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F-Indices and its coindices of some classes of graphs

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ABSTRACT. In this paper, first we investigate the basic properties of the F-index and its coindex of graph. Next we obtain the exact expression of F-indices and its coindices for bridge graph, chain graph and transformation of graph. Using some of these results, we have obtained the value of these indices for some important classes of chemical graphs.

1. INTRODUCTION

A chemical graph is a graph whose vertices denote atoms and edges denote bonds between those atoms of any underlying chemical structure. A topological index for a (chemical) graph G is a numerical quantity invariant under automorphisms of G and it does not depend on the labeling or pictorial representation of the graph. Topological indices and graph invariants based on the distances between vertices of a graph or vertex degrees are widely used for characterizing molecular graphs, establishing relationships between structure and properties of molecules, predicting biological activity of chemical compounds, and making their chemical applications. These indices may be used to derive quantitative structure-property or structure-activity relationships (QSPR/QSAR).

For a (molecular) graph G, the *first Zagreb index* $M_1(G)$ is the equal to the sum of the squares of the degrees of the vertices, and the *second Zagreb index* $M_2(G)$ is the equal to the sum of the products of the degrees of pairs of adjacent vertices, that is, $M_1(G) = \sum\limits_{u\in V(G)} d_G^2(u) = \sum\limits_{uv\in E(G)} (d_G(u)+d_G(v))$, $M_2(G) = \sum\limits_{uv\in E(G)} d_G(u)d_G(v)$. The *first* and *second Zagreb coindices* were first introduced by Ashrafi et al. [1]. They are defined as follows: $\overline{M}_1(G) = \sum\limits_{uv\notin E(G)} (d_G(u)+d_G(v))$, $\overline{M}_2(G) = \sum\limits_{uv\notin E(G)} d_G(u)d_G(v)$.

The *forgotten topological index*or *F-index* was introduced by Furtula and Gutman [5], and it is defined as $F = F(G) = \sum\limits_{u\in V(G)} d_G^3(u) = \sum\limits_{uv\in E(G)} (d_G^2(u)+d_G^2(v))$. In this sequence, the *forgotten topological coindex* or *F-coindex* is defined as $\overline{F}(G) = \sum\limits_{uv\notin E(G)} (d_G^2(u)+d_G^2(v))$. Khalifeh et al. [9] obtained the first and second Zagreb indices of the Cartesian, join, composition, disjunction and symmetric difference of two graphs. Ashrafi et al. [1] obtained the first and second Zagreb coindices of the Cartesian, join, composition, disjunction and symmetric difference of two graphs. Some topological indices of bridge and chain graphs have been computed previously [3, 10, 12]. The properties and several invariants of the transformation graphs are discussed in [7, 16, 17]. Zagreb indeices and its coindices some composite graphs are given in [13]. Some lower bounds for Zagreb index and its application are discussed in [15]. Zagreb indices of unicyclic graphs with given degree sequences are presented in [11]. Zagreb indices of four new sums of graphs are given in [2]. Some more detail in Zagreb indices see [14, 8, 18, 6, 4]. Furtula and Gutman [5], established a

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few basic properties of the F-index and show that it can significantly enhance the physico-chemical applicability of the first Zagreb index. In this paper, first we investigate the basic properties of the F-index and its coindex of graph. Next we obtain the exact expression of F-indices and its coindices for bridge graph, chain graph and transformation of graph. Using some of these results, we have obtained the value of these indices for some important classes of chemical graphs.

2. Basic properties

Let G be a graph on n vertices and m edges. The complement of G, denoted by $\overline{G}$, is a simple graph on the same set of vertices of G in which two vertices u and v are adjacent in $\overline{G}$ if and only if they are nonadjacent in G. Obviously, $E(G) \cup E(\overline{G}) = E(K_n)$ and $\overline{m} = \left|E(\overline{G})\right| = \frac{n(n-1)}{2} - m$. The degree of a vertex v in G is denoted by $d_G(v)$; the degree of the same vertex in $\overline{G}$ is given by $d_{\overline{G}}(v) = n-1-d_G(v)$. Let P_n and C_n be the path and cycle on n vertices, respectively. Then $M_1(P_n) = 4n-6$,$M_1(C_n) = 4n$,$F(P_n) = 8n-14$ and $F(C_n) = 8n$.

Lemma 2.1. *Let G be a connected graph with n vertices and m edges. Then $F(\overline{G}) = n(n-1)^3 - F(G) - 6(n-1)^2m + 3(n-1)M_1(G)$.*

Proof.

$$\begin{aligned} F(\overline{G}) &= \sum_{u\in V(\overline{G})} d^3_{\overline{G}}(u) = \sum_{u\in V(G)} \Big(n-1-d_G(u)\Big)^3 \\ &= \sum_{u\in V(G)} \Big((n-1)^3 - d^3_G(u) - 3(n-1)^2 d_G(u) + 3(n-1)d^2_G(u)\Big) \\ &= n(n-1)^3 - F(G) - 6(n-1)^2m + 3(n-1)M_1(G). \end{aligned}$$

□

Lemma 2.2. *Let G be a connected graph with n vertices and m edges. Then $\overline{F}(G) = (n-1)M_1(G) - F(G)$.*

Proof.

$$\begin{aligned} \overline{F}(G) &= \sum_{uv\notin E(G)} \Big(d^2_G(u) + d^2_G(v)\Big) \\ &= \sum_{uv\notin E(G)} \Big((n-1-d_G(u)-(n-1))^2 + (n-1-d_G(v)-(n-1))^2\Big) \\ &= \sum_{uv\in E(\overline{G})} \Big((d_{\overline{G}}(u)-(n-1))^2 + (d_{\overline{G}}(v)-(n-1))^2\Big) \\ &= \sum_{uv\in E(\overline{G})} \Big(d^2_{\overline{G}}(u) + (n-1)^2 - 2(n-1)d_{\overline{G}}(u) + d^2_{\overline{G}}(v) + (n-1)^2 - 2(n-1)d_{\overline{G}}(v)\Big) \\ &= \sum_{uv\in E(\overline{G})} \Big(d^2_{\overline{G}}(u) + d^2_{\overline{G}}(v)\Big) + 2(n-1)^2 \sum_{uv\in E(\overline{G})} (1) - 2(n-1) \sum_{uv\in E(\overline{G})} \Big(d_{\overline{G}}(u) + d_{\overline{G}}(v)\Big) \\ &= F(\overline{G}) - 2(n-1)M_1(\overline{G}) + 2(n-1)^2\overline{m}. \end{aligned}$$

By Lemma 2.1 and the expression $M_1(\overline{G}) = M_1(G) + 2(n-1)(\overline{m} - m)$ we obtain:
$\overline{F}(G) = (n-1)M_1(G) - F(G)$. □

3. Bridge graphs

In this section, we compute the forgotten topological index and coindex of the bridge graphs.

Type-I Let $\{G_i\}_{i=1}^n$ be a set of finite pairwise disjoint graphs with distinct vertices $v_i \in V(G_i)$. The *bridge graph* $H_1 = B(G_1, G_2, \dots, G_n; v_1, v_2, \dots, v_n)$ of $\{G_i\}_{i=1}^n$ with respect to the vertices $\{v_i\}_{i=1}^n$ is the graph obtained from the graphs $G_1, G_2, \dots, G_n$ by connecting the vertices v_i and v_{i+1} by an edge for all $i = 1, 2, \dots, n-1$, see Fig.1. The following lemma is easily follows from the structure of the graph H_1.

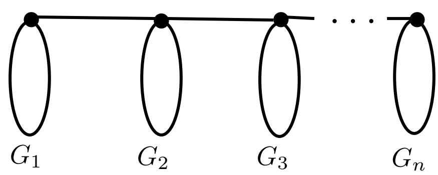

Fig.1. The bridge graph H_1

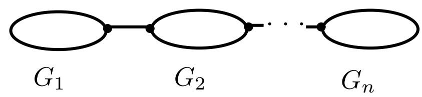

Fig.2. The bridge graph H_2

Lemma 3.3. *Let $H_1 = B(G_1, G_2, \dots, G_n; v_1, v_2, \dots, v_n)$, $n \geq 2$ be a bridge graph. Then the degree of an arbitrary vertex u in H_1 is given by* $d_{H_1}(u) = \begin{cases} d_{G_i}(u) \ if \ u \in V(G_i) - \{v_i\}, \ 1 \leq i \leq n, \\ d_{G_i}(v_i) + 1 \ if \ u = v_i, \ i = 1 \ and \ i = n, \\ d_{G_i}(v_i) + 2 \ if \ u = v_i, \ 2 \leq i \leq n-1. \end{cases}$

Theorem 3.1. *Let H_1 be a bridge graph. Then $F(H_1) = \sum\limits_{i=1}^n F(G_i) + 6\sum\limits_{i=1}^n r_i^2 + 12\sum\limits_{i=1}^n r_i - 3r_1(r_1+3) - 3r_n(r_n+3) + 8n - 14$, where $r_i = d_{G_i}(v_i)$ for $1 \leq i \leq n$.*

Proof. By Lemma 3.3, we obtain

$$F(H_1) = \sum_{i=1}^n \sum_{u \in V(G_i)-\{v_i\}} d_{G_i}^3(u) + (d_{G_1}(v_1)+1)^3 + \sum_{i=2}^{n-1}(d_{G_i}(v_i)+2)^3 + (d_{G_n}(v_n)+1)^3$$

$$\begin{aligned}
&= \sum_{i=1}^n \Big(F(G_i) - d_{G_i}^3(v_i)\Big) + \sum_{i=2}^{n-1}\Big(d_{G_i}^3(v_i) + 6d_{G_i}^2(v_i) + 12d_{G_i}(v_i) + 8\Big) \\
&\quad + \Big(d_{G_1}^3(v_1) + 3d_{G_1}^2(v_1) + 3d_{G_1}(v_1) + 1\Big) + \Big(d_{G_n}^3(v_n) + 3d_{G_n}^2(v_n) + 3d_{G_n}(v_n) + 1\Big) \\
&= \sum_{i=1}^n F(G_i) + 3\Big(2\sum_{i=1}^n \Big(d_{G_i}^2(v_i) - d_{G_1}^2(v_1) - d_{G_n}^2(v_n)\Big) \\
&\quad + 3\Big(4\sum_{i=1}^n d_{G_i}(v_i) - 3d_{G_1}(v_1) - 3d_{G_n}(v_n)\Big) + 8n - 14 \\
&= \sum_{i=1}^n F(G_i) + 6\sum_{i=1}^n r_i^2 + 12\sum_{i=1}^n r_i - 3r_1(r_1+3) - 3r_n(r_n+3) + 8n - 14.
\end{aligned}$$

□

For $G_i = G$ and $v_i = v$ for all $i = 1, 2, \dots, n$, in Theorem 3.1, we have the following corollary.

Corollary 3.1. *The F- index of $H_1 = B(G, G, \dots, G; v, v, \dots, v)$, $(n \geq 2 \ times)$ is $F(H_1) = nF(G) + 6r^2(n-1) + 6r(2n-3) + 8n - 14$.*

Theorem 3.2. *Let H_1 be a bridge graph. Then $\overline{F}(H_1) = (n-1)\sum\limits_{i=1}^n M_1(G_i) - \sum\limits_{i=1}^n F(G_i) - 6\sum\limits_{i=1}^n r_i^2 + 4(n-4)\sum\limits_{i=1}^n r_i + 3(r_1^2 + r_n^2) - (2n-11)(r_1+r_n) + 2(2n^2 - 9n + 10)$.*

Proof. The formula follows from Lemma 2.2, Theorem 3.1 and the expression $M_1(H_1) = \Big(\sum\limits_{i=1}^{n} M_1(G_i) + 4\sum\limits_{i=2}^{n} r_i\Big) + 2(r_1 + r_n + 2n - 3)$, where $r_i = d_{G_i}(v_i)$ for $1 \leq i \leq d$. □

The following corollary follows from Theorem 3.2 and $M_1(B(G, G, \dots, G; v, v, \dots, v)) = nM_1(G) + 4r(n-1) + 4n - 6,\ n \geq 2$.

Corollary 3.2. *If* $H_1 = B(G, G, \dots, G; v, v, \dots, v),\ (n \geq 2\ times)$, *then* $\overline{F}(H_1) = n(n-1)M_1(G) - nF(G) - 6r^2(n-1) + 4(n-4)rn - 2(2n-11)r + 2(2n^2 - 9n + 10)$.

Type-II The *bridge graph* $H_2 = B(G_1, G_2, \dots, G_n; u_1, v_1, u_2, v_2, \dots, u_n, v_n)$ of $\{G_i\}_{i=1}^{n}$ with respect to the vertices $\{u_i, v_i\}_{i=1}^{n}$ is the graph obtained from the graphs $G_1, G_2, \dots, G_n$ by connecting the vertices u_i and v_{i+1} by an edge for all $i = 1, 2, \dots, n-1$, see Fig.2. The following lemma is follows from the structure of the graph H_2.

Lemma 3.4. *Let* $H_2 = B(G_1, G_2, \dots, G_n; u_1, v_1, u_2, v_2, \dots, u_n, v_n),\ n \geq 2$ *be a bridge graph.*

Then $d_{H_2}(x) = \begin{cases} d_{G_i}(x)\ if\ x \in V(G_i) - \{v_i, u_i\},\ 2 \leq i \leq n-1, \\ d_{G_1}(x)\ if\ x \in V(G_1) - \{u_1\} \\ d_{G_n}(x)\ if\ x \in V(G_n) - \{v_n\} \\ d_{G_i}(u_i) + 1\ if\ x = u_i,\ i = 1 \leq i \leq n-1, \\ d_{G_i}(v_i) + 2\ if\ x = v_i,\ 2 \leq i \leq n. \end{cases}$

Theorem 3.3. *Let* $H_2 = B(G_1, G_2, \dots, G_n; u_1, v_1, u_2, v_2, \dots, u_n, v_n),\ n \geq 2$ *be a bridge graph. Then* $F(H_2) = \sum\limits_{i=1}^{n} F(G_i) + 3\sum\limits_{i=1}^{n-1}\Big(r_i^2 + r_i\Big) + 3\sum\limits_{i=2}^{n}\Big(h_i^2 + h_i\Big) + 2(n-1)$, *where* $r_i = d_{G_i}(u_i)$ *and* $h_i = d_{G_i}(v_i)$ *for* $1 \leq i \leq n$.

Proof. By Lemma 3.4, we obtain

$$\begin{aligned} F(H_2) &= \sum_{i=2}^{n-1}\ \sum_{x \in V(G_i)-\{u_i, v_i\}} d_{G_i}^3(x) + \sum_{x \in V(G_1)-\{u_1\}} d_{G_1}^3(x) + \sum_{x \in V(G_n)-\{v_n\}} d_{G_n}^3(x) \\ &\quad + \sum_{i=1}^{n-1}(d_{G_i}(u_i) + 1)^3 + \sum_{i=2}^{n}(d_{G_i}(v_i) + 1)^3 \\ &= \sum_{i=2}^{n-1}\Big(F(G_i) - d_{G_i}^3(v_i) - d_{G_i}^3(u_i)\Big) + \Big(F(G_1) - d_{G_1}^3(u_i)\Big) + \Big(F(G_n) - d_{G_i}^3(v_n)\Big) \\ &\quad + \sum_{i=1}^{n-1}\Big(d_{G_i}^3(u_i) + 3d_{G_i}^2(u_i) + 3d_{G_i}(u_i) + 1\Big) + \sum_{i=2}^{n}\Big(d_{G_i}^3(v_i) + 3d_{G_i}^2(v_i) + 3d_{G_i}(v_i) + 1\Big) \\ &= \sum_{i=1}^{n} F(G_i) + 3\sum_{i=1}^{n-1}\Big(r_i^2 + r_i\Big) + 3\sum_{i=2}^{n}\Big(h_i^2 + h_i\Big) + 2(n-1). \end{aligned}$$

□

For $G_i = G$, $u_i = u\ v_i = v$ and for all $i = 1, 2, \dots, n$, in Theorem 3.3, we have the following corollary.

Corollary 3.3. *The F- index of* H_2 *is* $F(H_2) = nF(G) + 3(n-1)(r^2 + r) + 3(n-1)(h^2 + h) + 2(n-1)$.

Theorem 3.4. *If* H_2 *is a bridge graph, then* $\overline{F}(H_2) = (n-1)\sum\limits_{i=1}^{n} M_1(G_i) - \sum\limits_{i=1}^{n} F(G_i) - 3\Big(\sum\limits_{i=1}^{n-1} r_i^2 + \sum\limits_{i=2}^{n} h_i^2\Big) + (2n-5)\Big(\sum\limits_{i=1}^{n-1} r_i + \sum\limits_{i=2}^{n} h_i\Big) + 2(n^2 - 3n + 2)$.

Proof. The formula follows from Lemma 2.2, Theorem 3.3 and the expression $M_1(H_2) = \left(\sum_{i=1}^{n} M_1(G_i) + 2\sum_{i=1}^{n-1} r_i + 2\sum_{i=2}^{n} h_i\right) + 2(n-1)$. □

The following corollary follows from Theorem 3.4 and $M_1(B(G, G, \ldots, G; u, v, u, v, \ldots, u, v)) = nM_1(G) + 2(n-1)(r+h+1),\ n \geq 2$.

Corollary 3.4. $\overline{F}(H_2) = n(n-1)M_1(G) - nF(G) - 3(n-1)(r^2+h^2) + (2n-5)(n-1)(r+h) + 2(n^2-3n+2)$.

Example 3.1. An internal hexagon H in a polyphenyl chain is said to be an ortho-hexagon, meta-hexagon and para-hexagon, respectively, if two vertices of H incident with two edges which connect other two hexagons are in ortho-,met-, and para-position. A polyphenyl chain of h hexagons is ortho-PPC_h and is denoted by O_h if all its internal hexagons are ortho-hexagons. Similarly, we define meta-PPC_h (denoted by M_h) and para-PPC_h (denoted by L_h), see Fig.3. The polyphenyl chain O_h, M_h and L_h are the bridge graphs $B(C_6, c_6, \ldots, C_6, u, v, u, v, \ldots, u, v)(h\ times)$. By Corollaries 3.1,3.2,3.3 and 3.4 we have $F(O_h) = 104h - 74, F(M_h) = F(L_h) = 86h - 38$, $\overline{F}(O_h) = 36h^2 - 162h + 88$ and $\overline{F}(M_h) = \overline{F}(L_h) = 34h^2 - 100h + 9$.

Example 3.2. Consider the square comb lattice graph $C_q(N)$ with open ends, where $N = n^2$ is the number of its vertices, see Fig.4. Clearly, the graph $C_q(N)$ is the bridge graph $B(P_n, P_n, \ldots, P_n, u, u, \ldots, u)(n\ times)$. By corralaries 3.1 and 3.2, $F(C_q(N)) = 8n^2 + 12n - 38$ and $\overline{F}(C_q(N)) = 4n^3 - 10n^2 - 24n + 48$.

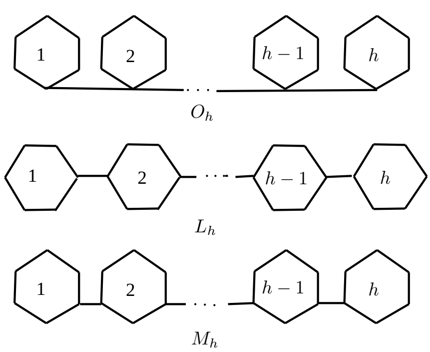

Fig.3. Ortho-,para-, and meta-polyphenyl chains with six hexagons.

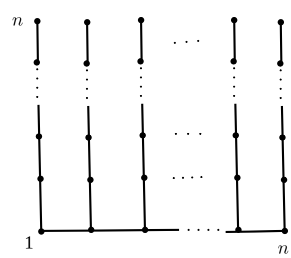

Fig.4. The square comb lattice graph with $N = n^2$ vertices

Example 3.3. Let D_n be the molecular graph of the nanostar dendrimer. One can see that the graph D_n is the bridge graph $B(G, G, \ldots, G; u, v, u, v, \ldots, u, v)$ $(n\ times)$, where G is the graph depicted in Fig.5. Then by Corollaries 3.3 and 3.4, $F(D_n) = 266n - 38$ and $\overline{F}(D_n) = 106n^2 - 382n + 48$.

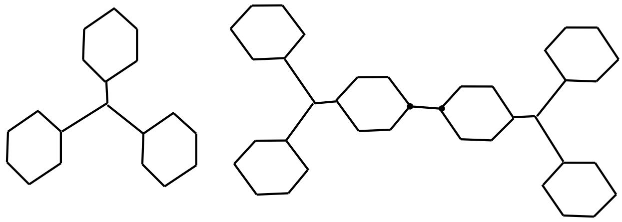

Fig.5. The graph of nanostar dendrimer D_n for $n = 1, 2$.

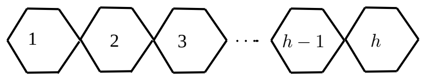

Fig.6. The spiro-chain of C_6

4. Chain graph

In this section, we compute the forgotten topological index and coindex of the chain graph. The *chain graph* $C = C(G_1, G_2, \ldots, G_n; u_1, v_1, u_2, v_2, \ldots, u_n, v_n)$ of $\{G_i\}_{i=1}^n$ with respect to the vertices $\{u_i, v_i\}_{i=1}^n$ is the graph obtained from the graphs $G_1, G_2, \ldots, G_n$ by identifying the vertices u_i and v_{i+1} for all $i = 1, 2, \ldots, n-1$. The following lemma is easily follows from the structure of the chain graph.

Lemma 4.5. *Let C be a chain graph.*

Then $d_C(x) = \begin{cases} d_{G_i}(x) \; if \; x \in V(G_i) - \{u_i, v_i\}, \; 2 \le i \le n-1, \\ d_{G_1}(x) \; if \; x \in V(G_1) - \{u_1\}, \\ d_{G_n}(x) \; if \; x \in V(G_n) - \{v_n\} \\ d_{G_i}(u_i) + d_{G_i}(v_{i+1}) \; if \; x = u_i = v_{i+1}, \; 1 \le i \le n-1. \end{cases}$

Next we determine the $F-$index of the chain graph.

Theorem 4.5. *Let $C = C(G_1, G_2, \ldots, G_n; u_1, v_1, u_2, v_2, \ldots, u_n, v_n)$, $n \ge 2$ be a chain graph. Then $F(C) = \sum\limits_{i=1}^{n} F(G_i) + 3\sum\limits_{i=1}^{n-1}\Big(r_i^2 h_{i+1} + r_i h_{i+1}^2\Big)$, where $r_i = d_{G_i}(u_i)$ and $h_i = d_{G_i}(v_i)$ for $1 \le i \le n$.*

Proof. By Lemma 4.5, we obtain

$$\begin{aligned} F(C) &= \sum_{i=2}^{n-1} \sum_{x \in V(G_i)-\{u_i,v_i\}} d_{G_i}^3(x) + \sum_{x \in V(G_1)-\{u_1\}} d_{G_1}^3(x) + \sum_{x \in V(G_n)-\{v_n\}} d_{G_n}^3(x) \\ &+ \sum_{i=1}^{n-1} \sum_{x = u_i = v_{i+1}} \Big(d_{G_i}(u_i) + d_{G_i}(v_{i+1})\Big)^3 \\ = & \sum_{i=2}^{n-1} \Big(F(G_i) - d_{G_i}^3(v_i) - d_{G_i}^3(u_i)\Big) + \Big(F(G_1) - d_{G_1}^3(u_i)\Big) + \Big(F(G_n) - d_{G_i}^3(v_n)\Big) \\ &+ \sum_{i=1}^{n-1} \Big(d_{G_i}^3(u_i) + d_{G_i}^3(v_{i+1}) + 3d_{G_i}(u_i)d_{G_i}^2(v_{i+1}) + 3d_{G_i}^2(u_i)d_{G_i}(v_{i+1})\Big) \\ = & \sum_{i=1}^{n} F(G_i) + 3\sum_{i=1}^{n-1}\Big(r_i^2 h_{i+1} + r_i h_{i+1}^2\Big). \end{aligned}$$

□

For $G_i = G$, $u_i = u$ $v_i = v$ and for all $i = 1, 2, \ldots, n$, in Theorem 4.5, we have the following corollary.

Corollary 4.5. *The F- index of the chain graph is $F(C) = nF(G) + 3(n-1)rh(r+h)$.*

Theorem 4.6. *If C is a bridge graph, then $\overline{F}(C) = (n-1)\Big(\sum_{i=1}^{n} M_1(G_i) + 2\sum_{i=1}^{n-1} r_i h_{i+1}\Big) - \sum_{i=1}^{n} F(G_i) - 3\sum_{i=1}^{n-1}\Big(r_i^2 h_{i+1} + r_i h_{i+1}^2\Big)$.*

Proof. The formula follows from Lemma 2.2, Theorem 4.5 and the expression $M_1(C) = \sum_{i=1}^{n} M_1(G_i) + 2\sum_{i=1}^{n-1} r_i h_{i+1}$, where $r_i = d_{G_i}(u_i)$ and $h_i = d_{G_i}(v_i)$ for $1 \le i \le n$. $\square$

The following corollary follows from Theorem 4.6 and $M_1(C(G, G, \ldots, G; u, v, u, v, \ldots, u, v)) = nM_1(G) + 2(n-1)rh,\ n \ge 2$.

Corollary 4.6. $\overline{F}(C) = n(n-1)M_1(G) - nF(G) + (n-1)rh\Big(2(n-1) - 3(r+h)\Big)$.

Example 4.4. The spiro-chain $S_n(C_t(k,\ell))$ of the graph $C_t(k,\ell)$(is the graph C_t, where k and ℓ are the numbers of the vertices u and v, respectively) is the chain graph $C(G, G, \ldots, G; u, v, u, v, \ldots, u, v)$, where $G = C_t(k,\ell)$, see Fig.6. By Corollaries 4.5 and 4.6, we have $F(S_n(C_t(k,\ell))) = 8nt + 48n - 48$ and $\overline{F}(S_n(C_t(k,\ell))) = 4n^2t - 12nt + 8n^2 - 64n - 56$.

5. Transformation graphs

In this section, we compute the forgotten topological indices and coindices of the transformation graphs.

Let G be a graph and x, y, z be three variables taking values $+$ or $-$. The *transformation graph* G^{xyz} is a graph whose vertex set is $V(G) \cup E(G)$ and for $\alpha, \beta \in V(G^{xyz})$ α and β are adjacent in G^{xyz} if and only if
(i) $\alpha, \beta \in V(G)$ α and β are adjacent in G if $x = +$ and α and β are not adjacent in G if $x = -$.
(ii) $\alpha, \beta \in E(G)$ α and β are adjacent in G if $y = +$ and α and β are not adjacent in G if $y = -$.
(iii) $\alpha \in V(G)$ and $\beta \in E(G)$, α and β are adjacent in G if $z = +$ and α and β are not adjacent in G if $z = -$.

Since there are eight distinct 3-permutations of $\{+, -\}$, there are eight different transformations of a given graph G. It is interesting to see that G^{+++} is just the total graph of G whereas G^{---} is the complement of the total graph of G.

The (a,b)-$Zagreb\,index$ of G is defined as $Z'_{a,b}(G) = \frac{1}{2}\sum_{uv \in E(G)}\Big(d_G(u)^a d_G(v)^b + d_G(u)^b d_G(v)^a\Big)$. For a positive integer k, define $N_k(G) = \sum_{uv \in E(G)}\Big(d_G(u)^{k-1} + d_G(v)^{k-1}\Big)$. If $k = 2$ and 3, then $N_2(G) = M_1(G)$ and $N_3(G) = F(G)$.

In this section, we obtain the exact expressions for the F- index and coindex of the above specified transformation graphs.

Theorem 5.7. *Let G be a graph on n vertices and m edges. Then*
$(i) F(G^{+++}) = 8F(G) + N_4(G) + 6Z'_{2,1}(G)$.
$(ii) F(G^{---}) = (3m + 3n - 11)F(G) + 3(m+n-1)\Big((5 - m - n)M_1(G) + 2M_2(G)\Big) - N_4(G) - 6Z'_{2,1}(G) + (m+n-1)^2\Big((m+n)(m+n-1) - 12m\Big)$.

Proof. One can observe that G^{+++} has $m+n$ vertices and $\frac{1}{2}M_1(G)+2m$ edges. For $u \in V(G^{+++}) \cap V(G)$, $d_{G^{+++}}(u) = 2d_G(u)$ and for $u \in V(G^{+++}) \cap E(G)$, $d_{G^{+++}}(u) = d_G(u) + d_G(v)$. Therefore

$$\begin{aligned} F(G^{+++}) &= \sum_{u\in V(G^{+++})\cap V(G)} d^3_{G^{+++}}(u) + \sum_{u\in V(G^{+++})\cap E(G)} d^3_{G^{+++}}(u) \\ &= \sum_{u\in V(G)} (2d_G(u))^3 + \sum_{uv\in E(G)} (d_G(u)+d_G(v))^3 \\ &= 8\sum_{u\in V(G)} d^3_G(u) + \sum_{uv\in E(G)} \Big(d^3_G(u)+d^3_G(v)+3d^2_G(u)d_G(v)+3d_G(u)d^2_G(v)\Big) \\ &= 8F(G) + \sum_{uv\in E(G)} \Big(d^3_G(u)+d^3_G(v)\Big) + 3\sum_{uv\in E(G)} \Big(d^2_G(u)d_G(v)+d_G(u)d^2_G(v)\Big) \\ &= 8F(G) + N_4(G) + 6Z'_{2,1}(G). \end{aligned}$$

Since G^{---} is the complement of G^{+++}. Therefore the second formula follows from Lemma 2.1 and $M_1(G^{+++}) = 4M_1(G) + 2M_2(G) + F(G)$. □

Theorem 5.8. *Let G be a graph on n vertices and m edges. Then*
$(i) F(G^{++-}) = 3(n-4)^2M_1(G) + 6(n-4)M_2(G) + 3(n-4)F(G) + N_4(G) + 6Z'_{2,1}(G) + nm^3 + m(n-4)^3$.
$(ii) F(G^{--+}) = 3(n-4)(2m+n+2)M_1(G) + 3(m+3)\Big(2M_2(G)+F(G)\Big) - N_4(G) - 6Z'_{2,1}(G) - nm^3 - m(n-4)^3 + (m+n-1)\Big(3mn(m+n-8) + 48m - 6(m+n-1)m_1 + (m+n)(m+n-1)^2\Big)$,
where m_1 is the number of edges in G^{--+}.

Proof. Note that G^{++-} has $m+n$ vertices. For $u \in V(G^{++-}) \cap V(G)$, $d_{G^{++-}}(u) = m$ and for $u \in V(G^{++-}) \cap E(G)$, $d_{G^{++-}}(u) = d_G(u) + d_G(v) + n - 4$. Therefore

$$\begin{aligned} F(G^{++-}) &= \sum_{u\in V(G^{++-})\cap V(G)} d^3_{G^{++-}}(u) + \sum_{u\in V(G^{++-})\cap E(G)} d^3_{G^{++-}}(u) \\ &= \sum_{u\in V(G)} m^3 + \sum_{uv\in E(G)} (n-4+d_G(u)+d_G(v))^3 \\ &= nm^3 + \sum_{uv\in E(G)} \Big((n-4)^3 + (d^3_G(u)+d^3_G(v)) + 3(d_G(u)^2d_G(v)+d_G(u)d^2_G(v)) \\ &\quad +3(n-4)(d^2_G(u)+d^2_G(v)) + 6(n-4)(d_G(u)d_G(v)) + 3(n-4)^2(d_G(u)+d_G(v))\Big) \\ &= 3(n-4)^2M_1(G) + 6(n-4)M_2(G) + 3(n-4)N_3(G) + N_4(G) + 6Z'_{2,1}(G) \\ &\quad +nm^3 + m(n-4)^3. \end{aligned}$$

One can see that $G^{--+} \cong \overline{G^{++-}}$. Therefore the second formula follows from Lemma 2.1 and $M_1(G^{++-}) = 2(n-4)M_1(G) + 2M_2(G) + F(G) + mn(m+n-8) + 16m$. □

Theorem 5.9. *Let G be a graph on n vertices and m edges. Then*
$(i) F(G^{-++}) = n(n-1)^3 + N_4(G) + 6Z'_{2,1}(G)$.
$(ii) F(G^{+--}) = 3(m+n-1)(F(G)+2M_2(G)) - N_4(G) - 6Z'_{2,1}(G) - n(n-1)^3 + (m+n-1)\Big(3n(n-1)^2 - 6(m+n-1)m_2 + (n+m)\Big)$, *where m_2 is the number of edges of G^{+--}.*

Proof. Since G^{-++} has $m+n$ vertices. For $u \in V(G^{-++}) \cap V(G)$, $d_{G^{-++}}(u) = n-1$ and for $u \in V(G^{-++}) \cap E(G)$, $d_{G^{-++}}(u) = d_G(u) + d_G(v)$. Similar argument for Theorem 5.7 and using the expression $M_1(G^{-++}) = n(n-1)^2 + F(G) + 2M_2(G)$, we arrive the required results. □

For $u \in V(G^{-+-}) \cap V(G)$, $d_{G^{-+-}}(u) = n+m-1-2d_G(u)$ and for $u \in V(G^{-+-}) \cap E(G)$, $d_{G^{-+-}}(u) = n-4+d_G(u)+d_G(v)$. Similar argument for Theorem 5.8 and using the expression $M_1(G^{-+-}) = 2(n-2)M_1(G) + 2M_2(G) + F(G) + (n+m-1)(n(n+m-1) - 8m) + m(n-4)^2$, we arrive the following results.

Theorem 5.10. *Let G be a graph on n vertices and m edges. Then*
$(i) F(G^{-+-}) = (3n^2+6m-18n+42)M_1(G) + (3n-20)F(G) + 6(n-4)M_2(G) + N_4(G) + 6Z'_{2,1}(G) + m(n-4)^3 + (m+n-1)^2(n(n+m-1)-12m)$.
$(ii) F(G^{+-+}) = (23n-3n^2-3m-49)M_1(G) + (3m+17)F(G) + 6(m+3)M_2(G) - N_4(G) - 6Z'_{2,1}(G) - m(n-4)^3 + (m+n-1)^2(2n(n+m-1)-12m-6m_3) + 3m(m+n-1)(n-4)^2 + (n+m)(n+m-1)^3$, *where m_3 is the number of edges of G^{+-+}.*

Apply Lemma 2.2 and Theorems 5.7 to 5.10 we can deduce expressions for the forgotten topological coindex of the transformation graphs G^{xyz}.

Corollary 5.7. *Let G be a graph on n vertices and m edges. Then*
$(i) \overline{F}(G^{+++}) = (m+n-1)\Big(M_1(G) + 2M_2(G)\Big) + (m+n-9)F(G) - N_4(G) - 6Z'_{2,1}(G)$.

$(ii) \overline{F}(G^{---}) = (m+n-1)(m+n-9)M_1(G) - 4(m+n+2)M_2(G) - 2(m+n-5)F(G) + N_4(G) + 6Z'_{2,1}(G) + (m+n-1)\Big((m+n)\big((m+n)^2-10m-2n+1\big) + 8m - (m+n-1)(m^2+n^2+2mn-n-13m)\Big)$.

$(iii) \overline{F}(G^{++-}) = (14n-n^2+2mn-8m-40)M_1(G) + 2(m-2n+11)M_2(G) + (m-2n+11)F(G) - N_4(G) - 6Z'_{2,1}(G) + (n+m-1)\Big(mn(m+n-8)+16m\Big) - nm^3 - m(n-4)^3$.

$(iv) \overline{F}(G^{--+}) = (20m-3n^2-2m^2-8mn+30)M_1(G) + (n-2m-10)(2M_2(G)+F(G)) + N_4(G) + 6Z'_{2,1}(G) - (n+m-1)\Big(3mn(m+n-8) + 48m - 6m_1(m+n-1) + (m+n)(m+n-1)^2 - n(n-1)^2 - m(m+3)^2\Big) + nm^3 + m(n-4)^3$.

$(v) \overline{F}(G^{-++}) = (n+m-1)\Big(F(G) + 2M_2(G)\Big) - N_4(G) - 6Z'_{2,1}(G) + mn(n-1)^2$.

$(vi) \overline{F}(G^{+--}) = N_4(G) + 6Z'_{2,1}(G) - 2(m+n-1)^2M_1(G) - 2(m+n-1)F(G) - 4(m+n-1)M_2(G) + 6(n+m-1)^2m_2 + n(n-1)^2(2-2n-3m) + (m+n-1)(3m^2n+m^3-2m^2+mn^2-2mn-n)$.

$(vii) \overline{F}(G^{-+-}) = (12n-n^2+2mn-10m-38)M_1(G) + 2(m-2n+11)M_2(G) + (m-2n+19)F(G) - N_4(G) - 6Z'_{2,1}(G) + 4(n+m-1)^2m + m(n-4)^2(m+3)$.

$(viii) \overline{F}(G^{+-+}) = (3n^2-2m^2+3m-25n-2mn+51)M_1(G) + 2(n-2m-10)M_2(G) + (n-2m-18)F(G) + N_4(G) + 6Z'_{2,1}(G) + (n+m-1)m(m+3)^2 + (n+m-1)^2(12m+6m_3) - (n+m-1)^3(3n+m) + m(n-4)^2(4-3m-3n)$.

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