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Exact and trajectory controllability of second order nonlinear differential equations with deviated argument

M. MUSLIM[1], AVADHESH KUMAR[2] and RAVI P. AGARWAL[3]

ABSTRACT. In this manuscript, we consider a control system governed by a second order nonlinear differential equations with deviated argument in a Hilbert space X. We used the strongly continuous cosine family of bounded linear operators and fixed point method to study the exact and trajectory controllability. Also, we study the exact controllability of the nonlocal control problem. Finally, we give an example to illustrate the application of these results.

1. INTRODUCTION

We consider a control problem represented by a second order nonlinear differential equation with deviated argument in a Hilbert space X:

$$\begin{aligned} &x''(t) = Ax(t) + Bu(t) + f(t, x(t), x[h(x(t), t)]), \quad t \in (0, T], \qquad (1.1)\\ &x(0) = x_0, \quad x'(0) = y_0, \end{aligned}$$

where $x : J(= [0, T]) \to X$ is the state function, $u(.) \in L^2(J, U)$ is the control function, U is a Hilbert space known as the control space, A is the infinitesimal generator of a strongly continuous cosine family of bounded linear operators $(C(t))_{t\in\mathbb{R}}$ on X, $B : U \to X$ is a bounded linear operator and $f : J \times X \times X \to X$ is a suitable continuous function to be specified later.

Controllability is one of the basic concepts in mathematical theory which was introduced by Kalman in 1960. This is a qualitative property of dynamical control systems and it is of particular importance in control theory. Roughly speaking, controllability means, that it is possible to steer dynamical control system from an arbitrary initial state to an arbitrary final state using the set of admissible controls. Many scientific and engineering problems are nonlinear in nature and can be described in infinite-dimensional spaces. So the study of controllability results for the control systems in infinite-dimensional spaces is very important. The controllability of nonlinear systems in finite-dimensional space has been studied extensively by many authors. Several authors [4, 8, 9, 10, 15, 16, 19] have extended the concept of controllability to infinite-dimensional systems and established sufficient conditions for the controllability of nonlinear systems in abstract spaces [18]. Among the various approaches to the study of the controllability of nonlinear systems, fixed-point techniques have been used effectively for these systems [2, 3, 25, 26]. In the fixed-point method, the controllability problem is transformed into a fixed-point problem for an appropriate nonlinear operator in a function space [11, 20, 21].

Several partial differential equations that arise in many problems connected with the transverse motion of an extensible beam, the vibration of hinged bars and many other physical phenomena can be formulated as the second order abstract differential equations

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Corresponding author: Ravi P. Agarwal; Ravi.Agarwal@tamuk.edu

in the infinite dimensional spaces. A useful tool for the study of second order abstract differential equations is the theory of strongly continuous cosine families of operators. For the initial study on the controllability of various kind of abstract second order differential equations we refer to [1, 7] and references cited in these papers. Sakthivel et al. [22] study the exact controllability of the following problem

$$\begin{aligned} &x''(t) = Ax(t) + Bu(t) + f(t, x(t), x'(t)), \quad t \in J = [0, b],\ t \neq t_k, \\ &x(0) = x_0, \quad x'(0) = y_0, \\ &\Delta x(t_k) = I_k^1(x(t_k)), \quad \Delta x'(t_k) = I_k^2(x(t_k^+)), \quad k = 1, 2...m, \end{aligned} \tag{1.2}$$

and extend the result for the nonlocal conditions but authors have not discussed trajectory controllability which is a stronger concept of exact controllability. Trajectory controllability has concern not only with initial and final points but also with trajectory which passes through these points. Chalishajar et al. [6] have shown trajectory controllability of integro-differential system in the finite and infinite dimensional spaces. The infinite dimensional version of the problem is

$$\begin{aligned} &w'(t) = Aw(t) + B(t, u(t)) + f\left(t, w(t), \int_0^t G(t, s, w(s))ds\right) \quad t \in J = [0, T], \\ &w(0) = w_0. \end{aligned} \tag{1.3}$$

In (4.1, [6]), authors have converted second order differential equation into the first order system of equations. In many cases it is advantages to treat second order abstract differential equations directly rather than convert them to first order systems. Hence, it is motivating to study trajectory controllability of second order differential problem.

In certain real world problems, delay depends not only on the time but also on the unknown quantity. The differential equations with deviated arguments are the generalization of delay differential equations. Gal [13] has considered a nonlinear abstract differential equations with deviated arguments and study the existence and uniqueness of solutions. Muslim et al. [17] have investigated exact controllability of first order system with deviated argument. As per author's knowledge, there are only few papers which discuss in detail both the exact controllability as well as trajectory controllability of the second order nonlinear differential. Therefore, motivated by [6, 17, 22], we consider a control problem described by a second order nonlinear differential equation with deviated argument. The plan of the paper is as follows. In the first and second section, we give the introduction, notations and results which are required for the later sections. In the third, fourth and fifth section, we study the exact controllability for the problem (1.1), integro-differential problem and nonlocal problem respectively. Trajectory controllability is discussed in the sixth section. In the last section, an example is given to show the application of these abstract results.

2. Preliminaries and assumptions

We briefly review some basic definitions and useful properties of the strongly continuous cosine family of bounded operators which will be used in the subsequent sections.

Definition 2.1. (see, [24]) A one parameter family $(C(t))_{t\in\mathbb{R}}$ of bounded linear operators mapping the Banach space X into itself is called a strongly continuous cosine family if and only if
(i) $C(s+t) + C(s-t) = 2C(s)C(t)$ for all $s, t \in \mathbb{R}$.
(ii) $C(0) = I$.
(iii) $C(t)x$ is continuous in t on $\mathbb{R}$ for each fixed point $x \in X$.

$(S(t))_{t\in\mathbb{R}}$ be the sine function associated with the strongly continuous cosine family, $(C(t))_{t\in\mathbb{R}}$ which is defined by

$$S(t)x = \int_0^t C(s)x\,ds, \quad x \in X, \quad t \in R.$$

Let M and $\tilde{M}$ are positive constants such that $\|C(t)\| \leq M$ and $\|S(t)\| \leq \tilde{M}$ for every $t \in J$. The infinitesimal generator of a strongly continuous cosine family $(C(t))_{t\in\mathbb{R}}$ is the operator $A : X \to X$ defined by

$$Ax = d^2/dt^2 C(0)x.$$

$D(A)$ be the domain of the operator A which is defined by

$$D(A) = \{x \in X : C(t)x \text{ is twice continuously differentiable in t}\}.$$

$D(A)$ is the Banach space endowed with the graph norm $\|x\|_A = \|x\| + \|Ax\|$ for all $x \in D(A)$. We define a set

$$E = \{x \in X : C(t)x \text{ is once continuously differentiable in t}\}$$

which is a Banach space endowed with norm $\|x\|_E = \|x\| + \sup_{0\leq t\leq 1} \|AS(t)x\|$ for all $x \in E$.

With the help of $C(t)$ and $S(t)$, we define a operator valued function

$$\bar{h}(t) = \begin{bmatrix} C(t) & S(t) \\ AS(t) & C(t) \end{bmatrix}.$$

Operator valued function $\bar{h}(t)$ is a strongly continuous group of bounded linear operators on the space $E \times X$ generated by the operator

$$\bar{A} = \begin{bmatrix} 0 & I \\ A & 0 \end{bmatrix}$$

defined on $D(A) \times E$. $AS(t) : E \to X$ is a bounded linear operator and that $AS(t)x \to 0$ as $t \to 0$, for each $x \in E$. If $x : [0, \infty) \to X$ is locally integrable function then

$$y(t) = \int_0^t S(t-s)x(s)ds$$

defines an E valued continuous function which is a consequence of the fact that

$$\int_0^t \bar{h}(t-s) \begin{bmatrix} 0 \\ x(s) \end{bmatrix} ds = \begin{bmatrix} \int_0^t S(t-s)x(s)ds \\ \int_0^t C(t-s)x(s)ds \end{bmatrix}$$

defines an $(E \times X)$ valued continuous function.

Proposition 2.1. *Let $(C(t))_{t\in R}$ be a strongly continuous cosine family in X. The following are true:*

(i) $C(t) = C(-t)$ *for all* $t \in \mathbb{R}$.
(ii) $C(s), S(s), C(t)$ *and* $S(t)$ *commute for all* $s, t \in \mathbb{R}$.
(iii) $S(t)x$ *is continuous in* t *on* $\mathbb{R}$ *for each fixed* $x \in X$.
(iv) $S(s+t) + S(s-t) = 2S(s)C(t)$ *for all* $s, t \in \mathbb{R}$.
(v) $S(s+t) = S(s)C(t) + S(t)C(s)$ *for all* $s, t \in \mathbb{R}$.
(vi) $S(t) = -S(-t)$ *for all* $t \in \mathbb{R}$.
(vii) there exist constants $K \geq 1$ *and* $\omega \geq 0$ *such that* $|C(t)| \leq Ke^{\omega|t|}$ *for all* $t \in \mathbb{R}$.
(viii) $|S(t_2) - S(t_1)| \leq K|\int_{t_1}^{t_2} e^{\omega|s|}ds|$ *for all* $t_1, t_2 \in \mathbb{R}$.

For additional details on cosine family theory, we refer to Travis & Webb [24].

Proposition 2.2. *Let $(C(t))_{t\in\mathbb{R}}$ be a strongly continuous cosine family in X with infinitesimal generator A. The following are true:*

(i) $D(A)$ *is dense in* X *and* A *is a closed operator in* X.

(ii) If $x \in X$ *and* $r, s \in \mathbb{R}$, *then define* $z = \int_r^s S(u)x\,du \in D(A)$ *and* $Az = C(s)x - C(r)x$.

(iii) If $x \in X$ *and* $r, s \in \mathbb{R}$, *then define* $z = \int_0^s \int_0^r C(u)C(v)x\,du\,dv \in D(A)$ *and* $Az = 2^{-1}(C(s+r)x - C(s-r)x)$.

(iv) If $x \in X$, *then* $S(t)x \in E$ *for every* $t \in \mathbb{R}$.

(v) If $x \in E$, *then* $S(t)x \in D(A)$, *and* $(d/dt)C(t)x = AS(t)x$ *for every* $t \in \mathbb{R}$.

(vi) If $x \in D(A)$, *then* $C(t)x \in D(A)$, *and* $(d^2/dt^2)C(t)x = AC(t)x = C(t)Ax$ *for every* $t \in \mathbb{R}$.

(vii) If $x \in E$ *then* $\lim_{t\to 0} AS(t)x = 0$.

(viii) $C(t+s) - C(t-s) = 2AS(t)S(s)$ *for all* $s, t \in \mathbb{R}$.

Proof. We refer to Fattorini [12] and Travis & Webb [23, 24]. □

Definition 2.2. Let $x_T(x_0, y_0; u)$ be the state value of(1.1) at time T corresponding to the control u and the initial value x_0 and y_0. The system(1.1) is said to be exactly controllable on the interval J if $\mathfrak{R}(T, x_0, y_0) = X$, where

$$\mathfrak{R}(T, x_0, y_0) = \{x_T(x_0, y_0; u) : u(.) \in L^2(J, U)\}.$$

We assume that the second order linear system

$$\begin{aligned} x''(t) &= Ax(t) + Bu(t), \qquad t \in J, \\ x(0) &= x_0, \qquad x'(0) = y_0 \end{aligned} \tag{2.4}$$

is exactly controllable on J. We define an operator

$$\Gamma_0^T = \int_0^T S(T-s)BB^*S^*(T-s)ds.$$

System (2.1) is exactly controllabe [14] iff there exists a $\delta > 0$ such that

$$\langle \Gamma_0^T x, x\rangle \geq \delta \|x\|^2,$$

for every $x \in X$ then $\|(\Gamma_0^T)^{-1}\| \leq \frac{1}{\delta}$.

Let $C([0,T], X)$ be the space of all continuous functions $x : [0,T] \to X$ which is a Banach space endowed with norm $\|x\|_{C(J,X)} = \sup_{t\in J} \|x(t)\|$. We define another set

$$C_L(J, X) = \{x \in C([0,T], X) : \|x(t) - x(s)\| \leq L|t-s|, \forall\, t,\ s \in J \text{ and}, L > 0\}.$$

Clearly $C_L(J, X)$ is a Banach space endowed with supremum norm.

Definition: A function $x(\cdot) \in C_L([0,T], X)$ is called a mild solution of the control problem (1.1) if $x(t)$ is the solution of the following integral equation

$$x(t) = C(t)x_0 + S(t)y_0 + \int_0^t S(t-s)[Bu(s) + f(s, x(s), x[h(x(s), s)])]ds. \tag{2.5}$$

In order to prove the exact controllability of the problem (1.1), we need the following assumptions:

(A1) $f : J \times X \times X \to X$ is a continuous function and there exists positive constants K_1 and K_2 such that

$$\|f(t, x_1, y_1) - f(t, x_2, y_2)\| \leq K_1(\|x_1 - x_2\| + \|y_1 - y_2\|)$$

for every $x_1, x_2, y_1, y_2 \in X$ and $\max_{t\in J} \|f(t, 0, x(0)\| = K_2$.

(A2) $h : X \times [0,T] \to \mathbb{R}^+$ is a uniformly continuous and there exists a positive constant $L_h = L_h(\alpha)$ such that

$$|h(x_1,s) - h(x_2,s)| \le L_h \|x_1 - x_2\|, \forall\, x_1, x_2 \in X \text{ whenever } 0 \le s \le \alpha$$

and satisfies $h(.,0) = 0$ for each $\alpha > 0$.

(A3) The linear system (2.4) is exactly controllable.

3. Exact controllability

Steering of a dynamical control system from an arbitrary initial state to an arbitrary final state using the set of admissible controls is called exactly controllable system. In this section, we investigate exact controllability of the system (1.1).

Theorem 3.1. *If $x_0 \in D(A)$, $y_0 \in E$ and all the assumptions* (**A1**)-(**A3**) *are satisfied. Then, the second order nonlinear control system (1.1) is locally exactly controllable on $[0,T_0]$.*

Proof. We take $K_3 = \|B\|$ and $\rho = \sup_{t\in[0,T_0]} \|AS(t)\|$. For a suitable $\delta_1 > 0$, we choose T_0, $0 < T_0 \le T$ such that

$$\frac{M\|x_0\| + \tilde{M}\|y_0\| + \tilde{M}K_3PT_0 + \tilde{M}K_1L^2L_hT_0^2 + \tilde{M}K_2T_0}{(1-\tilde{M}K_1T_0)} = \delta_1.$$

We define the feedback control function

$$u(t) = B^*S^*(T_0 - s)(\Gamma_0^{T_0})^{-1} \left[x_{T_0} - C(T_0)x_0 - S(T_0)y_0 - \int_0^{T_0} S(T_0-s)f(s,x(s),x[h(x(s),s)])ds\right]. \quad (3.6)$$

Hence, we get

$$\|u(t)\| \le P = K_3\tilde{M}\frac{1}{\delta}[\|x_{T_0}\| + M\|x_0\| + \tilde{M}\|y_0\| + \tilde{M}K_1T_0(\delta_1 + L^2L_hT_0) + \tilde{M}K_2T_0].$$

We choose $W = \{x(.) \in C_L([0,T_0],X) : \|x\|_{C([0,T_0],X)} \le \delta_1\}$. Clearly, W is a closed and bounded subset of $C_L([0,T_0],X)$. We define a map $F : W \to W$ given by

$$(Fx)(t) = C(t)x_0 + S(t)y_0 + \int_0^t S(t-s)[Bu(s) + f(s,x(s),x[h(x(s),s)])]ds.$$

First, we need to show that $Fx \in C_L([0,T_0],X)$ for any $x \in C_L([0,T_0],X)$. If $x \in C_L([0,T_0],X)$, $\quad T_0 > t_2 > t_1 > 0$, then we get

$$\begin{aligned} \|(Fx)(t_2) - (Fx)(t_1)\| &\le \|(C(t_2) - C(t_1))x_0\| + \|(S(t_2) - S(t_1))y_0\| \\ &+ \int_0^{t_1} \|S(t_2-s) - S(t_1-s)\|\|B\|\|u(s)\|ds \\ &+ \int_{t_1}^{t_2} \|S(t_2-s)\|\|B\|\|u(s)\|ds \\ &+ \int_0^{t_1} \|S(t_2-s) - S(t_1-s)\|\|f(s,x(s),x[h(x(s),s)])\|ds \\ &+ \int_{t_1}^{t_2} \|S(t_2-s)\|\|f(s,x(s),x[h(x(s),s)])\|ds \\ &\le \|I_1\| + \|I_2\| + \|I_3\| + \|I_4\| + \|I_5\| + \|I_6\|. \end{aligned} \quad (3.7)$$

We have,

$$\|I_1\| = \|(C(t_2) - C(t_1))x_0\| \;=\; \|\int_{t_1}^{t_2} AS(\tau)x_0 d\tau\| \leq C_1(t_2 - t_1), \tag{3.8}$$

where $C_1 = \rho\|x_0\|$. Similarlly, we have

$$\|I_2\| = \|(S(t_2) - S(t_1))y_0\| \;=\; \|K\int_{t_1}^{t_2} e^{\omega\tau} d\tau\|\|y_0\| \leq C_2(t_2 - t_1), \tag{3.9}$$

where $C_2 = K.e^{\omega T_0}\|y_0\|$. We calculate $\|I_3\|$ as follows

$$\begin{aligned}\|I_3\| &= \int_0^{t_1} \|S(t_2 - s) - S(t_1 - s)\|\|B\|\|u(s)\|ds \\ &\leq K_3 P\int_0^{t_1} \|K\int_{t_1-s}^{t_2-s} e^{\omega\tau}d\tau\|ds \\ &\leq C_3(t_2 - t_1),\end{aligned} \tag{3.10}$$

where $C_3 = K.e^{\omega T_0}K_3PT_0$. Fourth integral $\|I_4\|$ is calculated as follows

$$\|I_4\| = \int_{t_1}^{t_2} \|S(t_2 - s)\|\|B\|\|u(s)\|ds \leq C_4(t_2 - t_1), \tag{3.11}$$

where $C_4 = \tilde{M}K_3P$. Similarly, we calculate fifth and six part of inequality (3.7) as follows

$$\begin{aligned}&\|I_5\| = \int_0^{t_1} \|S(t_2 - s) - S(t_1 - s)\|\|f(s, x(s), x[h(x(s), s)])\|ds \\ &\leq N\int_0^{t_1} \|K\int_{t_1-s}^{t_2-s} e^{\omega\tau}d\tau\|ds \\ &\leq C_5(t_2 - t_1),\end{aligned} \tag{3.12}$$

where $C_5 = K.e^{\omega T_0}NT_0,\ N = [\tilde{M}K_1(\delta_1 + L^2L_hT_0) + \tilde{M}K_2]$ and

$$\|I_6\| = \int_{t_1}^{t_2} \|S(t_2 - s)\|\|f(s, x(s), x[h(x(s), s)])\|ds \leq C_6(t_2 - t_1), \tag{3.13}$$

where $C_6 = \tilde{M}N,\ N = [\tilde{M}K_1(\delta_1 + L^2L_hT_0) + \tilde{M}K_2]$.
We use the inequalities (3.8), (3.9), (3.10), (3.11), (3.12) and (3.13)) in inequality (3.7) and get the following inequality

$$\|(Fx)(t_2) - (Fx)(t_1)\| \leq L|t_2 - t_1|, \tag{3.14}$$

where $L = C_1+C_2+C_3+C_4+C_5+C_6$. Hence $Fx \in C_L([0, T_0], X)$ for any $x \in C_L([0, T_0], X)$
Our next task is to prove that $F : W \to W$. Now for $t \in (0, T_0]$ and $x \in W$, we have

$$\begin{aligned}\|(Fx)(t)\| &\leq \|C(t)x_0\| + \|S(t)y_0\| + \int_0^t \|S(t - s)\|\|B\|\|u(s)\|ds \\ &+ \int_0^t \|S(t - s)\|\|f(s, x(s), x[h(x(s), s)])\|ds \\ &\leq \|C(t)x_0\| + \|S(t)y_0\| + \tilde{M}K_3PT_0 + \tilde{M}K_1T_0(\delta_1 + L^2L_hT_0) + \tilde{M}K_2T_0.\end{aligned}$$

Thus, we get $\|(Fx)\|_{C([0,T_0],X)} \leq \delta_1$.

The feedback control (3.1) transfers the system (1.1) from the initial state to the final state provided that the mapping F has a fixed point. So if the mapping F has an unique

fixed point then the system (1.1) is exactly controllable.
For any x, y in $C_L([0,T_0],X)$, we have

$$\begin{aligned}
&\|(Fx)(t)-(Fy)(t)\| \\
&\leq \frac{1}{\delta}\tilde{M}^2K_3^2T_0\left[\int_0^T \tilde{M}\Big[K_1(\|x(t)-y(t)\|+\|x[h(x(s),s)]-y[h(y(s),s)]\|)\Big]ds\right] \\
&\quad+\int_0^t \tilde{M}\Big[K_1(\|x(t)-y(t)\|+\|x[h(x(s),s)]-y[h(y(s),s)]\|)\Big]ds.
\end{aligned}$$

Thus we have,

$$\|(Fx)-(Fy)\|_{C([0,T_0],X)} \leq \lambda\|x-y\|_{C([0,T_0],X)},$$

where $\lambda = \Big[\tilde{M}K_1T_0(1+\frac{1}{\delta}\tilde{M}^2K_3^2T_0)(2+LL_h)\Big]$. We choose T_0 in such a way that $\lambda < 1$. Hence, F is a contraction mapping. Therefore, F has a unique fixed point $x(.)$ in $C_L([0,T_0],X)$ which is the mild solution of the equation (1.1). □

4. Controllability of integro-differential equations

In this section, we consider a control system represented by an integro-differential equation in the Hilbert space X:

$$\begin{aligned}
x''(t) &= Ax(t)+Bu(t)+f(t,x(t),x[h(x(t),t)]) \\
&\quad+\int_0^t k(t-s)g(s,x(s))ds, \quad t\in J=(0,T], \\
x(0) &= x_0, \qquad x'(0)=y_0,
\end{aligned} \tag{4.15}$$

where x is the state function, $u(.) \in L^2(J,U)$ is the control function, U is the control space, A is the infinitesimal generator of a strongly continuous cosine family of bounded linear operators $(C(t))_{t\in\mathbb{R}}$ on X, $B : U \to X$ is a bounded linear operator. In order to prove the controllability of the integro-differential equation (4.15), we need the following conditions:
(A4.) $\kappa_T = \int_0^t |\kappa(s)|ds$.
(A5.) $\|g(t,u(t))-g(t,v(t))\| \leq L_g\|u(t)-v(t)\|$, where L_g is a positive number.

Theorem 4.2. *If $x_0 \in D(A)$, $y_0 \in E$ and all the conditions* (**A1**)*-*(**A5**) *are satisfied then the problem (4.15) is locally exactly controllable on* $[0,T_0]$.

Proof. The proof of the theorem is the consequence of the theorem (3.1). □

5. Nonlocal problems

The nonlocal condition is a generalization of the classical initial condition. Nonlocal conditions are more realistic than the classical initial conditions because they appear in many physical systems. The first results concerning the existence and uniqueness of mild solutions to the Cauchy problems with nonlocal conditions were studied Byszewski [5].

We consider the following second order nonlocal differential problem with deviated argument in a Hilbert space X :

$$\begin{aligned}
x''(t) &= Ax(t)+Bu(t)+f(t,x(t),x[h(x(t),t)]), \quad t\in J=(0,T], \\
x(0) &= x_0+p(x), \qquad x'(0)=y_0+q(x).
\end{aligned} \tag{5.16}$$

Definition 5.3. A function $x(.) \in C_L([0,T],X)$ is called a mild solution of the nonlocal control problem (5.16) if $x(t)$ is the solution of the following integral equation

$$\begin{aligned} x(t) &= C(t)(x_0+p(x))+S(t)(y_0+q(x)) \\ &+ \int_0^t S(t-s)[Bu(s)+f(s,x(s),x[h(x(s),s)])]ds. \end{aligned}$$

We need the following assumptions on the functions f, p and q to show the exact controllability of the system (5.16)

(A6): The function $f : J \times X \times X \to X$ holds the following conditions:
(i)The function $f(t,.) : X \times X \to X$ is continuous $a.e. \, t \in J$.
(ii)The function $f(.,x,y) : J \to X$ is strongly measurable for each $(x,y) \in X \times X$.
(iii)There exists a constant c_f such that

$$\|f(t,x_1,y_1)-f(t,x_2,y_2)\| \le \tilde{c_f}(\|x_1-x_2\|+\|y_1-y_2\|)$$

for every $x_1,x_2,y_1,y_2 \in X$.

(A7): The functions $p,q : C(J,X) \to X$ are continuous and there exist positive constants c_p and c_q such that

$$\begin{aligned} \|p(x_1)-p(x_2)\| &\le \tilde{c_p}\|x_1-x_2\|, \\ \|q(x_1)-q(x_2)\| &\le \tilde{c_q}\|x_1-x_2\|. \end{aligned}$$

Theorem 5.3. *Let $x_0 \in D(A)$, $y_0 \in E$. If the assumptions* (**A2**)*-*(**A3**) *and* (**A6**)*-*(**A7**) *are satisfied, then the second order nonlocal control system (5.16) is locally exactly controllable on* $[0,T_0]$.

Proof. Define the feedback control function for the nonlocal problem (5.16) as

$$\begin{aligned} u(t) &= B^*S^*(T_0-s)(\Gamma_0^{T_0})^{-1}\Big[x_{T_0}-C(T_0)(x_0+p(x))-S(T_0)(y_0+q(x)) \\ &- \int_0^{T_0} S(T_0-s)f(s,x(s),x[h(x(s),s)])ds\Big]. \end{aligned}$$

The proof of this theorem is the consequence of the theorem (3.1) in the previous section. □

Remark 5.1. By using the similar technique as in theorem (3.1), we can prove the exact controllability result for the following nonlocal integro-differential equation

$$\begin{aligned} x''(t) &= Ax(t)+Bu(t)+f(t,x(t),x[h(x(t),t)]) \\ &+ \int_0^t k(t-s)g(s,x(s))ds, \quad t \in J=(0,T], \\ x(0) &= x_0+p(x), \qquad x'(0)=y_0+q(x). \end{aligned} \tag{5.17}$$

6. TRAJECTORY CONTROLLABILITY

In exact controllability we steer the system from initial state to the desired final state without concern of path or trajectory but practically it may be desirable to steer the system from initial state to the desired final state along a prescribed trajectory. It may optimize certain factors involved in the steering system and leads to the motivation for the study of T-controllability.

Definition 6.4. Let $\mathfrak{T}$ be the set of all functions $z(.)$ defined on $J := [0,T]$ which are twice continuously differentiable such that $z(0)=x_0$, and $z'(0)=y_0$. The system (1.1) is called Trajectory controllable (T-controllable) if for any $z \in \mathfrak{T}$ there exists a control u such that the corresponding solution $x(.)$ of equation (1.1) satisfies $x(t)=z(t)$ almost everywhere. We call $\mathfrak{T}$ the set of all feasible trajectories for the system (1.1).

Theorem 6.4. *If the bounded linear operator B has left inverse, then the second order nonlinear control system (1.1) is T-controllable on J.*

Proof. Let $z(t)$ be a given trajectory in $\mathfrak{T}$. We look for a control function $u(t)$ satisfying

$$z(t) - C(t)x_0 - S(t)y_0 - \int_0^t S(t-s)f(s, z(s), z[h(z(s), s)])ds = \int_0^t S(t-s)Bu(s)ds.$$

Differentiating both sides with respect to t, we get

$$z'(t) - AS(t)x_0 - C(t)y_0 - \int_0^t C(t-s)f(s, z(s), z[h(z(s), s)])ds = \int_0^t C(t-s)Bu(s)ds.$$

Again differentiating both sides with respect to t, we get

$$\begin{aligned} z''(t) - AC(t)x_0 \quad &- \quad AS(t)y_0 - \int_0^t AS(t-s)f(s, z(s), z[h(z(s), s)])ds \\ &- \quad f(t, z(t), z[h(z(t), t)]) = \int_0^t AS(t-s)Bu(s)ds + Bu(t). \end{aligned} \tag{6.18}$$

Equation (6.18) can be rewritten in the form given below

$$\tilde{y}(t) = \int_0^t k(t,s)\tilde{y}(s)ds + \tilde{y}_0(t), \tag{6.19}$$

where $\tilde{y}(t) = Bu(t),\ k(t,s) = -AS(t-s)$ and $\tilde{y}_0(t)$ is the left hand side of (6.18). Define an operator $\tilde{K} : L^2(J, H) \to L^2(J, H)$ by

$$(\tilde{K}\tilde{y})(t) = \int_0^t k(t,s)\tilde{y}(s)ds. \tag{6.20}$$

It is not difficult to prove that $\tilde{K}$ is a bounded linear operator. Also it can be proved that $\tilde{K}^n$ is a contraction mapping for large n. Hence by generalized Banach principle, there exists a unique solution $\tilde{y}(.)$ for the equation (6.19) for given $\tilde{y}_0 \in L^2(J, H)$. Therefore T-controllability follows if we can extract $u(t)$ from the relation

$$Bu(t) = \tilde{y}(t).$$

It can be done by taking left inverse of B. □

7. Application

Example 7.1. Let $X = L^2(0,1)$. We consider the control system governed by the following partial differential equations with deviated argument,

$$\begin{cases} \partial_{tt}H(t,y) = \partial_{yy}H(t,y) + f_2(y, H(t,y)), +f_3(t, y, H(t,y)) \\ \qquad\qquad +b(y)W_1(t,y), \quad y \in (0,1),\ t > 0, \\ \ H(t,0) = H(t,1) = 0,\ t \in J := [0,T],\ 0 < T < \infty, \\ \ H(0,y) = x_0,\ y \in (0,1), \\ \ \partial_t H(0,y) = y_0,\ y \in (0,1), \end{cases} \tag{7.21}$$

where

$$f_3(y, H(t,y)) = \int_0^y \bar{K}(y,s)H(s, h(t)(a_1|H(t,s)| + b_1|H(t,s)|))ds.$$

We assume that $a_1, b_1 \geq 0, (a_1, b_1) \neq (0,0), h : [0,T] \to \mathbb{R}_+$ is locally Hölder continuous in t with $h(0) = 0$ and $\bar{K} : [0,1] \times [0,1] \to \mathbb{R}, b \in X$

We define an operator A, as follows,

$$Ax = x'' \quad \text{with} \quad x \in D(A) = \{x \in H_0^1(0,1) \cap H^2(0,1) : \; x'' \in X\}. \tag{7.22}$$

Here, clearly the operator A is the infinitesimal generator of a strongly continuous cosine family of operators on X. Let $B \in L(U, X)$ be defined by

$$Bu(t)(y) = b(y)W_1(t,y), \;\; 0 < y < 1, \;\; b(y) \in L^2(0,1).$$

The equation (7.21) can be reformulated as the following abstract equation in $X = L^2(0,1)$:

$$\begin{aligned} x'' &= Ax(t) + Bu(t) + f(t, x(t), x[h(x(t),t)]), \quad t > 0, \\ x(0) &= x_0, \;\; x'(0) = y_0, \end{aligned}$$

where $x(t) = H(t,.)$ that is $x(t)(y) = H(t,y)$, $y \in (0,1)$. The operator A is same as in equation (7.22).

The function $f : [0,T] \times X \times X \to X$, is given by

$$f(t,\psi,\xi)(y) = f_2(y,\xi) + f_3(t,y,\psi),$$

where $f_2 : [0,1] \times X \to H_0^1(0,1)$ is given by

$$f_2(y,\xi) = \int_0^y \bar{K}(y,x)\xi(x)dx,$$

and

$$\|f_3(t,y,\psi)\| \le Q(t,y)(1 + \|\psi\|_{H^2(0,1)})$$

with $Q(t,y) \in X$ and Q is continuous in its first argument. For more details see [13]. Thus, the theorem (3.1) can be applied to the problem (7.21).

We can choose the functions $p(x)$ and $q(x)$ as given below

$$p(x) = \sum_{k=1}^{n} c_k \, x(t_k), \;\; t_k \in J \;\text{ for all } n \in \mathbb{N},$$

$$q(x) = \sum_{k=1}^{n} d_k \, x(t_k), \;\; t_k \in J \;\text{ for all } n \in \mathbb{N},$$

where c_k and d_k are constants.

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[1,2]INDIAN INSTITUTE OF TECHNOLOGY MANDI
SCHOOL OF BASIC SCIENCES
175 005, KAMAND (H.P.), INDIA
Email address: malikiisc@gmail.com
Email address: soni.iitkgp@gmail.com

[3]TEXAS A AND M UNIVERSITY- KINGSVILLE
DEPARTMENT OF MATHEMATICS
700 UNIVERSITY BLVD., MSC 172 KINGSVILLE, TX 78363-8202 USA
Email address: Ravi.Agarwal@tamuk.edu