Xu's inequality.*

Lemma 2.1. [12] *For any real numbers $q > 1$ and $r > 0$, a Banach space E is uniformly convex if and only if there exists a continuous strictly increasing convex function $g : [0, +\infty) \to [0, +\infty)$ with $g(0) = 0$ such that*

$$\|tx + (1-t)y\|^q \leq t\|x\|^q + (1-t)\|y\|^q - \omega(q, t)g(\|x - y\|), \tag{2.3}$$

for all $x, y \in B_r(0) = \{x \in E; \|x\| \le r\}$ *and* $t \in [0,1]$*, where* $\omega(q,t) = t^q(1-t) + t(1-t)^q$*. In particular, take* $q = 2$ *and* $t = \frac{1}{2}$*,*

$$\left\|\frac{x+y}{2}\right\|^2 \le \frac{1}{2}\|x\|^2 + \frac{1}{2}\|y\|^2 - \frac{1}{4}g(\|x-y\|). \tag{2.4}$$

Lemma 2.2. [11] *Let* K *be a nonempty closed convex subset of a reflexive Banach space* E*. Assume that* $\rho : K \to R$ *is a proper convex lower semi-continuous and coercive function. Then the function* ρ *attains its minimum on* K*, that is, there exists* $x \in K$ *such that*

$$\rho(x) = \inf_{y \in K} \rho(y).$$

Lemma 2.3. [8] *A Banach space* E *is said to satisfy Opial's condition if, whenever any sequence* $\{x_n\}$ *in* E *converges weakly to a point* x*,*

$$\limsup_{n\to\infty} ||x_n - x|| < \limsup_{n\to\infty} ||x_n - y||$$

for any $y \in E$ *such that* $y \neq x$*.*

Definition 2.1. *Let* K *be a nonempty closed subset of an ordered Banach space* $(E, \le)$*. A mapping* $T : K \to E$ *is said to be :*

(1) *monotone* α*-nonexpansive if* T *is monotone and, for some* $\alpha < 1$*,*

$$\|Tx - Ty\|^2 \le \alpha\|Tx - y\|^2 + \alpha\|Ty - x\|^2 + (1-2\alpha)\|x-y\|^2,$$

for all $x, y \in K$ *with* $x \le y$*, by* [10, Lemma 2.2] *which is equivalent to*

$$\begin{aligned}\|Tx - Ty\|^2 \le \|x-y\|^2 &+ \frac{2\alpha}{1-\alpha}\|Tx - x\|^2 \\ &+ \frac{2|\alpha|}{1-\alpha}\|Tx - x\|\left[\|x-y\| + \|Tx - Ty\|\right].\end{aligned} \tag{2.5}$$

(2) *monotone quasi-nonexpansive if* T *is monotone,* $F(T) \neq \emptyset$ *and* $\|Tx - p\| \le \|x - p\|$ *for all* $p \in F(T)$ *and* $x \in K$ *with* $x \le p$ *or* $p \le x$*.*

3. Main results

3.1. **Monotone** (α, β)**-nonexpansive mappings.**

In this section, we define the notion of a monotone (α,β)-nonexpansive mapping and introduce a lemma in a uniformly convex Banach space $(E, \le)$ as follows:

Definition 3.2. Let K be a nonempty closed subset of an ordered Banach space $(E, \le)$. A mapping $T : K \to K$ is said to be *monotone* (α,β)*-nonexpansive* if T is monotone and

$$\|Tx - Ty\|^2 \le \alpha\|Tx - y\|^2 + \beta\|Ty - x\|^2 + (1-(\alpha+\beta))\|x-y\|^2, \tag{3.6}$$

for all $x, y \in K$ with $x \le y$ and α, $\beta < 1$.

Remark 3.1. (1) If $\alpha = \beta$, then α-nonexpansive implies (α,β)-nonexpansive mapping and converse is true;

(2) Every nonexpansive mapping is a 0-nonexpansive mapping and $(0,0)$-nonexpansive mapping;

(3) Every α-nonexpansive and (α,β)-nonexpansive mappings with $F(T) \neq \emptyset$ are a quasi-nonexpansive mapping.

Now, we introduce a mapping T which is a monotone (α,β)-nonexpansive mapping, but not a monotone α-nonexpansive mapping as follows:

Example 3.1. Let $T : [0,1] \to [0,1]$ be defined by

$$Tx = \begin{cases} 0.25 & if \quad x \neq 1, \\ 0.5 & if \quad x = 1. \end{cases}$$

Next, we will show that T is a monotone $(0.8, 0.2)$-nonexpansive mapping where $\|x-y\|$ is defined by $|x-y|$. The proof is divided into two parts.
First, we prove that T is monotone. The proof is divided into four cases.

Case 1. If $x = y = 1$ then $Tx = 0.5 = Ty$,

Case 2. if $x \neq 1$ and $y \neq 1$ then $Tx = 0.25 = Ty$,

Case 3. if $x \neq 1$ and $y = 1$ then $Tx = 0.25 < 0.5 = Ty$,

Case 4. if $y \neq 1$ and $x = 1$ then $Ty = 0.25 < 0.5 = Tx$.

Therefore T is monotone.
Second, we prove that T is monotone $(0.8, 0.2)$-nonexpansive. We divide the proof into four cases.

Cases1. $x = y = 1$. We have

$$\|Tx - Ty\|^2 = 0 \leq 0.225 = 0.8\|0.5 - 1\|^2 + 0.2\|0.5 - 1\|^2$$

Case 2. $x \neq 1, y \neq 1$. Then,

$$\|Tx - Ty\|^2 = 0 \leq 0.8\|0.25 - y\|^2 + 0.2\|0.25 - x\|^2$$

Case 3. $x = 1, y \neq 1$. We get

$$\begin{aligned} \|Tx - Ty\|^2 = 0.0625 &\leq 0.8\|0.5 - y\|^2 + 0.2\|0.25 - 1\|^2 \\ &= 0.1125 + 0.8\|0.5 - y\|^2 \end{aligned}$$

Case 4. $x \neq 1, y = 1$. So,

$$\begin{aligned} \|Tx - Ty\|^2 = 0.0625 &\leq 0.8\|0.25 - 1\|^2 + 0.2\|0.5 - x\|^2 \\ &= 0.45 + 0.2\|0.5 - x\|^2 \end{aligned}$$

From these four cases, the mapping T is is satisfied (3.6).
Hence, T is monotone $(0.8, 0.2)$-nonexpansive mapping. However, T is not a monotone 0.8-nonexpansive mapping. Note that if $x = 1, y = 0 \in [0,1]$ and $\alpha = \beta = 0.8$, we have

$$\|Tx - Ty\|^2 = 0.0625 \nleq 0.05 = 0.8\|0.5 - 0\|^2 + 0.8\|0.25 - 1\|^2 - 0.6\|1 - 0\|^2.$$

This show that T is not a monotone 0.8-nonexpansive mapping.

Before proving the main results, we need the following:

Lemma 3.4. *Let K be a nonempty closed convex subset of an ordered Banach space $(E, \leq)$ and $T : K \to K$ be a monotone (α, β)-nonexpansive mapping. Then we have:*

(1) *T is monotone quasi-nonexpansive;*

(2) *for all $x, y \in K$ and $\alpha, \beta < 1$ with $x \leq y$,*

$$\begin{aligned} \|Tx - Ty\|^2 \leq \|x - y\|^2 &+ \frac{\alpha + \beta}{1 - \beta}\|Tx - x\|^2 \\ &+ \frac{2}{1 - \beta}\|Tx - x\|\big[|\alpha|\|x - y\| + |\beta|\|Tx - Ty\|\big]. \end{aligned} \tag{3.7}$$

Proof. (1) It follows that

$$\begin{aligned} \|Tx - p\|^2 &= \|Tx - Tp\|^2 \\ &\leq \alpha\|Tx - p\|^2 + \beta\|Tp - x\|^2 + (1 - (\alpha + \beta))\|x - p\|^2 \\ &= \alpha\|Tx - p\|^2 + (1 - \alpha)\|x - p\|^2 \end{aligned}$$

and so $\|Tx - p\|^2 \leq \|x - p\|^2$.

(2) From Definition 3.2, we consider the following cases:
(a) If $0 \leq \alpha, \beta < 1$, then we have

$$\begin{aligned}
\|Tx - Ty\|^2 &\leq \alpha\|Tx - y\|^2 + \beta\|Ty - x\|^2 + (1 - (\alpha + \beta))\|x - y\|^2 \\
&\leq \alpha\big[\|Tx - x\| + \|x - y\|\big]^2 + \beta\big[\|Ty - Tx\| + \|Tx - x\|\big]^2 \\
&\quad + (1 - (\alpha + \beta))\|x - y\|^2 \\
&= \alpha\|Tx - x\|^2 + 2\alpha\|Tx - x\|\|x - y\| + \alpha\|x - y\|^2 + \beta\|Ty - Tx\|^2 \\
&\quad + 2\beta\|Ty - Tx\|\|Tx - x\| + \beta\|Tx - x\|^2 + (1 - (\alpha + \beta))\|x - y\|^2 \\
&= (\alpha + \beta)\|Tx - x\|^2 + (1 - \beta)\|x - y\|^2 + \beta\|Ty - Tx\|^2 \\
&\quad + 2\|Tx - x\|\big[\alpha\|x - y\| + \beta\|Tx - Ty\|\big]
\end{aligned}$$

and so

$$\begin{aligned}
\|Tx - Ty\|^2 \leq \|x - y\|^2 &+ \frac{\alpha + \beta}{1 - \beta}\|Tx - x\|^2 \\
&+ \frac{2}{1 - \beta}\|Tx - x\|\big[\alpha\|x - y\| + \beta\|Tx - Ty\|\big].
\end{aligned}$$

(b) If $0 \leq \alpha < 1$ and $\beta < 0$, then we have

$$\begin{aligned}
\|Tx - Ty\|^2 &\leq \alpha\|Tx - y\|^2 + \beta\|Ty - x\|^2 + (1 - (\alpha + \beta))\|x - y\|^2 \\
&\leq \alpha\big[\|Tx - x\| + \|x - y\|\big]^2 + \beta\big[\|Ty - Tx\| - \|Tx - x\|\big]^2 \\
&\quad + (1 - (\alpha + \beta))\|x - y\|^2 \\
&= \alpha\|Tx - x\|^2 + 2\alpha\|Tx - x\|\|x - y\| + \alpha\|x - y\|^2 + \beta\|Ty - Tx\|^2 \\
&\quad - 2\beta\|Ty - Tx\|\|Tx - x\| + \beta\|Tx - x\|^2 + (1 - (\alpha + \beta))\|x - y\|^2 \\
&= (\alpha + \beta)\|Tx - x\|^2 + (1 - \beta)\|x - y\|^2 + \beta\|Ty - Tx\|^2 \\
&\quad + 2\|Tx - x\|\big[\alpha\|x - y\| - \beta\|Tx - Ty\|\big]
\end{aligned}$$

and so

$$\begin{aligned}
\|Tx - Ty\|^2 \leq \|x - y\|^2 &+ \frac{\alpha + \beta}{1 - \beta}\|Tx - x\|^2 \\
&+ \frac{2}{1 - \beta}\|Tx - x\|\big[\alpha\|x - y\| - \beta\|Tx - Ty\|\big].
\end{aligned}$$

(c) If $\alpha < 0$ and $0 \leq \beta < 1$, then we have

$$\begin{aligned}
\|Tx - Ty\|^2 &\leq \alpha\|Tx - y\|^2 + \beta\|Ty - x\|^2 + (1 - (\alpha + \beta))\|x - y\|^2 \\
&\leq \alpha\big[\|Tx - x\| - \|x - y\|\big]^2 + \beta\big[\|Ty - Tx\| + \|Tx - x\|\big]^2 \\
&\quad + (1 - (\alpha + \beta))\|x - y\|^2 \\
&= \alpha\|Tx - x\|^2 - 2\alpha\|Tx - x\|\|x - y\| + \alpha\|x - y\|^2 + \beta\|Ty - Tx\|^2 \\
&\quad + 2\beta\|Ty - Tx\|\|Tx - x\| + \beta\|Tx - x\|^2 + (1 - (\alpha + \beta))\|x - y\|^2 \\
&= (\alpha + \beta)\|Tx - x\|^2 + (1 - \beta)\|x - y\|^2 + \beta\|Ty - Tx\|^2 \\
&\quad + 2\|Tx - x\|\big[-\alpha\|x - y\| + \beta\|Tx - Ty\|\big]
\end{aligned}$$

and so

$$\|Tx - Ty\|^2 \le \|x - y\|^2 + \frac{\alpha+\beta}{1-\beta}\|Tx - x\|^2 + \frac{2}{1-\beta}\|Tx - x\|\big[-\alpha\|x - y\| + \beta\|Tx - Ty\|\big].$$

(d) If $\alpha, \beta < 0$, then we have

$$\begin{aligned}\|Tx - Ty\|^2 &\le \alpha\|Tx - y\|^2 + \beta\|Ty - x\|^2 + (1 - (\alpha+\beta))\|x - y\|^2\\ &\le \alpha\big[\|Tx - x\| - \|x - y\|\big]^2 + \beta\big[\|Ty - Tx\| - \|Tx - x\|\big]^2\\ &\quad + (1 - (\alpha+\beta))\|x - y\|^2\\ &= \alpha\|Tx - x\|^2 - 2\alpha\|Tx - x\|\|x - y\| + \alpha\|x - y\|^2 + \beta\|Ty - Tx\|^2\\ &\quad - 2\beta\|Ty - Tx\|\|Tx - x\| + \beta\|Tx - x\|^2 + (1 - (\alpha+\beta))\|x - y\|^2\\ &= (\alpha+\beta)\|Tx - x\|^2 + (1-\beta)\|x - y\|^2 + \beta\|Ty - Tx\|^2\\ &\quad + 2\|Tx - x\|\big[-\alpha\|x - y\| - \beta\|Tx - Ty\|\big]\end{aligned}$$

and so

$$\|Tx - Ty\|^2 \le \|x - y\|^2 + \frac{\alpha+\beta}{1-\beta}\|Tx - x\|^2 + \frac{2}{1-\beta}\|Tx - x\|\big[-\alpha\|x - y\| - \beta\|Tx - Ty\|\big].$$

Thus it follow that, for all x, $y \in K$ and $\alpha, \beta < 1$ with $x \le y$,

$$\|Tx - Ty\|^2 \le \|x - y\|^2 + \frac{\alpha+\beta}{1-\beta}\|Tx - x\|^2 + \frac{2}{1-\beta}\|Tx - x\|\big[|\alpha|\|x - y\| + |\beta|\|Tx - Ty\|\big].$$

This completes the proof. □

3.2. **The existence of fixed points.** In this section, we consider the Ishikawa type iteration defined by

$$\begin{cases} y_n = (1 - s_n)x_n + s_n Tx_n,\\ x_{n+1} = (1 - s_n)x_n + s_n T(y_n)\end{cases} \tag{3.8}$$

for each $n \ge 1$, where $\{s_n\}$ is the sequences in $[0, 1]$. We denote

$$F_{\le}(T) = \{p \in F(T) : p \le x_1\}, \quad F_{\ge}(T) = \{p \in F(T) : x_1 \le p\}.$$

Note that, since the partial order $\le$ is defined by the closed convex cone P, it is obvious that both $F_{\le}(T)$ and $F_{\ge}(T)$ are closed convex.

Now, we introduce the following lemma to find fixed points of a monotone (α, β)-nonexpansive mapping in Banach space E:

Lemma 3.5. *Let K be a nonempty closed comvex subset of a Banach space $(E, \le)$. Let $T : K \to K$ be a monotone mapping and assume that the sequence $\{x_n\}$ defined by Ishikawa type iteration (3.8) and $x_1 \le Tx_1$ (or $Tx_1 \le x_1$). Then we have*

(1) $x_n \le y_n \le x_{n+1}$ *and* $x_n \le y_n \le Tx_n \le Ty_n$ *(or* $x_n \ge y_n \ge x_{n+1}$ *and* $x_n \ge y_n \ge Tx_n \ge Ty_n$*);*

(2) $x_n \le x$ *(or* $x \le x_n$*) for all* $n \le 1$ *if* $\{x_n\}$ *weakly converges to a point* $x \in K$.

Proof. (1) Let $k_1, k_2 \in K$ such that $k_1 \leq k_2$. Then we have

$$k_1 \leq (1-\alpha)k_1 + \alpha k_2 \leq k_2$$

for all $\alpha \in [0,1]$ since order intervals are convex. By the assumption, we have $x_1 \leq Tx_1$ and so the inequality is true for $n = 1$. Assume that $x_k \leq Tx_k$ for $k \geq 2$. By convexity and monotonicity, we have

$$x_k \leq (1-s_k)x_k + s_k Tx_k = y_k \leq (1-s_k)Tx_k + s_k Tx_k = Tx_k,$$

i.e., $x_k \leq y_k \leq Tx_k$. By monotonicity, we get $x_k \leq y_k \leq Tx_k \leq Ty_k$.
From $x_k \leq Ty_k$, by convexity

$$x_k \leq (1-s_k)x_k + s_k Ty_k = x_{k+1} \leq Ty_k$$

i.e., $x_k \leq x_{k+1} \leq Ty_k$. And again, we have

$$y_k = (1-s_k)x_k + s_k Tx_k \leq (1-s_k)x_k + s_k Ty_k = x_{k+1},$$

so we get $x_k \leq y_k \leq x_{k+1}$. By monotonicity, we have $Ty_k \leq Tx_{k+1}$.
So $x_{k+1} \leq Tx_{k+1}$. By induction, we can conclude that $x_n \leq Tx_n$ is true for all $n \geq 1$.
Now we have $x_n \leq Tx_n$ for all $n \geq 1$ by convexity

$$x_n \leq (1-s_n)x_n + s_n Tx_n = y_n \leq Tx_n,$$

since T is monotonicity $Tx_n \leq Ty_n$, that is $x_n \leq y_n \leq Tx_n \leq Ty_n$. And

$$y_n = (1-s_n)x_n + s_n Tx_n \leq (1-s_n)x_n + s_n Ty_n \leq x_{n+1},$$

i.e., $x_n \leq y_n \leq x_{n+1}$.
Hence, we conclude that $x_n \leq y_n \leq x_{n+1}$ and $x_n \leq y_n \leq Tx_n \leq Ty_n$.
On the other hand, if we assume $Tx_1 \leq x_1$, then we can show that $x_n \geq y_n \geq x_{n+1}$ and $x_n \geq y_n \geq Tx_n \geq Ty_n$.

(2) From Dehaish and Chamsi [3, Lemma 3.1]), we have the conclusion. This completes the proof. □

Next, we show some existence theorems of fixed points of monotone (α, β)-nonexpansive mappings in a uniformly convex ordered Banach space $(E, \leq)$.

Theorem 3.1. *Let K be a nonempty closed convex subset of a uniformly convex ordered Banach space $(E, \leq)$ and $T : K \to K$ be a monotone (α, β)-nonexpansive mapping. Assume that $x_1 \leq Tx_1$ and the sequence $\{x_n\}$ defined by Ishikawa type iteration (3.8) is bounded with $x_n \leq w$ for some $w \in K$ and $\liminf_{n\to\infty} \|x_n - Tx_n\| = 0$. Then $F_{\geq}(T) \neq \emptyset$.*

Proof. From $\liminf_{n\to\infty} \|x_n - Tx_n\| = 0$, it follows that there exists a subsequence $\{x_{n_k}\} \subset \{x_n\}$ such that

$$\liminf_{k\to\infty} \|x_{n_k} - Tx_{n_k}\| = 0.$$

From Lemma 3.5, we have $x_1 \leq x_{n_k} \leq x_{n_{k+1}}$. Let $C_k = \{z \in K : x_{n_k} \leq z\}$ for all $k \geq 1$. Then C_k is closed convex and $w \in C_k$ and so C_k is nonempty. Let $C^* = \bigcap_{n=1}^{\infty} C_n$. Then C^* is a nonempty closed convex subset of K. Since $\{x_n\}$ is bounded, we can define a function $\rho : C^* \to [0, +\infty)$ by

$$\rho(z) = \limsup_{n\to\infty} \|x_n - z\|^2,$$

for all $z \in C^*$. From Lemma 2.2, it follows that there exists $z^* \in C^*$ such that

$$\rho(z^*) = \inf_{z \in C^*} \rho(z). \tag{3.9}$$

By the definition of C^*, we have

$$x_1 \leq x_{n_1} \leq x_{n_2} \leq \cdots \leq x_{n_k} \leq x_{n_{k+1}} \leq \cdots \leq z^*.$$

Since T is monotone, it follows from Lemma 3.5 that

$$x_{n_k} \leq Tx_{n_k} \leq Tz^*,$$

for each $k \geq 1$, which means that $Tz^* \in C^*$ and hence $\frac{z^*+Tz^*}{2} \in C^*$. Thus, by (3.9), we have

$$\rho(z^*) \leq \rho\Big(\frac{z^*+Tz^*}{2}\Big), \quad \rho(z^*) \leq \rho(Tz^*). \tag{3.10}$$

On the other hand, it follows from Lemma 3.4 that

$$\begin{aligned}\|Tx_{n_k} - Tz^*\|^2 &\leq \|x_{n_k} - z^*\|^2 + \frac{\alpha+\beta}{1-\beta}\|Tx_{n_k} - x_{n_k}\|^2 \\ &\quad + \frac{2}{1-\beta}\|Tx_{n_k} - x_{n_k}\|\big[|\alpha|\|x_{n_k} - z^*\| + |\beta|\|Tx_{n_k} - Tz^*\|\big].\end{aligned}$$

Since the sequence $\{x_n\}$ is bounded and $\liminf\limits_{k\to\infty} \|x_{n_k} - Tx_{n_k}\| = 0$, we have

$$\|Tx_{n_k} - Tz^*\|^2 \leq \|x_{n_k} - z^*\|^2,$$

and then

$$\limsup_{k\to\infty} \|Tx_{n_k} - Tz^*\|^2 \leq \limsup_{k\to\infty} \|x_{n_k} - z^*\|^2. \tag{3.11}$$

Thus, using (3.11), we have

$$\begin{aligned}\rho(Tz^*) &= \limsup_{k\to\infty} \|x_{n_k} - Tz^*\|^2 \\ &= \limsup_{k\to\infty} \|Tx_{n_k} - Tz^*\|^2 \\ &\leq \limsup_{k\to\infty} [\|x_{n_k} - z^*\|^2 \\ &= \rho(z^*).\end{aligned} \tag{3.12}$$

Now, we show that $z^* = Tz^*$. It follows from Lemma 2.1 with $q = 2$ and $t = \frac{1}{2}$ and (3.12) that

$$\begin{aligned}\rho\Big(\frac{z^*+Tz^*}{2}\Big) &= \limsup_{k\to\infty} \Big\|x_{n_k} - \frac{z^*+Tz^*}{2}\Big\|^2 \\ &= \limsup_{k\to\infty} \Big\|\frac{x_{n_k} - z^*}{2} + \frac{x_{n_k} - Tz^*}{2}\Big\|^2 \\ &\leq \limsup_{k\to\infty} \Big(\frac{1}{2}\|x_{n_k} - z^*\|^2 + \frac{1}{2}\|x_{n_k} - Tz^*\|^2 - \frac{1}{4}g(\|z^* - Tz^*\|)\Big) \\ &\leq \frac{1}{2}\rho(z^*) + \frac{1}{2}\rho(Tz^*) - \frac{1}{4}g(\|z^* - Tz^*\|) \\ &= \rho(z^*) - \frac{1}{4}g(\|z^* - Tz^*\|).\end{aligned}$$

By Lemma 2.1, we have

$$\frac{1}{4}g(\|z^* - Tz^*\|) \leq \rho(z^*) - \rho\Big(\frac{z^*+Tz^*}{2}\Big) \leq 0.$$

Thus we have $g(\|z^* - Tz^*\|) = 0$ and so $z^* = Tz^*$ by the property of g. This completes the proof. $\square$

Theorem 3.2. *Let K be a nonempty closed convex subset of a uniformly convex ordered Banach space $(E, \leq)$ and $T : K \to K$ be a monotone (α, β)-nonexpansive mapping. Assume that $Tx_1 \leq x_1$ and the sequence $\{x_n\}$ defined by Ishikawa type iteration (3.8) is bounded with $w \leq x_n$ for some $w \in K$ and $\liminf_{n\to\infty} \|x_n - Tx_n\| = 0$. Then $F_{\leq}(T) \neq \emptyset$.*

Proof. It follow from $\liminf_{n\to\infty} \|x_n - Tx_n\| = 0$ that there exists a subsequence $\{x_{n_k}\} \subset \{x_n\}$ such that

$$\liminf_{k\to\infty} \|x_{n_k} - Tx_{n_k}\| = 0.$$

From Lemma 3.5, it follows that $x_{n_{k+1}} \leq x_{n_k} \leq x_1$. Let $C_k = \{z \in K : z \leq x_{n_k}\}$ for all $k \geq 1$. Then C_k is closed convex and $w \in C_k$ and so C_k is nonempty. Let $C^* = \bigcap_{n=1}^{\infty} C_n$. Then C^* is a nonempty closed convex subset of K. Since $\{x_n\}$ is bounded, define a function $\rho : C^* \to [0, +\infty)$ by

$$\rho(z) = \limsup_{n\to\infty} \|x_n - z\|^2,$$

for all $z \in C^*$. From Lemma 2.2, it follows that there exists $z^* \in C^*$ such that

$$\rho(z^*) = \inf_{z\in C^*} \rho(z). \tag{3.13}$$

By the definition of C^*, we have

$$x_1 \geq x_{n_1} \geq x_{n_2} \geq \cdots \geq x_{n_k} \geq x_{n_{k+1}} \geq \cdots \geq z^*.$$

Since T is monotone, it follows Lemma 3.5 that

$$x_{n_k} \geq Tx_{n_k} \geq Tz^*,$$

for each $k \geq 1$, which means that $Tz^* \in C^*$ and hence $\frac{z^*+Tz^*}{2} \in C^*$. Thus, by (3.13), we have

$$\rho(z^*) \leq \rho\Big(\frac{z^* + Tz^*}{2}\Big), \quad \rho(z^*) \leq \rho(Tz^*). \tag{3.14}$$

On the other hand, it follows from Lemma 3.4 that

$$\begin{aligned}\|Tx_{n_k} - Tz^*\|^2 &\leq \|x_{n_k} - z^*\|^2 + \frac{\alpha+\beta}{1-\beta}\|Tx_{n_k} - x_{n_k}\|^2 \\ &\quad + \frac{2}{1-\beta}\|Tx_{n_k} - x_{n_k}\|\big[|\alpha|\|x_{n_k} - z^*\| + |\beta|\|Tx_{n_k} - Tz^*\|\big].\end{aligned}$$

Since the sequence $\{x_n\}$ is bounded and $\liminf_{k\to\infty} \|x_{n_k} - Tx_{n_k}\| = 0$, we have

$$\|Tx_{n_k} - Tz^*\|^2 \leq \|x_{n_k} - z^*\|^2,$$

and then

$$\limsup_{k\to\infty} \|Tx_{n_k} - Tz^*\|^2 \leq \limsup_{k\to\infty} \|x_{n_k} - z^*\|^2. \tag{3.15}$$

Thus, using (3.15), we have

$$\begin{aligned}\rho(Tz^*) &= \limsup_{k\to\infty} \|x_{n_k} - Tz^*\|^2 \\ &= \limsup_{k\to\infty} \|Tx_{n_k} - Tz^*\|^2 \\ &\leq \limsup_{k\to\infty} [\|x_{n_k} - z^*\|^2 \\ &= \rho(z^*).\end{aligned} \tag{3.16}$$

Now, we show that $z^* = Tz^*$. It follows from Lemma 2.1 with $q = 2$ and $t = \frac{1}{2}$ and (3.16) that

$$\begin{aligned}
\rho\Big(\frac{z^*+Tz^*}{2}\Big) &= \limsup_{k\to\infty}\Big\|x_{n_k} - \frac{z^*+Tz^*}{2}\Big\|^2 \\
&= \limsup_{k\to\infty}\Big\|\frac{x_{n_k}-z^*}{2} + \frac{x_{n_k}-Tz^*}{2}\Big\|^2 \\
&\le \limsup_{k\to\infty}\Big(\frac{1}{2}\|x_{n_k}-z^*\|^2 + \frac{1}{2}\|x_{n_k}-Tz^*\|^2 - \frac{1}{4}g(\|z^*-Tz^*\|)\Big) \\
&\le \frac{1}{2}\rho(z^*) + \frac{1}{2}\rho(Tz^*) - \frac{1}{4}g(\|z^*-Tz^*\|) \\
&= \rho(z^*) - \frac{1}{4}g(\|z^*-Tz^*\|).
\end{aligned}$$

Thus, by Lemma 2.1, we have

$$\frac{1}{4}g(\|z^*-Tz^*\|) \le \rho(z^*) - \rho\Big(\frac{z^*+Tz^*}{2}\Big) \le 0$$

and hence $g(\|z^* - Tz^*\|) = 0$. So $z^* = Tz^*$ by the property of g. This completes the proof. □

3.3. **The convergence of Ishikawa type iteration.** In this section, we prove some convergence theorems of Ishikawa type iteration for a monotone (α, β)-nonexpansive mapping in an ordered Banach space E.

Theorem 3.3. *Let K be a nonempty closed convex subset of a uniformly convex ordered Banach space $(E, \le)$ and $T : K \to K$ be a monotone (α, β)-nonexpansive mapping. Assume that the sequence $\{x_n\}$ is defined by Ishikawa type iteration (3.8) with $x_1 \le Tx_1$ (or $Tx_1 \le x_1$) and $F_{\ge}(T) \ne \emptyset$ (or $F_{\le}(T) \ne \emptyset$). Then we have*

(1) *the sequence $\{x_n\}$ is bounded;*
(2) *$\|x_{n+1} - p\| \le \|x_n - p\|$ and $\lim_{n\to\infty}\|x_n - p\|$ exists for all $p \in F_{\ge}(T) \ne \emptyset$ (or $F_{\le}(T) \ne \emptyset$);*
(3) *$\liminf_{n\to\infty}\|x_n - Tx_n\| = 0$ provided $\limsup_{n\to\infty} s_n(1 - s_n) > 0$;*
(4) *$\lim_{n\to\infty}\|x_n - Tx_n\| = 0$ provided $\liminf_{n\to\infty} s_n(1 - s_n) > 0$.*

Proof. Without loss of generality, we assume that $x_1 \le p \in F_{\ge}(T) \ne \emptyset$. Now, we claim $x_n \le p$ for all $n \ge 1$. In fact, since T is monotone, we have $x_1 \le Tx_1 \le Tp = p$ and $x_1 \le y_1 \le Tx_1 \le p$ then we have $y_1 \le p$. Again from T is monotone, we get $Ty_1 \le Tp = p$ from $x_1 \le Ty_1$. By convex we can get $x_2 \le p$, and so $x_1 \le x_2 \le p$. Suppose that $x_k \le p$ for some $k \ge 2$. Then $Tx_k \le Tp = p$ by monotonicity, from the condition (1) of Lemma 3.5 we have $x_k \le y_k \le Tx_k \le Ty_k$ and $x_k \le y_k \le Tx_k \le p$. Since $y_k \le p$ then $Ty_k \le Tp = p$. And $x_k \le Ty_k$ by convexity

$$x_k \le (1 - s_k)x_k + s_kTy_k = x_{k+1} \le Ty_k.$$

That is, we get $x_{k+1} \le p$. Hence we conclude $x_n \le p$ for all $n \le 1$.
It follows from Lemma 3.5 that $\|Tx_n - p\| \le \|x_n - p\|$ for all $n \ge 1$ and so

$$\begin{aligned}
\|y_n - p\| &= \|(1 - s_n)x_n + s_nTx_n - p\| \\
&\le (1 - s_n)\|x_n - p\| + s_n\|T(x_n) - p\| \\
&\le (1 - s_n)\|x_n - p\| + s_n\|x_n - p\| \\
&= \|x_n - p\|.
\end{aligned}$$

Consequently, we have

$$\begin{aligned}
\|x_{n+1} - p\| &= \|(1 - s_n)x_n + s_n T(y_n) - p\| \\
&\leq (1 - s_n)\|x_n - p\| + s_n\|Ty_n - p\| \\
&\leq (1 - s_n)\|x_n - p\| + s_n\|y_n - p\| \\
&\leq (1 - s_n)\|x_n - p\| + s_n\|x_n - p\| \\
&= \|x_n - p\| \\
&\ \ \cdots \\
&\leq \|x_1 - p\|.
\end{aligned}$$

Then the sequence $\{\|x_n - p\|\}$ is non-increasing and bounded and hence the conclusions (1) and (2) hold.

Now, we show that the conclusion (3) and (4) hold. It follows from Lemma 2.1 with $q = 2$, $t = s_n$ and Lemma 3.5 that

$$\begin{aligned}
\|x_{n+1} - p\|^2 &= \|(1 - s_n)x_n + s_n Ty_n - p\|^2 \\
&\leq \|(1 - s_n)(x_n - p) + s_n(Ty_n - p)\|^2 \\
&\leq (1 - s_n)\|x_n - p\|^2 + s_n\|Ty_n - p\|^2 - s_n(1 - s_n)g(\|x_n - Ty_n\|) \\
&\leq (1 - s_n)\|x_n - p\|^2 + s_n\|y_n - p\|^2 - s_n(1 - s_n)g(\|x_n - Ty_n\|) \\
&\leq (1 - s_n)\|x_n - p\|^2 + s_n\|x_n - p\|^2 - s_n(1 - s_n)g(\|x_n - Ty_n\|) \\
&\leq \|x_n - p\|^2 - s_n(1 - s_n)g(\|x_n - Tx_n\|)
\end{aligned}$$

which implies that

$$s_n(1 - s_n)g(\|x_n - Tx_n\|) \leq \|x_n - p\|^2 - \|x_{n+1} - p\|^2.$$

Then it follows from the conclusion (2) that

$$\limsup_{n\to\infty} s_n(1 - s_n)g(\|x_n - Tx_n\|) = 0.$$

From the conclusion (3), since $\limsup_{n\to\infty} s_n(1 - s_n) > 0$,

$$\left(\limsup_{n\to\infty} s_n(1 - s_n)\right)\left(\liminf_{n\to\infty} g(\|x_n - Tx_n\|)\right) \leq \limsup_{n\to\infty} s_n(1 - s_n)g(\|x_n - Tx_n\|),$$

we have

$$\liminf_{n\to\infty} g(\|x_n - Tx_n\|) = 0.$$

Hence we have

$$\liminf_{n\to\infty} \|x_n - Tx_n\| = 0,$$

by the properties of g. From the conclusion (4), since $\liminf_{n\to\infty} s_n(1 - s_n) > 0$,

$$\left(\liminf_{n\to\infty} s_n(1 - s_n)\right)\left(\limsup_{n\to\infty} g(\|x_n - Tx_n\|)\right) \leq \limsup_{n\to\infty} s_n(1 - s_n)g(\|x_n - Tx_n\|),$$

we have

$$\lim_{n\to\infty} g(\|x_n - Tx_n\|) = \limsup_{n\to\infty} g(\|x_n - Tx_n\|) = 0.$$

Hence we have

$$\lim_{n\to\infty} \|x_n - Tx_n\| = 0,$$

by the properties of g. This completes the proof. □

Theorem 3.4. *Let K be a nonempty closed convex subset of a uniformly convex ordered Banach space $(E, \leq)$ and $T : K \to K$ be a monotone (α, β)-nonexpansive mapping. Assume that E satisfies Opial's condition and the sequence $\{x_n\}$ is defined by Ishikawa type iteration (3.8) with $x_1 \leq Tx_1$ (or $Tx_1 \leq x_1$). If $F_{\geq}(T) \neq \emptyset$ (or $F_{\leq}(T) \neq \emptyset$) and $\liminf_{n\to\infty} s_n(1 - s_n) > 0$, then the sequence $\{x_n\}$ converges weakly to a fixed point z of T.*

Proof. It follows from Theorem 3.3 that $\{x_n\}$ is bounded and $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$. Then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\{x_{n_k}\}$ converges weakly to a point $z \in K$. From Lemma 3.5, it follows that $x_1 \leq x_{n_k} \leq z$ (or $z \leq x_{n_k} \leq x_n$) for all $k \geq 1$.

On the other hand, the condition (2) of Lemma 3.4 means that

$$\begin{aligned}\|Tx - Ty\|^2 \leq \|x - y\|^2 &+ \frac{\alpha + \beta}{1 - \beta}\|Tx - x\|^2 \\ &+ \frac{2}{1 - \beta}\|Tx - x\|\big[|\alpha|\|x - y\| + |\beta|\|Tx - Ty\|\big].\end{aligned}$$

Since the sequence $\{x_n\}$ is bounded and $\lim_{k\to\infty} \|x_{n_k} - Tx_{n_k}\| = 0$, we have

$$\limsup_{k\to\infty} \|Tx_{n_k} - Tz\|^2 \leq \limsup_{k\to\infty} \|x_{n_k} - z\|^2$$

and hence

$$\limsup_{k\to\infty} \|Tx_{n_k} - Tz\| \leq \limsup_{k\to\infty} \|x_{n_k} - z\|. \tag{3.17}$$

Now, we prove that $z = Tz$. In fact, suppose that $z \neq Tz$. Then, by (3.17) and Opial's condition, we have

$$\begin{aligned}\limsup_{k\to\infty} \|x_{n_k} - z\| &\leq \limsup_{k\to\infty} \|x_{n_k} - Tz\| \\ &\leq \limsup_{k\to\infty}(\|x_{n_k} - Tx_{n_k}\| + \|Tx_{n_k} - Tz\|) \\ &\leq \limsup_{k\to\infty} \|x_{n_k} - z\|,\end{aligned}$$

which is a contraction. This implies that $z \in F_{\geq}(T)$ (or $z \in F_{\leq}(T)$). Using the conclusion (2) of Theorem 3.3, $\lim_{n\to\infty} \|x_n - z\|$ exists.

Now, we show that the sequence $\{x_n\}$ converge weakly to the point z. Suppose that this does not hold. Then there exists a subsequence $\{x_{n_j}\}$ to converge weakly to a point $x \in K$ and $z \neq x$. Similarly, we must have $x = Tx$ and $\lim_{n\to\infty} \|x_n - x\|$ exists. It follows from Opial's condition that

$$\lim_{n\to\infty} \|x_n - z\| < \lim_{n\to\infty} \|x_n - x\| = \limsup_{j\to\infty} \|x_{n_j} - x\| < \lim_{n\to\infty} \|x_n - z\|,$$

which is a contradiction and hence we get $x = z$. This completes the proof. □

Theorem 3.5. *Let K be a nonempty compact and closed convex subset of a uniformly convex ordered Banach space $(E, \leq)$ and $T : K \to K$ be a monotone (α, β)-nonexpansive mapping. Assume that the sequence $\{x_n\}$ is defined by Ishikawa type iteration (3.8) with $x_1 \leq Tx_1$. If $\limsup_{n\to\infty} s_n(1 - s_n) > 0$, then the sequence $\{x_n\}$ converges strongly to a fixed point $p \in F_{\geq}(T)$.*

Proof. Since K is compact, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\{x_{n_k}\}$ converges strongly to a point $p \in K$. From Lemma 3.5, it follows that $x_1 \leq x_{n_k} \leq p$ for all $k \geq 1$. By Theorem 3.1, we have $F_{\geq}(T) \neq \emptyset$ and it follows from Theorem 3.3 that $\{x_n\}$ is bounded and

$$\liminf_{n\to\infty} \|x_n - Tx_n\| = 0.$$

Assume that

$$\liminf_{k\to\infty} \|x_{n_k} - Tx_{n_k}\| = 0.$$

On the other hand, the condition (2) of Lemma 3.4 means that

$$\|Tx - Tp\|^2 \le \|x - p\|^2 + \frac{\alpha+\beta}{1-\beta}\|Tx - x\|^2 + \frac{2}{1-\beta}\|Tx - x\|\big[|\alpha|\|x-p\| + |\beta|\|Tx - Tp\|\big].$$

Since the sequence $\{x_{n_k}\}$ is bounded and

$$\lim_{k\to\infty} \|x_{n_k} - p\| = 0, \quad \lim_{k\to\infty} \|x_{n_k} - Tx_{n_k}\| = 0,$$

we have

$$\limsup_{k\to\infty} \|Tx_{n_k} - Tp\|^2 \le 0$$

and hence

$$\lim_{k\to\infty} \|Tx_{n_k} - Tp\| = 0. \tag{3.18}$$

Therefore, we have

$$\limsup_{k\to\infty} \|x_{n_k} - Tp\| \le \limsup_{k\to\infty}(\|x_{n_k} - Tx_{n_k}\| + \|Tx_{n_k} - Tp\|) = 0$$

and so $\lim_{k\to\infty} \|x_{n_k} - Tp\| = 0$, which implies that $p \in F_{\ge}(T)$. Using the conclusion (2) of Theorem 3.3, $\lim_{k\to\infty} \|x_{n_k} - p\|$ exists and so $\lim_{k\to\infty} \|x_n - p\| = 0$. This completes the proof. □

Theorem 3.6. *Let K be a nonempty compact and closed convex subset of a uniformly convex ordered Banach space $(E, \le)$ and $T : K \to K$ be a monotone (α, β)-nonexpansive mapping. Assume that the sequence $\{x_n\}$ is defined by Ishikawa type iteration (3.8) with $x_1 \le Tx_1$. If $\liminf_{n\to\infty} s_n(1-s_n) > 0$, then the sequence $\{x_n\}$ converges strongly to a fixed point $p \in F_{\ge}(T)$.*

Proof. Since K is compact, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\{x_{n_k}\}$ converges strongly to a point $p \in K$. From Lemma 3.5, it follows that $x_1 \le x_{n_k} \le p$ for all $k \ge 1$. By Theorem 3.1, we have $F_{\ge}(T) \neq \emptyset$ and it follows from Theorem 3.3 that $\{x_n\}$ is bounded and

$$\liminf_{n\to\infty} \|x_n - Tx_n\| = 0.$$

Without loss of generality, we can assume that

$$\liminf_{k\to\infty} \|x_{n_k} - Tx_{n_k}\| = 0.$$

On the other hand, the condition (2) of Lemma 3.4 means that

$$\|Tx - Tp\|^2 \le \|x - p\|^2 + \frac{\alpha+\beta}{1-\beta}\|Tx - x\|^2 + \frac{2}{1-\beta}\|Tx - x\|\big[|\alpha|\|x-p\| + |\beta|\|Tx - Tp\|\big].$$

Since the sequence $\{x_{n_k}\}$ is bounded and

$$\lim_{k\to\infty} \|x_{n_k} - p\| = 0, \quad \lim_{k\to\infty} \|x_{n_k} - Tx_{n_k}\| = 0,$$

we have

$$\liminf_{k\to\infty} \|Tx_{n_k} - Tp\|^2 \le 0$$

and hence

$$\lim_{k\to\infty} \|Tx_{n_k} - Tp\| = 0. \tag{3.19}$$

Therefore, we have

$$\liminf_{k\to\infty} \|x_{n_k} - Tp\| \le \liminf_{k\to\infty} (\|x_{n_k} - Tx_{n_k}\| + \|Tx_{n_k} - Tp\|) = 0$$

and so $\lim_{k\to\infty} \|x_{n_k} - Tp\| = 0$, which implies that $p \in F_{\ge}(T)$. Using the conclusion (2) of Theorem 3.3, $\lim_{k\to\infty} \|x_{n_k} - p\|$ exists and so $\lim_{k\to\infty} \|x_n - p\| = 0$. This completes the proof. □

Similarly, the following theorem can be proved:

Theorem 3.7. *Let K be a nonempty compact and closed convex subset of a uniformly convex ordered Banach space $(E, \le)$ and $T : K \to K$ be a monotone (α, β)-nonexpansive mapping. Assume that the sequence $\{x_n\}$ is defined by Ishikawa type iteration (3.8) with $Tx_1 \le x_1$. If either $\liminf_{n\to\infty} s_n(1 - s_n) > 0$ or $\limsup_{n\to\infty} s_n(1 - s_n) > 0$, then the sequence $\{x_n\}$ converges strongly to a fixed point $p \in F_{\le}(T)$.*

From Theorem 3.6, we have the following:

Corollary 3.1. *Let K be a nonempty closed convex subset of a uniformly convex ordered Banach space $(E, \le)$ and $T : K \to K$ be a monotone α-nonexpansive mapping. Assume that $x_1 \le Tx_1$ (or $Tx_1 \le x_1$) and the sequence $\{x_n\}$ defined by Mann's iteration is bounded with $x_n \le y$ (or $y \le x_n$) for some $y \in K$ and $\liminf\limits_{n\to\infty} \|x_n - Tx_n\| = 0$. Then $F_{\ge}(T) \ne \emptyset$.*

3.4. **A numerical example.** Now, we give numerical example to illustrate Example 3.2 in this paper. This section, the numerical solution of this example is presented in Figure 2,3 and Table 1.

Example 3.2. Let $T : [0, 1] \to [0, 1]$ be a mapping defined by

$$Tx = \begin{cases} 0.25 & if \quad x \ne 1, \\ 0.5 & if \quad x = 1. \end{cases}$$

for any $x \in [0, 1]$. Then T is a $(0.8, 0.2)$-nonexpansive mapping. Define the sequences $\{s_n\}$ and $\{t_n\}$ by $s_n = \frac{1}{4} + \frac{1}{n^2}$ for each $n \ge 1$, then $\limsup_{n\to\infty} s_n(1 - s_n) = \limsup_{n\to\infty}(\frac{1}{4} + \frac{1}{n^2})(\frac{3}{4} + \frac{1}{n^2}) = \frac{3}{16} > 0$. Then all the conditions of Theorem 3.5 are satisfied. Also, 0.25 is a fixed point of T (see Figure 1, 2, 3 and Table 1).

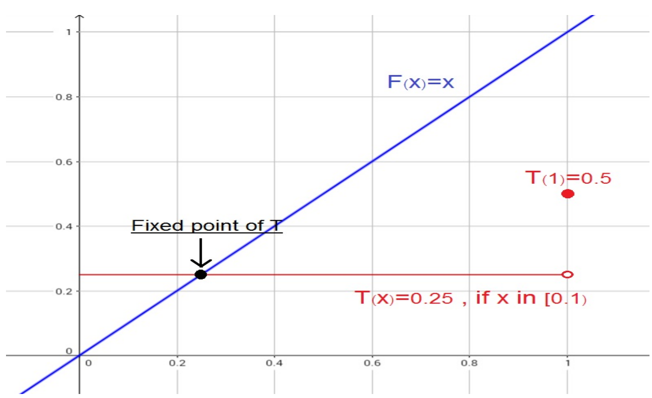

FIGURE 1. A fixed point of T is 0.25

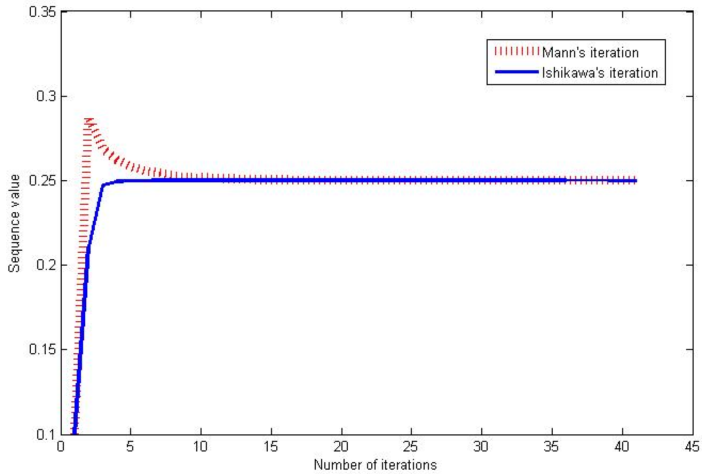

FIGURE 2. The numerical solution in Example 3.2 for $s_n = \frac{1}{4} + \frac{1}{n^2}$ and $x_0 = 0.1$

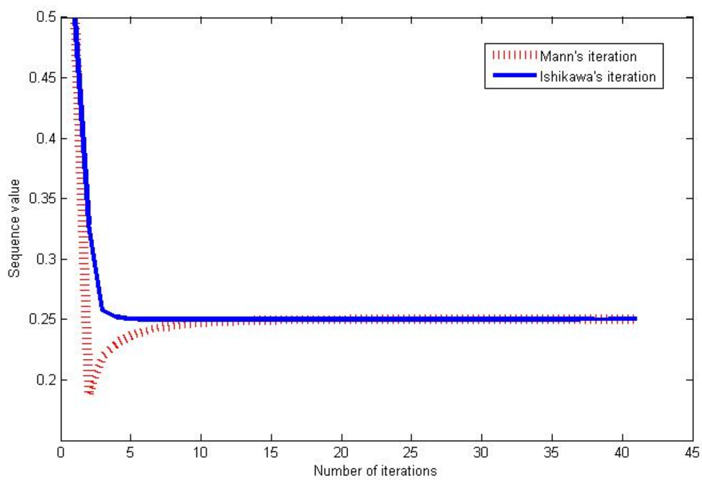

FIGURE 3. The numerical solution in Example 3.2 for $s_n = \frac{1}{4} + \frac{1}{n^2}$ and $x_0 = 0.5$

TABLE 1. The convergent step of $\{x_n\}$ for Example 3.2 with $s_n = \frac{1}{4} + \frac{1}{n^2}$

Number of Iterations	*Sequence value of Mann*		*Sequence value of Ishikawa*	
	$x_0 = 0.1$	$x_0 = 0.5$	$x_0 = 0.1$	$x_0 = 0.5$
1	*0.1000000*	*0.5000000*	*0.1000000*	*0.5000000*
2	*0.2875000*	*0.1875000*	*0.2107903*	*0.3294046*
4	*0.2619791*	*0.2300347*	*0.2493380*	*0.2518192*
6	*0.2558473*	*0.2402544*	*0.2499226*	*0.2502132*
8	*0.2530811*	*0.2448648*	*0.2499882*	*0.2500322*
10	*0.2516690*	*0.2472181*	*0.2499980*	*0.2500053*
12	*0.2509161*	*0.2484731*	*0.2499996*	*0.2500009*
14	*0.2505065*	*0.2491557*	*0.2499999*	*0.2500001*
16	*0.2502813*	*0.2495311*	*0.2499999*	*0.2500000*
18	*0.2501566*	*0.2497388*	*0.2499999*	*0.2500000*
20	*0.2500874*	*0.2498542*	*0.2499999*	*0.2500000*
22	*0.2500488*	*0.2499185*	*0.2499999*	*0.2500000*
24	*0.2500273*	*0.2499544*	*0.2499999*	*0.2500000*
26	*0.2500153*	*0.2499744*	*0.2499999*	*0.2500000*
28	*0.2500085*	*0.2499856*	*0.2499999*	*0.2500000*
30	*0.2500048*	*0.2499919*	*0.2499999*	*0.2500000*
32	*0.2500026*	*0.2499955*	*0.2499999*	*0.2500000*
34	*0.2500015*	*0.2499974*	*0.2499999*	*0.2500000*
36	*0.2500008*	*0.2499985*	*0.2500000*	*0.2500000*
38	*0.2500004*	*0.2499992*	*0.2500000*	*0.2500000*
40	*0.2500002*	*0.2499995*	*0.2500000*	*0.2500000*

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